On the poles of a class of 2D linear systems

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Abstract

Repetitive processes are a distinct class of 2D systems of both practical and theoretical interest. In this paper we use recent work in behavioral theory for nD linear systems to characterize poles for the case of so-called discrete linear repetitive processes. A unique feature is that the resulting poles lead to a physically based interpretation of stability for these processes.

1 The repetitive process model

In this paper we apply the general theory of nD system pole structure, as developed under the behavioral approach in [2], to a class of linear multipass processes. A repetitive or multipass process is characterized by repeated sweeps, termed passes, through a set of dynamics defined over a finite duration, where the output ("pass profile") from one pass is fed forward to modify the output of the next pass. Applications include long-wall coal cutting and metal rolling operations and classes of linear iterative learning schemes. A detailed treatment of repetitive processes, including the underlying stability theory, can be found in [1].

We consider repetitive processes with the following state-space model:

$$\begin{array}{rcl} x_{k+1}(t+1) & = & Ax_{k+1}(t)+Bu_{k+1}(t)+B_0y_k(t) \\ y_{k+1}(t) & = & Cx_{k+1}(t)+D_0y_k(t) \end{array} \quad (1)$$

Here k denotes the pass number, t measures the time or position within a given pass, and $x_k(t) \in \mathbb{R}^n, y_k(t) \in \mathbb{R}^m$ and $u_k(t) \in \mathbb{R}^l$ are respectively the pass profile (output), state, and control input vectors. To complete the process description it is necessary to specify the initial conditions

$$\begin{array}{rcl} x_{k+1}(0) & = & d_{k+1}, k \geq 0 \\ y_0(t) & = & y(t), 0 \leq t \leq \alpha \end{array} \quad (2)$$

where the $n \times 1$ vector d_{k+1} has constant entries and the entries in the $n \times 1$ vector $y(t)$ are known functions of $t \in [0,\alpha]$. Clearly this model has a two-dimensional, or 2D, structure, i.e. information propagation occurs along a given pass (t direction) and from pass to pass (k direction).

The unique stability problem for repetitive processes is that the sequence of pass profiles $\{y_k\}_{k\geq 1}$ generated can contain oscillations that increase in amplitude in the pass to pass (k direction) which cannot be removed by standard control action. Instead, the stability theory of [1] must be used. This consists of the distinct concepts of asymptotic stability and stability along the pass. In effect, asymptotic stability demands bounded-input bounded-output (BIBO) stability over the (finite and constant) pass length and stability along the pass demands this property independently, i.e. independent of the pass length.

In general, it is the stronger property of stability along the pass which is required. The following set of necessary and sufficient conditions for this property is the essential starting point for this paper.

Theorem 1 *The model S generated by (1) and (2) is stable along the pass if, and only if the two variable polynomial*

$$p(z_1,z) = det\begin{pmatrix} z_1I_n - A & -B_0 \\ -C & zI_m - D_0 \end{pmatrix} \quad (3)$$

satisfies

$$p(a_1,a) \neq 0, \textit{for}\, |a_1| \geq 1\, \textit{and}\, |a| \geq 1 \quad (4)$$

2 The poles of a repetitive process

We now study the poles of the repetitive process (1) as defined in the behavioral approach [2]. Since the state in pass 0 plays no role, it is convenient to re-label the state trajectories $x_{k+1}(t) \mapsto x_k(t)$ (keeping of course the same interpretation). The repetitive process is now described by the kernel representation

$$\begin{pmatrix} z_1I_n - A & -zB & -B_0 \\ -C & 0 & zI_m - D_0 \end{pmatrix}\begin{pmatrix} x \\ u \\ y \end{pmatrix} = 0, \quad (5)$$

where z_1 and z denotes the shift operators along the pass and from pass to pass respectively, i.e.

$$(z_1x_k)(t) := x_k(t+1), \qquad zx_k(t) := x_{k+1}(t) \quad (6)$$

The components of the solutions of the system can be considered as functions from $\mathbb{N}^2$ to $\mathbb{R}$, though for purposes of interpretation they are cut off in one dimension at the pass length α.

The **poles** of the system are essentially the 2D frequencies which can arise in the state and output when the input vanishes. The behavior to study is therefore the behavior of all trajectories with $u = 0$, which is described by the matrix

$$\begin{pmatrix} z_1 I_n - A & -B_0 \\ -C & zI_m - D_0 \end{pmatrix}, \tag{7}$$

Applying [2, Thm/Defn 4.4] we can formally define a pole as a point at which (7) has less than full column rank. In other words, the poles are given by the set

$$V \; := \; \{(a_1, a) \in \mathbb{C}^2 \mid p(a_1, a) = 0\}, \tag{8}$$

where $p(z_1, z)$ is the polynomial given in (3). The set V is called the **pole variety** of the system.

