A Note on the Solution of a Class of BMIs for H_∞ Problems

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Abstract
This note deals with the solution of a particular class of Bilinear Matrix Inequalities arising in the synthesis of H_∞ *fixed-order controllers*. It is well known that BMI optimisation problems are non convex and NP hard to solve. In this paper s*ufficient* conditions are provided which allow to convert the controller design problem into a Linear Matrix Inequality feasibility problem.

1. Introduction

Today it is well known to the automatic control scientific community that many control problems can be successfully solved if formulated in terms of LMIs (Linear Matrix Inequalities, [1]) for which efficient algorithms are available. Other control problems, like the synthesis of *fixed order controllers* we deal with in this paper, cannot be formulated in terms of LMIs falling into a wider class of problems denoted as BMIs (Bilinear Matrix Inequalities, [2], [3]).

The BMI optimisation problems are non convex and NP-hard to solve [4], [5], however many *ad hoc* numerical algorithms can be found in the literature of the past decade giving *local* or *global solutions* to them.

Among the local optimisation algorithms, in [6] the use of the so-called *method of centres algorithm* is proposed. In [2] an approach based on a two stage optimisation process is analysed. This process, reminiscent of the *D-K* iteration used in the μ synthesis context, is called by the authors *V-K* iteration and is based on the alternate solution of two convex optimisation problems formulated in terms of LMIs.

As for the *global optimisation* problem, in [6] a branch and bound method is proposed which was improved in [7] using new upper and lower bounds computations and in [8] using some geometric properties of the BMIs. In [9] a global optimisation algorithm, based on the generalised *Benders decomposition* is proposed. Unfortunately all the above mentioned algorithms are only guaranteed to converge in a finite number of iterations, but it has not been yet clarified how much computational effort is required. In general, the number of iterations required to find a global solution within a specified tolerance depends on the given problem and may become unreasonable.

In this paper we focus on the synthesis of *fixed order controllers* with H_∞ performance which can be formulated in terms of BMIs having a particular structure.

2. Problem Description

Consider the following LTI (Linear Time Invariant) system strictly proper on the controlled output:

$$\dot{\tilde{\boldsymbol{x}}} = \tilde{\boldsymbol{A}}\boldsymbol{x} + \tilde{\boldsymbol{B}}\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{B}}_w\boldsymbol{w} \tag{1a}$$

$$\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{C}}_y\tilde{\boldsymbol{x}} \tag{1b}$$

$$\boldsymbol{z} = \tilde{\boldsymbol{C}}_z\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{D}}_{zu}\tilde{\boldsymbol{u}} + \boldsymbol{D}_{zw}\boldsymbol{w} \tag{1c}$$

where $\tilde{\boldsymbol{x}} \in \Re^{\tilde{n}}$, $\tilde{\boldsymbol{u}} \in \Re^{\tilde{m}_1}$, $\boldsymbol{w} \in \Re^{m_2}$, $\tilde{\boldsymbol{y}} \in \Re^{\tilde{p}_1}$ and $\boldsymbol{z} \in \Re^{p_2}$. Consider the following LTI output feedback dynamic controller

$$\dot{\boldsymbol{x}}_c = \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{B}_c\tilde{\boldsymbol{y}} \tag{2a}$$

$$\tilde{\boldsymbol{u}} = \boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_c\tilde{\boldsymbol{y}} \tag{2b}$$

where $\boldsymbol{x}_c \in \Re^{n_c}$, n_c being a pre-assigned order. The closed loop system obtained connecting in feedback the dynamic controller (2) with system (1) can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{B}_w\boldsymbol{w} \tag{3a}$$

$$\boldsymbol{y} = \boldsymbol{C}_y\boldsymbol{x} \tag{3b}$$

$$\boldsymbol{z} = \boldsymbol{C}_z\boldsymbol{x} + \boldsymbol{D}_{zu}\boldsymbol{u} + \boldsymbol{D}_{zw}\boldsymbol{w} \tag{3c}$$

$$\boldsymbol{u} = \boldsymbol{K}_c\boldsymbol{y} \tag{3d}$$

where $\boldsymbol{x} = \begin{bmatrix} \tilde{\boldsymbol{x}} \\ \boldsymbol{x}_c \end{bmatrix} \in \Re^n$, $\boldsymbol{y} \in \Re^{p_1}$; $\boldsymbol{u} \in \Re^{m_1}$, and

$$\boldsymbol{A} = \begin{bmatrix} \tilde{\boldsymbol{A}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} & \tilde{\boldsymbol{B}} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \ \boldsymbol{B}_w = \begin{bmatrix} \tilde{\boldsymbol{B}}_w \\ \boldsymbol{0} \end{bmatrix},$$

$$\boldsymbol{C}_y = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \tilde{\boldsymbol{C}}_y & \boldsymbol{0} \end{bmatrix}, \ \boldsymbol{D}_{zu} = \begin{bmatrix} \boldsymbol{0} & \tilde{\boldsymbol{D}}_{zu} \end{bmatrix}, \ \boldsymbol{K}_c = \begin{bmatrix} \boldsymbol{A}_c & \boldsymbol{B}_c \\ \boldsymbol{C}_c & \boldsymbol{D}_c \end{bmatrix}.$$

Hence the fixed structure controller synthesis problem can be formulated as a static output feedback problem on the fictitious system (3a-3c). The controller design problem is determining the matrix gain $\boldsymbol{K}_c$ such that the closed loop system guarantees stability and assigned properties.

