H_2/H_∞ Signal Reconstruction in Noisy Filter Bank Systems

Jingxin Zhang **Cishen Zhang**‡

† School of Engineering and Technology, Deakin University, Pigdons Rd, Geelong Vic 3217, Australia
Fax: +61-3-52272696 Email: jingxz@deakin.edu.au
‡ Department of Electrical and Electronic Engineering, University of Melbourne, Parkville, Vic 3052, Australia

Abstract: This paper investigates signal reconstruction in the multirate filter bank systems with noisy subband paths. It casts the problem of signal reconstruction into a problem of mixed H_2/H_∞ control, and uses the standard results of mixed H_2/H_∞ synthesis to design an IIR synthesis filter bank.

1 Introduction

Depicted in Fig 1 is a multirate filter bank system, where $H_i(z)$ and $S_i(z)$, $i = 1, 2$, are transfer functions of discrete linear time invariant (LTI) filters, $\downarrow$ and $\uparrow$ denote respectively downsampler and upsampler by a factor of two. The input signal $x(k)$ is first decomposed by the analysis filters into band limited signals and downsampled, then quantized and transmitted over a communication channel to a receiver. Due to quantization, roundoff and/or corruption in transmission channels, the signals arriving at the receiver side are corrupted by the subband noises. The subband noises can be regarded as additive white or colored noises [9, 3, 4], which are modeled by $n_i(k)$, $i = 1, 2$, in Fig 1. To reconstruct $x(k)$ at the receiver side, the upsamplers and synthesis filters $S_0(z)$ and $S_1(z)$ are used to reverse the downsampling and decomposition processes. The reconstructed signal $\hat{x}(k)$ is considered to be perfect [9] if $\hat{x}(k) = x(k - d)$, for some $d \geq 0$. This is modeled by the block z^{-d} in Fig 1, which represents the desired transfer function from $x(k)$ to $\hat{x}(k)$. The performance of reconstruction is measured by the reconstruction error $e(k) := x(k-d) - \hat{x}(k)$ and is determined by the $S_i(z)$s used for reconstruction. If the perfect reconstruction is achieved, then $e(k) = 0$.

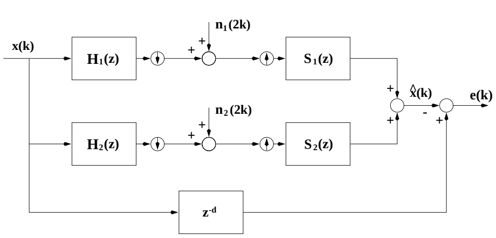

Fig. 1 Multirate filter bank with noisy subband paths

These systems have gained a great popularity in many applications such as subband coding of audio/speech and image signals and spectral analysis and have been extensively studied in recent years, see e.g. [9, 1, 3, 4] and the references therein. Two classes of methods have been developed for the design of $S_i(z)$s. The first class which has dominated the literature of multirate filter bank design consider only the noise free case and seeks a perfect or an optimal approximation of perfect reconstruction, e.g. [9, 1, 6]. The latest development in this direction is the introduction of H_∞ optimization techniques in designing the $S_i(z)$s which minimize the l_2 induced norm from $x(k)$ to $e(k)$ [1, 8, 6].

The second class of design methods which appears recently in the literature, e.g. [3, 4], incorporates subband noise attenuation into design. Assuming full knowledge (model) of the input signal $x(k)$, a multirate Kalman filter is proposed in [4] to replace the conventional filters in the places of $S_i(z)$s and to estimate $x(k)$ from the corrupted subband signals. Using explicitly the estimated second-order statistics of $x(k)$ and $n_i(2k)$s, an FIR filter design method is proposed in [3] which minimizes the power of $e(k)$. The $S_i(z)$s designed in [3, 4] are explicit functions of the model parameters or second-order statistics of $x(k)$ and $n_i(k)$s. Therefore their performance rely largely on the precision of the model parameters or second-order statistics. This poses a serious robustness problem in practice, where the precise estimates of $x(k)$ and $n_i(k)$s are often unavailable, and limits their applications.

This paper adds a new design method to the second class. The method is derived from solving the problem defined below.

