When are Product Systems Controllable?

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Abstract

A product system consists of a finite number of independent control systems. We introduce the notion of 'controllability with selectable time' in order to investigate the controllability of product systems. If all factors are controllable and at most one factor is not controllable with selectable time, then the product system is controllable. For locally accessible control affine systems, the converse is true as well.

1 Introduction

In the present paper we investigate the controllability of systems, which are decoupled into independent subsystems. We will call those systems *product systems.*

In contrast to *linear* control systems, the controllability of *nonlinear* control systems is not transferred from two independent systems to the product system.

As a simple example we can think of two independent clocks, both without adjustment. Each clock performs a periodic motion, say on the unit sphere $S^1 = R\text{mod}2\pi$, given by the constant flow $\dot{x}(t) = 1$. Each of the two independent flows is controllable on S^1 in the control theoretic sense: Any two points of the state space S^1 can be connected with a trajectory. But the product system, consisting of both systems on the torus $T^2 = S^1 \times S^1$ as state space, is far away from being controllable. Intuitively, the controllability of the product is related to the synchronization of the controllability times of the two factors. In the simple example above, synchronization is not possible, since the sets of possible controllability times, which one needs to steer from one point to another, are of the form $\{s, 2\pi + s, 3\pi + s, \ldots\}$, with an $s \in [0, 2\pi)$, and are much too thin. Obviously the picture changes, when we equip one clock with an adjustment. Mathematically we add a control action to the system's equation on S^1, say $\dot{x}(t) = 1 + u(t)$, where the control function u takes values in $[0, 1]$. By the presence of the control, we can select, to some extend, the time that we need to steer from one point to the other. This subsystem has stronger controllability properties, it is what we call *controllable with selectable time*: The set of possible controllability times contains an infinite interval. It easily follows that the product system on $T^2 = S^1 \times S^1$ becomes controllable, the two clocks can be synchronized. We elaborate to what extend this simple example of two independent clocks is typical with respect to the controllability of product systems.

Our main result states that, for smooth control affine product systems on compact product manifolds, controllability with selectable time of the subsystems, with possible exception of at most one subsystem, is necessary and sufficient for the controllability of the product system, provided that all subsystems are controllable and the accessibility algebra has full rank in one point of the product space.

2 Controllability with selectable time

We consider a nonlinear control system $\Sigma = (X, \Omega, f)$

$$\dot{x}(t) = f(x(t), u(t))$$

on a manifold X. For $t \in [0, \infty)$ let $t \mapsto x(t, x, u) \in X$ be the trajectory of the system Σ, produced by an initial value $x = x(0, x, u) \in X$ and by a control function $u : [0, \infty) \to \Omega$. As usual, we define the attainable set for $(t, x) \in [0, \infty) \times X$ by

$$A(t, x) = \bigcup_{u:[0,t]\to\Omega} \{x(t, x, u)\},$$

and say that the system Σ is controllable, if for all $x \in X$ the equality

$$\bigcup_{t\in[0,\infty)} A(t, x) = X$$

is valid. We introduce a stronger controllability notion, which plays a fundamental role in connection with product systems.

Definition. The system Σ is controllable with selectable time, if there is a time $T \geq 0$ such that for all $t \geq T$ and for all $x \in X$ the equality

$$A(t, x) = X$$

is valid.

Throughout the paper we will assume:

• The state space X is a compact smooth manifold of finite dimension $n \in N$.
• The control range is a compact metric space.
• The control functions $u : [0, \infty) \to \Omega$ are measurable.
• The vector fields $f(\cdot, \omega) : X \to TX$ are uniformly Lipschitz on X for any $\omega \in \Omega$.
• The map $f(\cdot, \cdot) : X \times \Omega \to TX$ is continuous.
• The right-hand sides $\bigcup_{\omega \in \Omega} \{f(x, \omega)\} \subset T_x X$ are convex.