In this paper we focus on the H_∞ problem which can be formulated in terms of the following conditions involving BMIs.

Condition *Fixed Order Controller with an H_∞ norm bound*
System (1) is stabilisable, via the fixed structure controller (2), and has an H_∞ norm bound less than a positive number γ_b on the $\boldsymbol{w}$-$\boldsymbol{z}$ input-output channel, if there exist a positive definite symmetric matrix $\boldsymbol{P}$ and a controller gain matrix $\boldsymbol{K}_c$ such that the following inequality is satisfied

$$\begin{bmatrix} (A+BK_cC_y)^T P + P(A+BK_cC_y) & PB_w & \gamma_b^{-1}(C_z^T + C_y^T K_c^T D_{zu}^T) \\ B_w^T P & -I & \gamma_b^{-1} D_{zw}^T \\ \gamma_b^{-1}(C_z + D_{zu}K_cC_y) & \gamma_b^{-1} D_{zw} & -I \end{bmatrix} < 0 \quad (4) \quad \blacksquare$$

3. Main Result

In the following section we state two sufficient conditions which, at the price of some conservatism, transform the BMI problem (4) into LMI problems

Theorem 1

In the hypothesis that $D_{zu} = \mathbf{0}$, the BMI feasibility problem (4) admits a solution if there exist two positive definite matrices $P_1 \in \Re^{m_{r1}\times m_{r1}}$ and $P_2 \in \Re^{(n-m_{r1})\times(n-m_{r1})}$ and a gain matrix $W_y \in \Re^{m_{r1}\times p_1}$, m_{r1} being the rank of matrix B, such that the following inequality is satisfied

$$\begin{bmatrix} \hat{A}^T P + \hat{C}_y^T \begin{bmatrix} W_y & 0 \\ 0 & 0 \end{bmatrix}^T + P\hat{A} + \begin{bmatrix} W_y & 0 \\ 0 & 0 \end{bmatrix}\hat{C}_y & P\hat{B}_w & \gamma_b^{-1}\hat{C}_z^T \\ \hat{B}_w^T P & -I & \gamma_b^{-1} D_{zw}^T \\ \gamma_b^{-1}\hat{C}_z & \gamma_b^{-1} D_{zw} & -I \end{bmatrix} < 0 \quad (5)$$

where

$\hat{A}(p) = T^{-1}A(p)T$; $\hat{C}_y(p) = C_y(p)T$;

$\hat{C}_z(p) = C_z(p)T$; $\hat{B}_w(p) = T^{-1}B_w(p)$; $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$,

and T is a nonsingular transformation matrix such that

$$T^{-1}B = \begin{bmatrix} I_{m_{r1}} & 0 \\ 0 & 0 \end{bmatrix} \quad (6).$$

A possible value for the controller matrix gain is $K_c = \begin{bmatrix} P_1^{-1} & 0 \end{bmatrix}\begin{bmatrix} W_y \\ 0 \end{bmatrix} \in \Re^{m_1\times p_1}$, $\blacksquare$

Theorem 2

The BMI problem (4) problem admits a solution if there exist two positive definite symmetric matrices $Q_1 \in \Re^{p_{r1}\times p_{r1}}$ and $Q_2 \in \Re^{(n-p_{r1})\times(n-p_{r1})}$, and a gain matrix $W_y \in \Re^{m_1\times p_{r1}}$, p_{r1} being the rank of matrix $\tilde{C}_y$, such that the following inequality is satisfied

$$\begin{bmatrix} \hat{A}Q + Q\hat{A}^T + \hat{B}\begin{bmatrix} W_y^T & 0 \\ 0 & 0 \end{bmatrix}^T + \begin{bmatrix} W_y^T & 0 \\ 0 & 0 \end{bmatrix}\hat{B}^T & \hat{B}_w & \gamma_b^{-1}\left(Q\hat{C}_z^T + \begin{bmatrix} W_y^T & 0 \\ 0 & 0 \end{bmatrix}\tilde{D}_{zu}^T\right) \\ \hat{B}_w^T & -I & \gamma_b^{-1} D_{zw}^T \\ \gamma_b^{-1}\left(\hat{C}_z Q + \tilde{D}_{zu}\begin{bmatrix} W_y^T & 0 \\ 0 & 0 \end{bmatrix}^T\right) & \gamma_b^{-1} D_{zw} & -I \end{bmatrix} < 0 \quad (7)$$

where

$$\hat{A}(p) = T^{-1}A(p)T \text{ ; } \hat{B}(p) = T^{-1}B(p) \text{ ;}$$

$$\hat{B}_w(p) = T^{-1}B_w(p) \text{ ; } \hat{C}_z(p) = C_z(p)T \text{ ; } Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$$

and T is a nonsingular transformation matrix such that

$$C_y T = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \quad (8)$$

The following is a possible value for the output feedback matrix gain $K_c = \begin{bmatrix} W_y & 0 \end{bmatrix}\begin{bmatrix} Q_1^{-1} \\ 0 \end{bmatrix} \in \Re^{m_1\times p_1}$ $\blacksquare$

4. References

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