P: *Suppose $n_i(2k)$s in Fig 1 are white and $H_i(z)$ and d are given. Design IIR $S_i(z)$s for the system in Fig 1 that minimize $J = \sup_{x \in \mathcal{P}} \{\|e\|_p^2 - \gamma^2 \|x\|_p^2\}$, where $\|e\|_p := \lim_{N\to\infty} \frac{1}{N} \sum_{k=-N}^{N} e^2(k)$ [10] is the power seminorm of $e(k)$, $\mathcal{P}$ denotes the space of bounded power signals [10], and $\gamma > 0$ is a prescribed number.*

The physical meaning of the criterion J will be discussed in the next section. The work is inspired by [1, 8, 6] which cast the noise free case into an H_∞ optimization problem, but it differs from [1, 8, 6] in the problem investigated and the solution obtained. In fact the work of [1] is a special case of **P** where $n_i(2k) \equiv 0$ and $\gamma \equiv 0$.

2 Conversion to a problem of mixed H_2/H_∞ control

Due to presence of multirate, the system in Fig 1 is not LTI. Using the standard polyphase representations and lifting (blocking) technique [9, 1], it can be converted into an MIMO LTI system as shown in Fig 2. In the figure, $V := [x(2k)\ x(2k-1)]^T$, $N := [n_1(2k)\ n_2(2k)]^T$, $W := [e(2k)\ e(2k-1)]^T$ and $\hat{V} := [\hat{x}(2k)\ \hat{x}(2k-1)]^T$, which are lifted variables evolving at the slow rate $2k$; $E(z)$ and $R(z)$ are 2×2 polyphase matrices of $H_i(z)$ and $S_i(z)$, respectively, and satisfy the following equalities (see [9, 1] for details)

$$[H_1(z)\ \ H_2(z)]^T = E(z^2)[1\ \ z^{-1}]^T$$

$$[S_1(z)\ S_2(z)] = [z^{-1}\ 1]R(z^2) \qquad (1)$$

$M(z)$ is given by $M(z) = z^{-d}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, if d is an odd number, $M(z) = z^{-d}\begin{bmatrix} 0 & z \\ 1 & 0 \end{bmatrix}$ if d is an even number.

The systems in Figs 1 and 2 are equivalent in the sense that the power seminorms of V, W and $\hat{V}$ equal respectively those of x, e and $\hat{x}$, and that N is a vector representation of the original $n_i(2k)$, $i = 1, 2$, which evolves at the same rate $2k$.

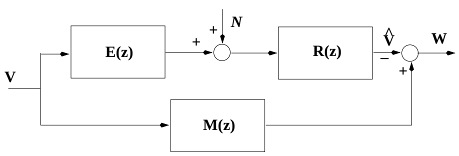

Fig. 2 Polyphase representation of the system in Fig 1

Define the generalized plant

$$G(z) := \begin{bmatrix} 0 & M(z) & -I \\ I & E(z) & 0 \end{bmatrix}$$

Then it is straightforward to show that the system in Fig 2 is equivalent to the generalized system represented by

$$\begin{aligned} \begin{bmatrix} E \\ Y \end{bmatrix} &= \begin{bmatrix} 0 & M(z) & -I \\ I & E(z) & 0 \end{bmatrix} \begin{bmatrix} N \\ V \\ \hat{V} \end{bmatrix} \\ \hat{V} &= R(z)Y \end{aligned} \qquad (2)$$

The polyphase matrix $R(z)$ now becomes the controller of the generalized plant $G(z)$, which, as shown in (1), determines uniquely the synthesis filters $S_i(z)$s to be designed. The design problem **P** is now equivalent to

P1: *Suppose N in Fig 2 is white and $E(z)$ and $M(z)$ are given. For a given prescribed $\gamma > 0$, find a controller $R(z)$ which minimizes $J_1 = \sup_{V\in\mathcal{P}}\{\|W\|_p^2 - \gamma^2\|V\|_p^2\}$.*

This is a standard problem of mixed H_2/H_∞ control considered in [2]. Since $\|e\|_p = \|W\|_p$ and $\|x\|_p = \|V\|_p$, $J \equiv J_1$. The physical meaning of J (J_1) is now clear: as explained in [10, 2], γ is a desired bound on $\|T_{VW}(z)\|_\infty$; and if V and N are orthogonal (not correlated), then for $\gamma > \|T_{VW}(z)\|_\infty$, $J = J_1 = \|T_{NW}(z)\|_2$; where $T_{VW}(z)$ and $T_{NW}(z)$ are the transfer functions from V and N to W, respectively, and $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are H_∞ and H_2 norms of a transfer function.

