A Probabilistic Approach to Robust Control Design

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Abstract: A new probabilistic approach to disturbance attenuation problem for LTI discrete-time systems is proposed. The performance is measured by a probability with respect to the stochastic noise of which the worst case 2-norm of the output against a class of deterministic signals with bounded 2-norm is less than a specified level. We first provides a matrix inequality characterization of the probability based on the Toeplitz form of the system and derive a lower bound of the probability. We then show that a guaranteed performance level can be computed by solving an LMI convex optimization problem.

1 Introduction

The mixed H_2/H_∞ control problem is recognized as one of interesting robust control problems. Since the H_∞ norm reflects the worst case performance while the H_2-norm relates the average performance, the mixed H_2/H_∞ control problem is more natural than the H_2 control or the H_∞ control for adopting the practical situations. Therefore, several types of mixed H_2/H_∞ control problems have been investigated [2, 3, 4, 5, 6].

In this paper, we propose a different type of control problem which is close to the mixed H_2/H_∞ control problem in some sense. The problem is a disturbance attenuation problem for LTI discrete-time systems with two types of exogenous inputs, a deterministic disturbance input and a stochastic noise. The performance is measured by a probability with respect to the stochastic noise of which the worst case 2-norm of the output against disturbance input in a class of deterministic signals with bounded 2-norm is less than a specified level. What is the major difference between the proposed problem and other existing mixed H_2/H_∞ control problems is that our measure is a probability which includes much information than the average or the expectation of the performance. In other words, our purpose here is to propose a new approach to robust control theory for practical control applications.

We use the Toeplitz formulae for representing the system and utilize the fundamental results for model set validation in [7], and we show that the analysis problem can be reduced to an LMI convex optimization problem.

Notation. $T_z\{a_i|_{i=0}^{n-1}\}$ is used to represent the lower Toeplitz matrix with its first column elements as $a_0, \cdots, a_{n-1}$. $\lambda_{min}(P)$ denotes the minimal eigenvalue of a symmetric matrix P. Prob$\{\star\}$ stands for the probability that phenomenon $\star$ occurs. $f \sim \mathcal{N}(m, \sigma^2)$ means that a random variable f is normally distributed, and its expectation and variance are respectively m and σ^2.

2 Problem formulation

We consider a stable LTI discrete-time system whose input/output relation is given by

$$z_i \quad = \quad \mathbf{g}(\lambda) w_i + \mathbf{h}(\lambda) v_i, \tag{1}$$

where $z = z_i|_{i=0}^{n-1}$ is the output of the system, $w = w_i|_{i=0}^{n-1}$ and $v = v_i|_{i=0}^{n-1}$ respectively denote the deterministic disturbance input and the stochastic noise.

Our attention here is the behavior in a finite time interval $i = 0, 1, \cdots, n-1$ instead of infinite time interval. Therefore, for the sake of simplicity, we assume that the transfer functions $\mathbf{g}(\lambda)$ and $\mathbf{h}(\lambda)$ are polynomials of the delay operator λ with coefficients g_i and h_i ; $i = 0 \sim n-1$ respectively. Sets of discrete-time signals w and v are defined as follows:

- **Set of Deterministic Disturbance inputs $\mathcal{W}_n$:**

$$\mathcal{W}_n \triangleq \{w = w_i|_{i=0}^{n-1} \;\; | \;\; ||w||_2 \triangleq \left(\sum_{i=0}^{n-1} w_i^2 \right)^{1/2} \leq 1\}$$

- **Set of Stochastic Noises $\mathcal{V}_n$:**

$$\mathcal{V}_n = \{v = v_i|_{i=0}^{n-1} \;\; | \;\; \tilde{v}_{r,i} \sim \mathcal{N}(0, \sigma_{r,i}^2), \; \tilde{v}_{j,i} \sim \mathcal{N}(0, \sigma_{j,i}^2)\}$$

where $\tilde{v}_{r,i} := \Re\left\{\bar{V}\left(e^{j\frac{i\pi}{2n}}\right)\right\}$, $\tilde{v}_{j,i} := \Im\left\{\bar{V}\left(e^{j\frac{i\pi}{2n}}\right)\right\}$,
$\bar{V}(\lambda) := \sum_{i=0}^{n-1} v_i \lambda^i / \sqrt{n}.$

