Optimal closed-loop assignment by static output feedback: a case of study.

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Abstract

This preliminary work studies the pole placement problem by static output feedback for linear time-invariant multivariable systems by considering the minimization of the control effort. The main tool for this analysis is an explicit and parametric expression of output feedback matrix. The minimization of the control effort is performed by a genetic optimization algorithm based on the value of the performance index.

Keywords: Linear Multivariable Systems, Static Output Feedback, Genetic Algorithm .

1 Introduction

The eigenvalue assignment problem for linear time-invariant (MIMO) n-th order systems, with m inputs and p outputs, using static output feedback has been widely investigated in the past years. Although several theoretical results have been obtained [4]-[6], the problem, as illustrated in [3], remains still open because the synthesis of the output gain-feedback matrix is related to solving a large set of non linear equations.

In [8]-[10], the pole placement problem by output feedback has been solved in terms of mp non linear equations. This allows to give a parametric expression for the feedback matrix K that assigns the desired closed-loop spectrum. Usually $m \times p \geq n$, so that the number of parameters is greater than necessary ones to assign the closed-loop eigenvalues; therefore, such "freedom" can be used to satisfy some other specifications [7]. One of the most important sector were the output feedback can be applied is the control of the aircraft. In this preliminary work, the idea consists in finding an output feedback matrix K such that a performance index can be minimized, namely the control effort.

2 Problem Formulation

Let us consider the linear multivariable system

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and $y \in \mathbf{R}^p$. We shall suppose that (A, B, C) is controllable and observable with B and C of full rank. If the control law $u = Ky$ is applied to (1), the closed-loop equation becomes

$$\dot{x} = (A + BKC)\,x. \tag{2}$$

Problem: Given a set of symmetric complex numbers $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, with $\Re[\lambda_i] \leq 0 \ \forall i$, $n = n_c + n_r$ where n_c is the number of complex eigenvalues and n_r the number of real eigenvalues; for any given symmetric and positive definite matrix $R \in \mathcal{R}^{m\times m}$, find a gain-output feedback matrix $K^* \in \mathcal{R}^{m\times p}$ such that

$$\begin{aligned} \mathcal{I}(K^*) &\leq \mathcal{I}(K), \qquad \forall\, K \neq K^*, & (3)\\ \sigma(A + BK^*C) &= \Lambda, & (4)\end{aligned}$$

where $\mathcal{I} = \int_0^\infty u^T(t)Ru(t)dt$ is the *control effort.*

As it is well know, this problem can be rewritten as: find $K^* \in \mathcal{R}^{m\times p}$ such that

$$\begin{aligned} trace(P(K^*)) &\leq trace(P(K)), \qquad \forall\, K \neq K^*,\\ \sigma(A + BK^*C) &= \Lambda,\end{aligned}$$

where P is the positive-definite solution of the Lyapunov equation

$$(A + BKC)^T P + P(A + BKC) = -C^T K^T RKC. \tag{5}$$

We observe that:

a) Under the assumption on the dimensions of the state, input and output, it is well-known [4] that is possible to determine a static output feedback that allows to assign the closed-loop eigenvalues;

b) there exists a positive-definite solution of the Lyapunov equation when the closed-loop matrix $A + BKC$ is stable.

In the paper [9], in order to assign a desired symmetric set of complex numbers, a method is reported that allows to characterize a set $\mathcal{K}$ in a parametric form, that is

$$K = K(\overline{\Theta}), \qquad \overline{\Theta} \in \mathcal{R}^{mp-n}.$$

Moreover, under the assumption $p \geq m$, it is proved that the gain-output feedback matrix K that assigns the spectrum of the closed-loop system (2) is given by the following expression

$$K = \left(V_m^T B\right)^{-1} [z_1|\ldots|z_p]\,[C\omega_1|\ldots|C\omega_p]^{-1}, \tag{6}$$

where

- $V_m^T = [v_{i_1}^T; v_{i_2}^T; \ldots; v_{i_m}^T]$ a set of m open-loop left eigenvectors

- $z_k = \mathrm{diag}\left[(\lambda_k - \pi_{i_1}),\ \ldots,(\lambda_k - \pi_{i_m})\right]\begin{bmatrix}\gamma_{ki_1}\\ \vdots \\ \gamma_{ki_m}\end{bmatrix}$

  $k = 1,\ldots,p.$
- $\pi_1,\ldots,\pi_n$ denote the eigenvalues of A
- $U = [u_1,\ \ldots,\ u_n]$ the open-loop right eigenvectors
- $\Omega = [\omega_1,\ \ldots,\ \omega_n]$ the closed-loop right eigenvectors
- γ_{ki_j} are the entries of the closed-loop eigenvectors in the basis U: $\Gamma = U^{-1}\Omega$

The expression (6) is parameterized with respect to the closed-loop eigenvectors $\omega_1,\ldots,\omega_p$, chosen in the corresponding characteristic subspaces

$$\left\{f \in \mathcal{C}^n : (\lambda I - A)^{-1} f \in \mathrm{Im}(B)\right\},\quad i=1,\ldots,p;$$

$$\omega_i = F(\lambda_i)\left[\theta_{i1}\ \ldots\ \theta_{i(m-1)}\ 1\right]^T,\quad i=1,\ldots,p.$$

Hence, the parameter vector can be used to minimize the performance index $\mathcal{I}(K)$. In this preliminary paper an genetic algorithm has been used in order to minimize the index. The next step is to study the properties of the trace of the matrix, solution of the Lyapunov equation (6).

