Characterization of the Hermite indices of the pair $(A+BF,B)$[1]

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Abstract

We study the problem of characterizing the Hermite indices of a Linear System $\dot{x}(t) = Ax(t) + Bu(t)$ when state feedback is performed. Namely, given the pair (A,B), we study the problem of the existence of a matrix F such that $(A+BF,B)$ has prescribed Hermite indices.

1 Introduction

Consider the following linear time-invariant system of differential equations with control:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{n\times m}$ and $\mathbb{F}$ is the field of the real or complex numbers. As usual we will identify this system with the matrix pair $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$.

Considering the controllability matrix of the pair, $C(A,B) = [B\ AB \dots\ A^{n-1}B]$. If rank $C(A,B) = r$ and we select from left to right the first r linearly independent colums [Kailath (1980), p. 427, scheme II], and we write them as $b_1, \dots, A^{l_1-1}b_1, \dots, b_m, \dots, A^{l_m-1}$, where $l_i = 0$ if b_i is absent, then we will say that $l_1, \dots, l_m$ are the generalized controllability indices of the system. Rearranging these indices in nonincreasing order, we obtain the controllability indices, $k_1, \dots, k_m$.

Considering the same colums of $C(A,B)$ but in the order in which they appear in $H(A,B) = [b_1, \dots, A^{n-1}b_1, \dots, b_m \dots, A^{n-1}b_m]$, if we select from left to right the firts r linearly independent colums [Kailath (1980), p. 426, scheme I] and we write them as $b_1, \dots, A^{h_1-1}b_1, \dots, b_m, \dots, A^{h_m-1}$, where, again $h_i = 0$ if b_i is absent, then $h_1, \dots, h_m$ are the Hermite indices of the system. These indices are the degrees of the polynomials appearing in the diagonal of the Hermite normal form of the right denominator of the transfer matrix $(sI_n - A)^{-1}B$ [Kailath (1980), p. 476]. This is why they were called Hermite indices in [Zaballa (1997)].

We recall that two matrix pairs (A,B), $(\overline{A},\overline{B}) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ are feedback equivalent if there are nonsingular matrices $P \in \mathbb{F}^{n\times n}$ and $Q \in \mathbb{F}^{m\times m}$ and a matrix $F \in \mathbb{F}^{m\times n}$ such that $(\overline{A},\overline{B}) = (PAP^{-1} + PBF, PBQ)$. It is well known that if the pair $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ is (completely) controllable, i.e. rank $C(A,B) = n$, then the controllability indices are a complete system of invariants for the feedback equivalence relation (see for example [Brunovsky (1970)]).

As shown in [Zaballa (1997)] the Hermite and controllability indices are significant for the study of the structural properties of linear control systems. Both are invariant under system similarity (see [Falb P. (1999)], [Kailath (1980)]) but the latter are not invariant under feedback equivalence. This presentation is part of a larger project about the possible Hermite indices that can be attained when a system is submitted to a transformation of the Feedback Group. Hermite indices appear also on the study of structured systems (see [Siparis et *al.* (1991)]).

Problem 1 *Let $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ be a controllable matrix pair and let $h_1, \dots, h_m$ be nonnegative integers. Under what conditions does there exist a matrix $F \in \mathbb{F}^{m\times n}$ such that $(A+BF,B)$ has $h_1, \dots, h_m$ as Hermite indices?*

Definition 1 *Given $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ two partitions of nonnegative integers, we say that $a\mathcal{M}b$ if*

$$\sum_{j=1}^{k} a_j \le \sum_{j=1}^{k} b_j, \quad 1 \le k \le n \text{ and } \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j.$$

It is easy to see that, if $(l_1, \dots, l_m)$ are the generalized controllability indices and $(h_1, \dots, h_m)$ are the Hermite indices of a pair (A,B) then

$$(l_1, \dots, l_m)\mathcal{M}(h_1, \dots, h_m). \tag{2}$$

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2 Equivalence Relation and Canonical form

In order to solve Problem 1, our first goal is to reduce the given pair (A, B).