Lemma. If the system Σ is controllable, then there is a time $R \geq 0$ such that for all $x \in X$ holds:

$$\bigcup_{t \in [0,R]} A(t, x) = X.$$

This lemma is easy to prove, it relies on Baire's category theorem. The following lemma relates controllability with selectable time and strong accessibility.

Lemma. There are equivalent:
(i) The system Σ is controllable with selectable time.
(ii) The system Σ is controllable and strongly accessible from a point $x \in X$, that is $\mathrm{int}A(S, x) \neq \emptyset$ for an $S \geq 0$.

This lemma gives some hints how to varify controllability with selectable time, provided the system is already controllable. First, we mention that strong accessibility for control affine systems, produced by smooth vector fields, locally can be varified by calculating the strong accessibility algebra (see e.g. [2, 3]). Secondly, controllable systems with equilibria, that is with roots of $f(x, \omega) = 0$, obviously have property (ii) of this lemma. This implies that for control systems on certain manifolds, which do not allow the existence of nonvanishing continuous vector fields, e.g. $X = S^2$, controllability with selectable time is equivalent to controllability.

3 Controllability of product systems

In this section we investigate the controllability of product systems. For $i = 1, \ldots, m$, $m \geq 2$, let $\Sigma_i = (X_i, \Omega_i, f_i)$ be *independent* control systems. Independence means that, for $i = 1, \ldots, m$, the trajectory of the system Σ_i only depends on the initial point $x_i \in X_i$ and on the control function $u_i : [0, \infty) \to \Omega_i$.
When is the product system $\Sigma = \Sigma_1 \times \ldots \times \Sigma_m$

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), u_1(t)), \\ &\vdots \\ \dot{x}_m(t) &= f_m(x_m(t), u_m(t)) \end{aligned}$$

controllable on $X = X_1 \times \ldots \times X_m$?
A sufficient condition easily can be formulated.

Proposition. The product system $\Sigma = \Sigma_1 \times \ldots \times \Sigma_m$ is controllable with selectable time on $X = X_1 \times \ldots \times X_m$ if and only if all systems Σ_i are controllable with selectable time on X_i, and it is controllable on $X = X_1 \times \ldots \times X_m$ if all systems Σ_i are controllable on X_i and at least $m - 1$ of the m systems Σ_i are controllable with selectable time on X_i.

4 Smooth control affine systems

Sofar we considered nonsmooth systems and only exploited continuity properties of the attainable sets. Thus the natural question arises, whether we can obtain sharper results for smooth systems, by making use of geometric control theory. In the following we will impose additional conditions.

• The control range Ω is a convex subset of R^k with $0 \in \mathrm{int}\Omega$.
• For fixed $\omega = (\omega^1, \ldots, \omega^k) \in \Omega$ the vector fields $f(\cdot, \omega)$ on X are of the form $f(\cdot, \omega) = f^0(\cdot) + \sum_{j=1}^{k} \omega^j f^j(\cdot)$ with smooth C^∞-vector fields $f^j(\cdot)$, $j = 0, \ldots, k$, on X.

As usual, we call the Lie algebra generated by the vector fields $f^j(\cdot)$, $j = 0, \ldots, k$, the *accessibility algebra* and the smallest subalgebra, which contains all control vector fields $f^j(\cdot)$, $j = 1, \ldots, k$ and which is invariant under the Lie product with the drift vector field $f^0(\cdot)$, the *strong accessibility algebra* (see e.g. [3]).

Theorem. If there is a point $x \in X$, such that the accessibility algebra of Σ has full rank n in $x \in X$, then there are equivalent:
(i) The product system $\Sigma = \Sigma_1 \times \ldots \times \Sigma_m$ is controllable on $X = X_1 \times \ldots \times X_m$.
(ii) All the systems Σ_i are controllable on X_i and at most one of these systems is not controllable with selectable time.

The proof of this statement can be found in [1]. It relies on the fact that the rank of the accessibility algebra exceeds the rank of the strong accessibility algebra by at most *one*.

References

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[3] H. Nijmeijer, A.J. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer, New York, 1990