Suppose a stochastic noise is specified, we can define a modified H_∞-norm of the system as

$$\phi_\infty(v) \triangleq \sup_{w \in \mathcal{W}_n} ||z||_2. \tag{2}$$

$\phi_\infty(v)$ is the worst case 2-norm of the output z against disturbance input, w in $\mathcal{W}_n$, where z is contaminated by the noise v. Then, our probability measure is defined by

$$\mathrm{p}_\infty(\gamma) \triangleq \mathtt{Prob}\left\{ v \in \mathcal{V}_n \mid \phi_\infty(v) \leq \gamma \right\}.$$

3 Preliminaries

Note first that the Toeplitz form of Eq. (1) is given by

$$\mathsf{Z} \quad = \quad \mathsf{GW} + \mathsf{HV} \tag{3}$$

and that $\phi_\infty(v) \le \gamma$ is equivalent to

$$\begin{bmatrix} \gamma & (\mathsf{Gw}+\mathsf{Hv})^{\mathrm{T}} \\ \mathsf{Gw}+\mathsf{Hv} & \gamma \mathrm{I} \end{bmatrix} \ge 0; \quad \forall \begin{bmatrix} 1 & \mathbf{w}^{\mathrm{T}} \\ \mathbf{w} & \mathrm{I} \end{bmatrix} \ge 0, \qquad (4)$$

where the capital and small letters respectively denote the Toeplitz form and its first column of the corresponding functions, e.g., $\mathsf{G} = T_z\{g_i|_{i=0}^{n-1}\}$, $\mathsf{H} = T_z\{h_i|_{i=0}^{n-1}\}$, $\mathsf{W} = T_z\{w_i|_{i=0}^{n-1}\}$ and $\mathbf{w} = [w_0, w_1, \cdots, w_{n-1}]^{\mathrm{T}}$.

Using an S-procedure based on Eq. (4), we have a matrix inequality characterization of the probability $\mathrm{p}_\infty(\gamma)$.

Theorem 1

$$\mathrm{p}_\infty(\gamma) = \mathtt{Prob}\{\mathbf{v} \mid {}^\exists\tau > 0,\ P_\tau(\mathbf{v}) \ge 0\} \qquad (5)$$

where

$$P_\tau(\mathbf{v}) \triangleq M_\tau - \hat{V} \ge 0 \qquad (6)$$

with

$$M_\tau \triangleq \begin{bmatrix} \gamma^2 - \tau & 0 & 0 \\ 0 & \mathsf{H}^{-1}\mathsf{H}^{-\mathrm{T}} & \mathsf{H}^{-1}\mathsf{G} \\ 0 & \mathsf{G}^{\mathrm{T}}\mathsf{H}^{-\mathrm{T}} & \tau \mathrm{I} \end{bmatrix}, \qquad (7)$$

$$\hat{V} \triangleq \begin{bmatrix} 0 & \mathbf{v}^{\mathrm{T}} & 0 \\ \mathbf{v} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (8)$$

What is important in Theorem 1 is that $P_\tau(\mathbf{v})$ has a special structure on $\mathbf{v}$ as in Eq. (6), i.e., $P_\tau(\mathbf{v}) = M_\tau - \hat{V}$, since the form of $\hat{V}$ in Eq. (8) plays a key role in the derivations of a lower bound of $\mathrm{p}_\infty(\gamma)$.

Lemma 1 *For a given* $\mathbf{v}^0 = [v_0^0, v_1^0, \cdots, v_{n-1}^0]^{\mathrm{T}}$ *and* $\tau > 0$, *define* $\alpha_\tau(\mathbf{v}^0)$ *as*

$$\alpha_\tau(\mathbf{v}^0) = \min_{0 \le i \le 2n}\{d_i^0\} + \frac{1}{2\sqrt{n}}\lambda_{min}(P_\tau(\mathbf{v}^0))$$

where

$$d_i^0 := \Re\left\{ \frac{\lambda^n}{\sqrt{n}} \sum_{k=0}^{n-1} v_k^0 \lambda^k \bigg|_{\lambda = e^{j\frac{i\pi}{2n}}} \right\}; \quad i = 0, 1, \cdots, 2n .$$

Suppose $\alpha_\tau(\mathbf{v}^0)$ *is nonnegative, then we have*

$$\mathrm{p}_\infty(\gamma) \ge \mathrm{Prob}\{f < \alpha_\tau(\mathbf{v}^0)\}, \qquad (9)$$

where $f \sim \mathcal{N}(0, \sigma^2)$ *with*

$$\sigma = \max_{0 \le i \le 2n} |\Re\{\lambda^n(\sigma_{r,i} + j\sigma_{j,i})|_{\lambda = e^{j\frac{i\pi}{2n}}}\}| .$$

In order to obtain a good lower bound of $\mathrm{p}_\infty(\gamma)$, it is desirable to search a noise series $v_i^0|_{i=0}^{n-1}$ in the noise set $\mathcal{V}_n$, which maximizes α_τ. Fortunately, we can show that the maximum value of α_τ is given by $\mathbf{v}_\star^0 = 0$, i.e., $v_{\star i}^0 = 0$ $(i = 0, 1, \cdots, n-1)$. This fact leads to our main result on the analysis provided in the next section.