Definition 2 *Let* (A, B), $(\overline{A}, \overline{B}) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$. *We say that* (A, B) *is* Γ_S*-equivalent to* $(\overline{A}, \overline{B})$, *and we write* $(A, B) \overset{\Gamma_S}{\sim} (\overline{A}, \overline{B})$, *if there exist nonsingular matrices* $P \in \mathbb{F}^{n\times n}$ *and* $T \in \mathbb{F}^{m\times m}$, T *upper triangular, and a matrix* $F \in \mathbb{F}^{m\times n}$ *such that*

$$(\overline{A}, \overline{B}) = (PAP^{-1} + PBF, PBT).$$

In the next Lemma we give a complete system of invariants for the Γ_s-equivalence in the controllable case.

Lemma 1 *Let* (A, B), $(\overline{A}, \overline{B}) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ *be two controllable matrix pairs. Then* $(A, B) \overset{\Gamma_S}{\sim} (\overline{A}, \overline{B})$ *if and only if they have the same generalized controllability indices.*

In the following Lemma we can see that solving the Problem 1 for a pair (A, B) is the same as solving the Problem 1 for any pair in the same Γ_S-equivalence class.

Lemma 2 *Let* (A, B), $(\overline{A}, \overline{B}) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ *and let* $h_1, \ldots, h_m$ *be nonnegative integers. Suppose that* $(A, B) \overset{\Gamma_S}{\sim} (\overline{A}, \overline{B})$. *Then, there exists a matrix* $F_1 \in \mathbb{F}^{m\times n}$ *such that* $(A+BF_1, B)$ *has* $h_1, \ldots, h_m$ *as Hermite indices if and only if there exists a matrix* $F_2 \in \mathbb{F}^{m\times n}$ *such that* $(\overline{A} + \overline{B}F_2, B)$ *has* $h_1, \ldots, h_m$ *as Hermite indices.*

By the definitions of the generalized controllability indices and the Hermite indices, we can observe that if a generalized controllability index l_i is diferent from zero then the corresponding Hermite index h_i will also be diferent from zero. Therefore, we can assume without loss of generality that rank $B = m$. In the following Lemma we will give a canonical form for the Γ_S-equivalence relation.

Lemma 3 *Let* $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ *a controllable pair with* $l_1, \ldots, l_m$ *as generalized controllability indices. Then,* $(A, B) \overset{\Gamma_S}{\sim} (A_c, B_c)$ *where*

(i) $A_c = Diag(A_{11}, \ldots, A_{mm})$

$$A_{ii} = \left(\begin{bmatrix} 0 & 0 \\ I_{l_i-1} & 0 \end{bmatrix} \right) \in \mathbb{F}^{l_i\times l_i},\ 1 \leq i \leq m$$

(ii) $B_c = Diag(B_{11}, \ldots, B_{mm})$

$$B_{ii} = [1\ 0 \ldots\ 0]^T \in \mathbb{F}^{l_i\times 1},\ i = 1, \ldots, m$$

Attending to the previous Lemmas, from now on we can consider that the pair (A, B) has the form exhibited in Lemma 3.

3 Main result

Theorem 1 *Let* $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ *be a controllable pair. Let* $l_1, \ldots, l_m$ *be its generalized controllability indices and let* $h_1, \ldots, h_m$ *be nonnegative integers.Assume that* $l_1 \geq \ldots \geq l_m$. *Then there exists a matrix* $F \in \mathbb{F}^{m\times n}$ *such that* $(A+BF, B)$ *has* $h_1, \ldots, h_m$ *as Hermite indices if and only if the condition (2) holds*

Sketch of the proof.- Bearing in mind that the generalized controllability indices of (A, B) are invariant under state feedback [Popov (1972)] we have that (2) is a necessary condition.

It can be proved that it is also sifficient for the existence of matrices $A_{ij} \in \mathbb{F}^{l_i\times l_j}$, $i = 1, \ldots, m$ such that if

$$\overline{A} = \begin{bmatrix} A_{11} & 0 & \ldots & 0 \\ A_{21} & A_{22} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{bmatrix}$$

then $(\overline{A}, B)$ has $h_1, \ldots, h_m$ as Hermite indices.

Finally, if $l_1 \geq \ldots \geq l_m$ then there exists a matrix $F \in \mathbb{F}^{m\times n}$ such that $(\overline{A}, B)$ is similar to $(A + BF, B)$ and the theorem holds.

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