

Geometrical approach to parameter dependent Lyapunov functions

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Abstract

In this paper we consider the problem of constructing parameter dependent Lyapunov functions that guarantee the stability of the linear systems with an uncertain constant real parameter. First we formulate the surface on which all parameter dependent Lyapunov matrices for a given uncertain system exist. Next, after defining a Riemannian metric on the surface, we derive a method to obtain a parameter dependent Lyapunov matrix as a geodesic on the surface.

1 Introduction

1.1 problem

This paper is concerned with the systems of the form

$$\dot{\xi}(t) = A(\theta)\xi(t), \quad (1)$$

where $\xi : \mathfrak{R} \rightarrow \mathfrak{R}^n$ is the state vector, $\theta \in \mathfrak{R}$ is a uncertain constant real parameter and $A : \mathfrak{R} \rightarrow \mathcal{M}(n, n)$ is a matrix valued function of θ , where we denote the set of all real $n \times m$ matrices by $\mathcal{M}(n, m)$. We assume $A(\theta)$ is twice continuously differentiable. We also assume that the range of θ is known as follows:

$$\theta \in [\underline{\theta}, \bar{\theta}]. \quad (2)$$

The system (1)(2) is stable if and only if there exists a Lyapunov function of the form $V_\theta(\xi) := \xi^T P(\theta)\xi$, where $P(\theta)$ is a parameter dependent Lyapunov matrix (PDLM). In this paper, we propose a method to prove the existence of $P(\theta)$.

1.2 notation

We summarize the notations we use in the following sections. Let $N := n(n+1)/2$. $\text{PD}(n)$ denotes the set of all $n \times n$ symmetric positive definite matrices. We denote the set of all $n \times n$ Hurwitz matrices by $\mathcal{H}(n)$. $P : \mathfrak{R}^{n(n+1)/2} \rightarrow \mathcal{M}(n, n)$ is a matrix valued function defined by

$$P([a_1, a_2, \dots, a_{\frac{n(n+1)}{2}}]^T)$$

$$:= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_{n+1} & \cdots & a_{2n-1} \\ & & \cdots & \\ a_n & a_{2n-1} & \cdots & a_{\frac{n(n+1)}{2}} \end{bmatrix}. \quad (3)$$

For a symmetric matrix X , $\text{cs}X$ denotes a vector whose components consists of all independent components of X , namely

$$\begin{aligned} \text{cs} \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{12} & X_{22} & \cdots & X_{2n} \\ & & \cdots & \\ X_{1n} & X_{2n} & \cdots & X_{nn} \end{bmatrix} \\ := [X_{11}, \dots, X_{1n}, X_{22}, \dots, X_{2n}, \dots, X_{nn}]^T. \end{aligned} \quad (4)$$

We define a mapping $H : \mathfrak{R}^N \times \mathfrak{R} \rightarrow \mathfrak{R}^N$ by

$$H(x, \theta) := \text{cs}[A(\theta)P(x) + P(x)A(\theta)^T]. \quad (5)$$

Next we define some sets.

$$\begin{aligned} \mathcal{D} &:= \mathfrak{R}^N \times \mathfrak{R}, \quad \mathcal{S} := \mathcal{D} \times \mathfrak{R}^N, \\ \mathcal{P} &:= \{(x, \theta, y) \in \mathcal{S} | P(y) \in \text{PD}(n)\}, \\ \mathcal{L} &:= \{(x, \theta, y) \in \mathcal{S} | y = -H(x, \theta)\}, \\ \mathcal{L}^+ &:= \mathcal{L} \cap \mathcal{P}. \end{aligned}$$

We also define a set valued function $\mathcal{B} : \mathfrak{R} \rightarrow \text{Pow}(\mathcal{S})$ by

$$\mathcal{B}(\tau) := \{(x, \theta, y) \in \mathcal{S} | \theta = \tau\}, \quad (6)$$

where $\text{Pow}(\mathcal{S})$ denotes the power set of \mathcal{S} .

2 A stability condition

Theorem 2.1 *The system (1) is stable for the uncertain parameter that satisfies (2) if and only if both of (a) and (b) below holds. (a) There exists $\theta^* \in [\underline{\theta}, \bar{\theta}]$ such that $A(\theta^*) \in \mathcal{H}(n)$. (b) Suppose E_{θ^*} is the arcwise connected component of \mathcal{L}^+ including (x^*, θ^*, y^*) where $x^*, y^* \in \mathfrak{R}^N$ are the vectors that satisfies*

$$\begin{aligned} A(\theta^*)P(x^*) + P(x^*)A(\theta^*)^T + P(y^*) &= 0, \\ P(y^*) &\in \text{PD}(n). \end{aligned} \quad (7)$$

Then

$$E_{\theta^*} \cap \mathcal{B}(\underline{\theta}) \neq \emptyset \text{ and } E_{\theta^*} \cap \mathcal{B}(\bar{\theta}) \neq \emptyset \quad (8)$$

holds.

If E_{θ^*} is the arcwise connected component that satisfies the conditions of the theorem 2.1, the projection on \mathcal{D} of a curve on E_{θ^*} that connects $E_{\theta^*} \cap \mathcal{B}(\underline{\theta})$ and $E_{\theta^*} \cap \mathcal{B}(\bar{\theta})$ is a parameter dependent Lyapunov matrix (PDLM) for the system (1)(2).

Example 2.1 Consider the following system :

$$\dot{\xi} = A(\theta)\xi, \quad \theta \in [\underline{\theta}, \bar{\theta}], \quad (9)$$

where $A(\theta) = \theta - 1$, $\underline{\theta} = 0$ and $\bar{\theta} = 2$.

