Generalized Lyapunov Equations for Stable Singular System

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Abstract

Linear time-invariant singular system $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ is treated. Two generalized Lyapunov equations for the stable system, one for controllability and the other one for observability, are constructed. The sufficient and necessary conditions for the existence of unique, positive definite solutions to the two equations are derived.

Key words: Lyapunov equation, stablility, singular system.

1 Introduction

In this paper, we study the generalized Lyapunov equations of singular system of the form:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where E is an $n \times n$ constant singular matrix.

Many theories of controllability and observability of singular system have been achieved, interested readers can refer to [2]. We can see that in many respects the singular system has common properties with the normal system. As we know for a stable normal system there are Lyapunov equations for controllability and observability respectively and many equivalent results can be achieved through it. But the equations like Lyapunov equations for singular system have not been established yet. In this paper we are to provide generalized Lyapunov equation without discomposing it into fast and slow subsystems.

For a stable normal system of the form

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \\ &y = Cx(t), \end{aligned} \tag{2}$$

there are two Lyapunov equations

$$AX_1 + X_1A^\top + BB^\top = 0, \tag{3}$$

$$AX_2 + X_2A^\top + C^\top C = 0, \tag{4}$$

related to controllability and observability respectively. The relations are that the system (2) is controllable and observable if and only if there exists positive definit solution X_1 and X_2 to (3) and (4) respectively.

One of the focal points of our exploration is to establish generalized Lyapunov equations for a regular, stable singular system without decomposing it. The other point is to establish two symmetric matrices, their positive definiteness is equivalent to the controllability and observability of the system, also the two matrices are solutions to the generalized Lyapunov equations mentioned above.

This paper is organized as follows. First, in section 2, the concepts of regular singular system and basic results are reviewed for convenience. Second, in section 3 we establish generalized Lyapunov equations for the singular system and prove their equivalence to the controllability and observability of the system. This is the main section. Finally, in section 4, we make some comments on the results in this paper.

2 Main Results

In this section we will find generalized Lyapunov equations for stable singular system.

First we consider the controllability of stable singular system.

Proposition 1 *When the singular system (1) is stable, then it is controllable if and only if*

$$rank\left[\sigma I - EA^{-1}, B\right] = n, \quad \sigma \in C. \tag{5}$$

Furthermore, it is observable if and only if

$$rank\left[\sigma I - \left(EA^{-1}\right)^T, C^T\right] = n, \quad \sigma \in C. \tag{6}$$

Proof We only prove the controllability. The other can be done in the same way.

When the singular system (1) is stable, A is a nonsingular matrix. So A^{-1}exists. Then

$$rank\left[\ \sigma I - EA^{-1}, \quad B\ \right] = rank[\sigma E - A, B] = n, \sigma \neq 0,$$

and

$$rank[\sigma I - EA^{-1}, B] = rank[E, B] = n, \sigma = 0.$$

Thus, the result holds. ■

According to the proposition we know that a stable singular system is controllable (observable) if and only if the normal system $\left(I, EA^{-1}, B, C\right)$ is controllable (observable).

By decomposition, we can get

$$EA^{-1} = Qdiag(A_1^{-1}, N)Q^{-1}.$$

This formula is true when the singular system (1) is stable. Then for arbitrary $\sigma > 0$, all the eigenvalues of

$$\left(EA^{-1} - \frac{\sigma}{2}I\right) = Qdiag(A_1^{-1} - \frac{\sigma}{2}I, N - \frac{\sigma}{2}I)Q^{-1},$$

lie in the open left half plane.

Now according to the discussion above, we can give the following result without proof.

Theorem 1 *The singular system (1) is stable if and only if the normal system* $\left(EA^{-1} - \frac{\sigma}{2}I, B\right)$ *is stable for arbitrary* $\sigma > 0$.

For a stable system we know that

$$\begin{aligned} X(\infty) &= \int_0^{+\infty} e^{\left(EA^{-1}-\frac{\sigma}{2}I\right)t} BB^\tau e^{\left(EA^{-1}-\frac{\sigma}{2}I\right)^\tau t} dt \\ &= \int_0^{+\infty} e^{-\sigma t} e^{EA^{-1}t} BB^\tau e^{\left(EA^{-1}\right)^\tau t} dt, \end{aligned}$$

exists (for $\forall\ \sigma > 0$), and it is the solution to

$$(EA^{-1})X + X(EA^{-1})^\tau + BB^\tau = \sigma X.$$

Especially when (1) is controllable and stable the solution is unique.

Now we directly have following statement.

Theorem 2 *If the singular system (1) is stable, the following statements are equivalent:*

(i)The system is controllable.

(ii)

$$X = \int_0^{+\infty} e^{-\sigma t} e^{EA^{-1}t} BB^\tau e^{[EA^{-1}]^\tau t} dt > 0.$$

(iii)There exists $X \in \mathbb{R}^{n\times n}$ such that

$$X > 0, (EA^{-1})X + X(EA^{-1})^\tau + BB^\tau = \sigma X.$$

(iv)

$$rank\left[B, EA^{-1}B, \cdots, \left(EA^{-1}\right)^{n-1} B\right] = n.$$

Now we consider the observability of stable singular system.

Because the observable mode is dual to the controllable mode, we immediately get the following result.

Corollary 1 *If the singular system (1) is stable, the following statements are equivalent:*

(i) The system is observable.

(ii)

$$Y = \int_0^{+\infty} e^{-\sigma t} e^{[EA^{-1}]t} C^\tau C e^{[EA^{-1}]^\tau t} dt > 0.$$

(iii)There exists $Y \in \mathbb{R}^{n\times n}$ such that

$$Y > 0, (EA^{-1})^T Y + Y(EA^{-1}) + C^\tau C = \sigma Y.$$

(iv)

$$rank\left[C^\tau, \left(EA^{-1}\right)^\tau C^\tau, \cdots, \left(\left(EA^{-1}\right)^T\right)^{n-1} C^\tau\right] = n.$$

3 Remarking Conclusion

We have derived generalized Lyapunov equations for stable singular system, and find a sufficient and necessary conditions for the existence of its unique solution. Furthermore, there is somewhere that could be improved. We should notice that if σ is sufficiently small (certainly $\sigma > 0$) all the results mentioned above can be achieved and we need not discuss other $\sigma's$. In fact if we denote $\delta = \min\{\|\mathrm{Re}(\lambda_i)\|,\ \lambda_i$ is eigenvalue of the system$\}$, theorem 2 can be changed into the following form:

Corollary 2 *If the singular system (1) is stable, the following statements are equivalent:*

(i)The system is controllable.

(ii)

$$X = \int_0^{+\infty} e^{-\sigma t} e^{EA^{-1}t} BB^\tau e^{[EA^{-1}]^\tau t} dt > 0, \sigma = \frac{\delta}{2}.$$

(iii)When $\sigma = \frac{\delta}{2}$, there exists $X > 0$ satisfying

$$(EA^{-1})X + X(EA^{-1})^\tau + BB^\tau = \sigma X.$$

(iv)

$$rank\left[B, \left(EA^{-1}\right)B, \cdots, \left(EA^{-1}\right)^{n-1} B\right] = n.$$

References

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