3 Computation of $R(z)$

Let $\{A_M, B_M, C_M, D_M\}$, $\{A_E, B_E, C_E, D_E\}$ and $\{A_R, B_R, C_R, D_R\}$ be the state-space realization of $M(z)$, $E(z)$ and $R(z)$, respectively. Then it is a routine to show that the state-space realization $\{A, B.C, D\}$ of the closed loop system is in the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B_V \ \vdots\ B_N \\ \hline C & D_V \ \vdots\ D_N \end{array}\right]$$

$$:= \left[\begin{array}{ccc|cc} A_M & 0 & 0 & B_M & \vdots\ 0 \\ 0 & A_E & 0 & B_E & \vdots\ 0 \\ 0 & B_R C_E & A_R & B_R D_E & \vdots\ B_R \\ \hline C_M & -D_R D_E & -C_R & D_M - D_R DE & \vdots\ -D_R \end{array}\right]$$

Following [7, 5], it can be shown that the solution to **P1** is given by the solution of the following LMIs:

$$\min_{X,S,A,B,C,D} Tr[S], \quad \text{subject to}$$

$$\begin{bmatrix} S & B_N^T X & D_N^T \\ XB_N & X & 0 \\ D_N & 0 & I \end{bmatrix} > 0, \ \begin{bmatrix} X - A^T XA & C^T \\ C & I \end{bmatrix} > 0,$$

$$\begin{bmatrix} A^T XA - X & A^T XB_V & C^T \\ B_V^T XA & B_V^T XB_V + I & D_V^T \\ C & D_V & -\gamma^2 I \end{bmatrix} < 0, \ X = X^T > 0.$$

Following [5] the above LMIs can be converted into a set of LMIs which are linear in the parameter matrices A_R, B_R, C_R and D_R. The LMIs thus obtained can be easily solved using MATLAB LMI toolbox.

References

[1] Chen, T and Francis B R, Design of multirate filter banks by H_∞ optimization, *IEEE Trans on Sig Proc*, Vol. 43, pp. 2822-2830, 1995.

[2] Doyle, J, Zhou, K, Glover, K and Bodenheimer, B, Mixed H_2 and H_∞ objectives II: optimal control, performance analysis, *IEEE TAC*, Vol.39, pp. 1575-1587, 1994.

[3] Dalopoulos, A and Kollias, S, Optimal filter banks for signal reconstruction from noise subband components, *IEEE Trans on Sig Proc*, Vol. 44, pp. 212-224, 1996.

[4] Chen, B, Lin, C and Chen, Y, Optimal signal reconstruction in noisy filter bank systems: Kalman synthesis filter approach, *IEEE Trans on Sig Proc*, Vol. 43, pp. 2496-2504, 1995.

[5] Gahinet, P. and P. Apkarian, A linear matrix inequality approach to H_∞ control, *Int J of Robust and Nonlinear Control*, Vol. 42, pp. 421-448, 1994.

[6] Fu, M. S. Dasgupta and G. A. Williamson, Algorithms for optimal multirate filter bank design, *Proc ISSPA*, 1996.

[7] Kaminer, I., P. Khargonekar and M. Rotea, Mixed H_2/H_∞ control for discrete-time systems via convex optimization, *Automatica*, Vol. 29, pp.57-70, 1993.

[8] Shenoy, R.G., D. Burnside and T. W. Parks, Linear periodic systems and multirate filter design, *IEEE Trans. on Sig. Proc.*, Vol. 42, pp. 22424-2256, 1994.

[9] Vaidyanathan, P, Multirate systems and filter banks, Prentice Hall, 1993.

[10] Zhou, K, Glover, K, Bodenheimer, B and Doyle, J, Mixed H_2 and H_∞ objectives I: robust performance analysis, *IEEE Trans. on Automa Contr.*, Vol.39, pp. 1564-1574, 1994.