4 Main result

The following theorem shows that a guaranteed performance level $\gamma^\sharp$ so that the probability is greater than a specified value $p^\sharp \ge 0.5$ can be easily calculated by solving an LMI convex optimization problem with only 3 variables γ^2, τ and κ, where κ relates a freedom appeared in the simultaneous replacements of $\mathbf{h}(\lambda)$ by $\mathbf{h}(\lambda)/\kappa$ and σ by $\kappa\sigma$.

Theorem 2 *For given* $p^\sharp \ge 0.5$, *let* $\theta^\sharp$ *be the unique solution of* $p^\sharp = \mathtt{Prob}\{f < 2\sqrt{n}\theta^\sharp\}$ *where* $f \sim \mathcal{N}(0, \sigma^2)$. *Then we have*

$$\mathrm{p}_\infty(\gamma^\sharp) \ge p^\sharp,$$

where $\gamma^\sharp$ *is the optimal value of the following LMI optimization problem:*

$$\text{minimize } \gamma^2$$

$$\text{s.t. } \gamma^2 - \kappa\theta^\sharp \ge \tau > 0, \quad \kappa > 0, \qquad (10)$$

$$\begin{bmatrix} \mathrm{I} & \mathsf{G} & \mathsf{H} \\ \mathsf{G}^{\mathsf{T}} & (\tau - \kappa\theta^\sharp)\mathrm{I} & 0 \\ \mathsf{H}^{\mathsf{T}} & 0 & (\kappa/\theta^\sharp)\mathrm{I} \end{bmatrix} \ge 0. \qquad (11)$$

Although we can obtain a concrete form of an upper bound of $\mathrm{p}_\infty(\gamma)$ by a similar manner as in Theorem 3 in [7] for the MSUP, it is not provided here due to the space limitation. However, the upper bound together with the derived lower bound yields the exact H_∞-norm γ_h satisfying $\mathrm{p}_\infty(\gamma_h) = 0.5$ by setting $\theta^\sharp = 0$. In this case, (11) can be rewritten as $\tau\mathrm{I} \ge \mathsf{G}^{\mathsf{T}}\mathsf{G}$. This together with $\gamma^2 \ge \tau$ leads to $\gamma_h = ||\mathbf{g}(\lambda)||_\infty$.

5 Conclusion

In this paper, we have proposed a probability measure for LTI discrete-time stable systems with two types of exogenous inputs, a deterministic disturbance input and a stochastic noise. We have shown that a guaranteed performance level satisfying the prescribed probability condition can be computed by solving an LMI convex optimization problem. We can show that the corresponding synthesis problem can be also reduced to an LMI optimization problem if we fix the denominator of the free parameter in the class of all stabilizing controllers.

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References

[1] D.R.Brillinger. Times series–data analysis and theory. (Expanded Edition) *Holden-Day, San Francisco, 1981.*

[2] D. S. Bernstein and W. H. Haddad. LQG control with an H_∞ performance bound: A Riccati equation approach. *IEEE Trans. Automatic Control, Vol.34, No.3, pp298∼305, 1989*

[3] J.Doyle, K. Zhou, K. Glover, and B.Bodenheimer. Mixed H_2 and H_∞ performance objectives II: Optimal control. *IEEE Trans. Automatic Control, Vol.39, No.8, pp.1575∼1587, 1994*

[4] P.Khargonekar and M. Rotea. Mixed H_2/H_∞ control: A convex optimization approach. *IEEE Trans. Automatic Control, Vol.36, pp.824∼837, 1991*

[5] F. Paganini. Frequency Domain Conditions for Robust H_2 Performance. *IEEE Trans. Automatic Control, Vol.44, No.1, pp.38∼49, 1999*

[6] K. Zhou, K. Glover, B. Bodenheimer and J. Doyle. Mixed H_2 and H_∞ performance objectives I: Robust performance analysis. *IEEE Trans. Automatic Control, Vol.39, pp.1564∼1574, 1994*

[7] T.Miyazato, T.Zhou and S.Hara. A Probabilistic Approach to Model Set Validation. *Proc. Conference on Decision and Control, Phoenix, Arizona, USA, 1999.*