3 Design Example

This section illustrates the benefits of the selection of the output feedback matrix minimizing the control energy with a pointwise closed loop eigenvalues through an aircraft L-1011 autopilot design. The example is taken from [2] and has been slightly simplified to focus on the relevant aspects with respect to the proposed technique. The linearized lateral axis model with the actuators dynamics removed is the following:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & -0.1540 & -0.0042 & 1.54 & 0\\ 0 & 0.249 & -1 & -5.2 & 0\\ 0.0386 & -0.996 & -0.0003 & -0.117 & 0\\ 0 & 0.5 & 0 & 0 & -0.5\end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0\\ -0.744 & -0.032\\ 0.3370 & -1.12\\ 0.02 & 0\\ 0 & 0\end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}$$

The state vector is represented by $x = [\phi\ ,\ r\ ,\ p_r\ ,\ \beta\ ,\ x_5]^T$, where ϕ is the bank angle (rad), r the yaw rate (rad/sec), pr the sideslip angle (rad) and x_5 the washed-out filter state. The inputs are the rudder deflection (rad) δ_r and the aileron deflection (rad) δ_a; while, the outputs are the washed-out yaw rate r_{wo}, the roll rate p_r, the sideslip angle β and the bank angle ϕ. In order to decouple the rolling and yawing motions of the aircraft, the closed-loop eigenvalues must be allocated in $\Lambda = \{-0.05, -2\pm1.5\,i, -1.5\pm1.5\,i\}$. In this case, the constraint equation, that guarantees the pole placement, is the following

$$\frac{N[\vartheta_{11},\vartheta_{41}]}{D[\vartheta_{11},\vartheta_{41}]} = -0.05 \qquad (7)$$

where

$$\begin{aligned} N[\vartheta_{11},\vartheta_{41}] \quad = \quad & 4.1404810^{-4} - 0.155631\,\vartheta_{11} + 14.2433\,\vartheta_{11}^2 + \\ & 0.166731\,\vartheta_{41} - 31.5803\,\vartheta_{11}\,\vartheta_{41} - 4.08752\,\vartheta_{11}^2\,\vartheta_{41} \\ & +17.6921\,\vartheta_{41}^2 + 4.98909\,\vartheta_{11}\,\vartheta_{41}^2 + 0.540721\,\vartheta_{11}^2\,\vartheta_{41}^2 \\ D[\vartheta_{11},\vartheta_{41}] \quad = \quad & 1.3536810^{-3} - 0.374296\,\vartheta_{11} + 22.9949\,\vartheta_{11}^2 + \\ & 0.408954\,\vartheta_{41} - 51.0428\,\vartheta_{11}\,\vartheta_{41} - 7.08143\,\vartheta_{11}^2\,\vartheta_{41} \\ & +28.7979\,\vartheta_{41}^2 + 8.81058\,\vartheta_{11}\,\vartheta_{41}^2 + \vartheta_{11}^2\,\vartheta_{41}^2 \end{aligned}$$

Now the entries of the parameterized gain matrix K have a structure of this kind

$$K_{ij} = \frac{a_{ij}\,\vartheta_{11}\,(1+\vartheta_{41}) + b_{ij}\,\vartheta_{41}\,(1+\vartheta_{41}) + c_{ij}\,(1+\vartheta_{41}+\vartheta_{41}^2)}{D[\vartheta_{11},\vartheta_{41}]},$$

$$i = 1,\ldots,m; j = 1,\ldots,p$$

where a_{ij}, b_{ij}, c_{ij} are non zero real coefficients. By solving the assignment condition (7) with respect to the parameter ϑ_{41} of the matrix K, the control effort index $\mathcal{I}$ only depends on the parameter ϑ_{11}. Moreover, equation (7) admits a real solutions when ϑ_{11} belongs to the interval $[0.01\ ,\ 0.035]$. The consequent parameterized expression of the gain matrix K involves certainly a complex structure of the index $\mathcal{I}$, but nevertheless it explicitly depends only on one parameter, namely ϑ_{11}. Hence, in order to obtain the minimum value of the objective function, we have used an optimization procedure based on the theory of genetic algorithms [1]. The genetic algorithms constitute a class of search and optimization methods, which imitate the principles of natural evolution. Our approach employs an evolution strategy that operates on vectors of real numbers which corresponds to parameter ϑ_{11} in the gain matrix K. The numerical minimum value of the index $\mathcal{I}$, where R is chosen equal to the identity matrix, is obtained for $\vartheta_{11}^* = 0.0170813$, which involves the following gain output feedback

$$K = \begin{bmatrix} 29.4212 & -1.59383 & -2.91074 & 2.3204\\ 659.21 & -33.304 & -207.55 & -52.5405\end{bmatrix}$$

4 Conclusions

In this preliminary paper, the free parameters of the output feedback matrix, which assigns the closed-loop eigenvalues, have been used to minimize the control effort. The methodology has been applied to synthesize a static regulator for the L-1011 aircraft. The next step is to study the properties of the optimization problem which it's under investigation.

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