Since in this case the pole variety is given by the vanishing of a single 2D non-unit polynomial, it is guaranteed to be a one-dimensional geometric set in 2D complex space, that is, a union of curves. In particular, the pole variety cannot be a finite set. Note also that the pole variety is a complex variety, even though the entries of the matrices A, B_0, C and D_0 are generally assumed to be real. This is essential in order to capture the full exponential-type dynamics of the system.

Poles can be interpreted in terms of exponential trajectories [2], which in the case of repetitive processes have a clear physical interpretation. Take therefore a point $(a_1 = r_1 e^{\imath\theta_1}, a = re^{\imath\theta}) \in \mathbb{C}^2$, where $\theta_1 = 0$ for $a_1 = 0$ and $\theta = 0$ for $a = 0$. Then (a_1, a) is a pole of the system if and only if there exists an “exponential trajectory” in the system having the form

$$\begin{aligned} \hat{x}_k(t) \; &= x^1_{00} r_1^t r^k \cos(\theta_1 t + \theta k) + x^2_{00} r_1^t r^k \sin(\theta_1 t + \theta k), && (9) \\ \hat{y}_k(t) \; &= y^1_{00} r_1^t r^k \cos(\theta_1 t + \theta k) + y^2_{00} r_1^t r^k \sin(\theta_1 t + \theta k) && (10) \\ \hat{u}_k(t) \; &= 0, && (11) \end{aligned}$$

where $x^1_{00}, x^2_{00} \in \mathbb{R}^n$, $y^1_{00}, y^2_{00} \in \mathbb{R}^m$.

In the case of a pole $(a_1, a) \in \mathbb{R}^2$, it is straightforward to construct such a trajectory. Take a_1 and a to be real numbers satisfying $p(a_1, a) = 0$. There must then exist a non-zero vector $(x_{00}, y_{00}) \in \mathbb{R}^{n+m}$ satisfying

$$\begin{pmatrix} a_1 I_n - A & -B_0 \\ -C & aI_m - D_0 \end{pmatrix} \begin{pmatrix} x_{00} \\ y_{00} \end{pmatrix} \; = \; 0 \tag{12}$$

Now extend (x_{00}, y_{00}) to a system trajectory by

$$\begin{aligned} \hat{x}_k(t) \; &= \; x_{00} a_1^t a^k, && (13) \\ \hat{y}_k(t) \; &= \; y_{00} a_1^t a^k, && (14) \\ \hat{u}_k(t) \; &= \; 0 && (15) \end{aligned}$$

It is easy to check that (13)–(15) is indeed a solution of the system.

Returning to the general case (9)–(11), we see that if $|a| = r > 1$ then we have a non-zero exponential (or sinusoidal) state-output trajectory in the system, which tends towards infinity as the pass number increases (but may remain stable along any given pass). Conversely, if $|a| = r \leq 1$ for all poles (a_1, a), then no trajectory tends to infinity for a given value of t as the pass number increases, but there may be trajectories tending to infinity along the pass. Thus we again run up against the distinction between asymptotic stability and stability along the pass. In order to avoid having trajectories of the form (9)–(11) which are unstable either along the pass or in the k-direction, we also need to avoid poles (a_1, a) with $|a_1| > 1$. In other words, we need that the characteristic variety (8) of the zero-input behavior lies in the closed unit polydisc

$$\overline{P}_1 \; = \; \{(a_1, a) \in \mathbb{C}^2 \;\mid\; |a_1| \leq 1, |a| \leq 1\} \tag{16}$$

Theorem 1 states that stability along the pass is equivalent to the condition that no poles of the system lie outside $\overline{P}_1$. Equivalently, with zero input there should be no exponential/ sinusoidal state-output trajectories which tend to infinity either in the pass-to-pass direction or along the pass.

Finally, note that poles can be decomposed into controllable and uncontrollable, observable and unobservable poles, as described in [2]. The unobservable poles give the (2D) frequencies which can occur in the state when both input and output vanish; these are easily found to be the rank-loss points in $\mathbb{C}^2$ of the matrix

$$\begin{pmatrix} z_1 I_n - A \\ -C \end{pmatrix} \tag{17}$$

and so indeed describe the defect of observability. Note that whether (a_1, a) is an unobservable pole depends entirely on a_1. Indeed if a_1 is say a real rank-loss point of (17), then there must be a vector $x_0 \in \mathbb{C}^n$ such that, for any function $f(k)$, the state evolution $x_k(t) = x_0 a_1^t f(k)$ can occur whilst both input and output remain zero. A similar problem will occur with unobservable poles having an imaginary part. Thus stability along the pass requires observability.

It is also possible to give an explicit form for the observable poles by eliminating the state variables from (1), but for brevity we omit this here.

References

[1] E. Rogers and D.H. Owens. *Stability Analysis for Linear Repetitive Processes*, volume 175 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, 1992.

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