We have $H(x, \theta) = (\theta - 1)x$. The system (9) becomes unstable when $\theta \geq 1$. In this case $\mathcal{S} = \mathbb{R}^3$ and \mathcal{L} is the surface defined by the equation $y = -H(x, \theta) = -(\theta - 1)x$ in \mathbb{R}^3 as shown in figure 1. \mathcal{L}^+ consists of two arcwise connected components, namely $\{(x, \theta, y) \in \mathcal{L} | x > 0, \theta < 1\}$ and $\{(x, \theta, y) \in \mathcal{L} | x < 0, \theta > 1\}$. The former is identical to E_{θ^*} for any $\theta^* \in [0, 1)$.

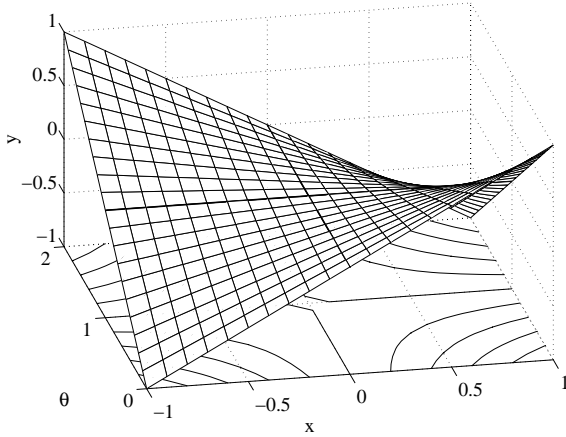


Figure 1: The surface \mathcal{L} of the example 2.1

3 PDLMs as geodesic curves

In this section we propose a method to obtain a PDLM for the system (1)(2) as a geodesic on \mathcal{L}^+ .

We define an embedding $\Phi : \mathcal{D} \rightarrow \mathcal{P}$ by $\Phi(x, \theta) := (x, \theta, -H(x, \theta))$.

Definition 3.1 Suppose $P(y) \in \text{PD}(n)$ and $P(u), P(v) \in \text{T}_{P(y)}\text{PD}(n)$. The inner product

$$g_y(u, v) := \text{Trace}(P(y)^{-1}P(u)P^{-1}(y)P(v)) \quad (10)$$

defines a Riemannian metric on $\text{PD}(n)$ [1].

Definition 3.2 Suppose $(x, \theta) \in \mathcal{D}$ and $\alpha, \beta \in \text{T}_{(x, \theta)}\mathcal{D}$. The inner product

$$h_{(x, \theta)}(\alpha, \beta) \quad (11)$$

$$\begin{aligned} &:= \hat{h}_{\Phi(x, \theta)}(d\Phi_{(x, \theta)}(\alpha), d\Phi_{(x, \theta)}(\beta)), \\ &:= \alpha \cdot \beta + g_{-H(x, \theta)}(dH_{(x, \theta)}(\alpha), dH_{(x, \theta)}(\beta)) \end{aligned}$$

defines a Riemannian metric on \mathcal{L}^+ , where \hat{h} is a metric on \mathcal{P} and g is the metric on $\text{PD}(n)$ defined in the definition 3.1.

Using this metric on \mathcal{L}^+ we can obtain a PDLM for the system (1)(2) as a geodesic on \mathcal{L}^+ .

Theorem 3.1 The system (1) is stable for the uncertain parameter that satisfies (2) if and only if both of (a) and (b) below holds. (a) There exists $\theta^* \in [\underline{\theta}, \bar{\theta}]$ such that $A(\theta^*) \in \mathcal{H}(n)$. (b) There exists a geodesic which connects $\mathcal{L}^+ \cap \mathcal{B}(\underline{\theta})$ and $\mathcal{L}^+ \cap \mathcal{B}(\bar{\theta})$.

Example 3.1 $A(\theta)$ and the range of θ are given as follows:

$$A(\theta) = \begin{bmatrix} \theta - 1 & 1 \\ 0 & \theta - 2 \end{bmatrix}, \quad \theta \in [0, 1). \quad (12)$$

We obtained the geodesic for this system numerically. The initial conditions are $x(0) = [0.5833, 0.0833, 0.25]^T$, $\theta(0) = 0$ and $\frac{dx}{ds}(0) = [0, 0, 0]^T$, $\frac{d\theta}{ds}(0) = 1$, where x and θ are the functions of s which is the arc length of the curve. The obtained geodesic is shown in figure 2 (the geodesic is projected to the (x_1, x_3, θ) subspace).

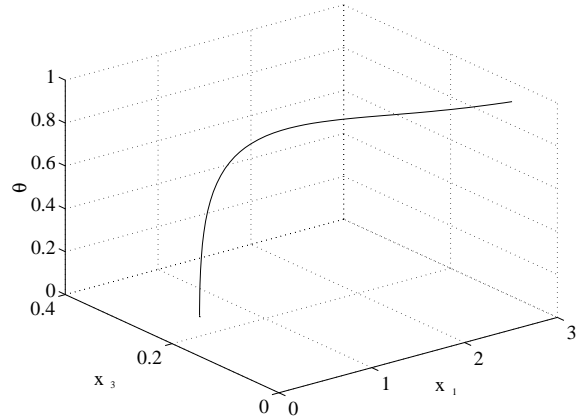


Figure 2: The geodesic for the system (12)

References

- [1] A. Ohara and T. Kitamori, "Distances on Gramians and their applications," Proceedings of the Int. Symp. MTNS-91, 449-454 (1992)