Stability condition of distributed delay systems based on the analytic solution of Lyapunov functional equations

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Abstract

Analytic solution of Lyapunov functional equations of distributed delay systems is derived. The analytic solution is computed using a matrix exponential function, while conventional computation has been relied on numerical approximations. Based on the analytic solution, a stability condition for distributed delay systems with unknown but bounded constant delay is proposed.

1 Introduction

Stability conditions of distributed delay systems fall into two categories: one is based on numerical computation of poles and the other is based on Lyapunov functionals. In the former stability conditions [1], the poles of distributed delay systems are numerically computed using numerical techniques of [2]. In the latter stability conditions, Lyapunov functional equations of distributed delay systems are solved using the infinite dimensional system theory [3]. The main problem to this method is that the Lyapunov equations are in the form of coupled partial differential equations. For a state delay system without distributed delay terms, the analytic solution was derived in [4]. But for distributed delay systems, no analytic solution to the Lyapunov equations has been known yet and its solution relies on numerical approximations.

In this paper, an analytic solution to the Lyapunov equations is derived for a certain type of distributed delay systems. The analytic solution can be computed by solving a simple linear equation and by computing a matrix exponential function. Based on the analytic solution, a stability condition is proposed for distributed delay systems with unknown but bounded constant delay. Notation: For a matrix $M = [m_{ij}] \in \mathbb{C}^{n\times n}$, the column string $\operatorname{cs} M$ is defined by

$$\operatorname{cs} M \triangleq [m_{11}\ m_{21}\ \cdots\ m_{n1} \mid m_{12}\ m_{22} \cdots\ m_{n2} \mid \cdots \mid m_{1n}\ m_{2n}\ \cdots\ m_{nn}]' \in \mathbb{C}^{n^2\times 1}.$$

2 Lyapunov Functional

Consider the distributed delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + \int_0^h F(r) x(t-h+r)\, dr \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is a state. In this paper, the integral kernel $F(r)$ is assumed to satisfy the following:

$$F(r) = F_1(r)F_3,\ \dot{F}_1(r) = F_1(r)F_2, \quad (2)$$

where F_2 and F_3 are constant matrices.

Consider a Lyapunov functional V for (1) defined by

$$\begin{aligned} V(\mathcal{P}) \triangleq\ & x(t)'P_{00}x(t) + x(t)'\textstyle\int_0^h P_{01}(s)x(t-h+s)\,ds \\ & + \textstyle\int_0^h x(t-h+r)'P_{10}(r)\,dr\,x(t) \\ & + \textstyle\int_0^h x(t-h+r)' \int_0^h P_{11}(r,s)x(t-h+s)\,ds\,dr \end{aligned} \quad (3)$$

where

$$P_{00} = P_{00}',\ P_{01}(r) = P_{10}(r)',\ P_{11}(r,s) = P_{11}(s,r)'.$$

The following theorem [3] states the stability of (1) using the Lyapunov functional (3).

Theorem 1 *Suppose $X = X' \in \mathbb{R}^{n\times n} > 0$ is given. The system (1) is stable if there exist $V(\mathcal{P})$ and $\epsilon > 0$ such that*

$$V(\mathcal{P}) \geq \epsilon x(t)'x(t), \quad (4)$$

$$\frac{d}{dt}V(\mathcal{P}) = -x(t)'Xx(t), \quad (5)$$

for all $x(t),\ t \geq 0$.

The Lyapunov equation (5) can be simplified in the following lemma [3].

Lemma 1 *Given $X = X'$, the solution $\mathcal{P}$ to (5) is given by the following equations:*

$$A_0'P_{00} + P_{00}A_0 + P_{01}(h) + P_{01}(h)' = -X \quad (6)$$

$$\frac{d}{ds}P_{01}(s) = A_0'P_{01}(s) + P_{11}(h,s) + P_{00}F(s) \quad (7)$$

$$\left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right) P_{11}(r,s) = F(r)'P_{01}(s) + P_{01}(r)'F(s) \quad (8)$$

where the initial conditions are given by

$$P_{01}(0) = P_{00}A_1,\ P_{11}(r,0) = P_{01}(r)'A_1. \quad (9)$$

Equations (6) $\sim$ (8) are coupled equations of P_{00}, P_{01}, and P_{11}, whose analytic solution is not known. From (6) $\sim$ (8), we obtain

$$A_0'P_{00} + P_{00}A_0 + P_{01}(h) + P_{01}(h)' = -X \quad (10)$$

$$\begin{aligned}\frac{d}{ds}P_{01}(s) = A_0'P_{01}(s) + P_{01}(h-s)'A_1 \\ + \int_0^s P_{01}(h-s+t)'F(t)\,dt \\ + \int_0^s F(h-s+t)'P_{01}(t)\,dt + P_{00}F(s)\end{aligned} \tag{11}$$

where

$$P_{01}(0) = P_{00}A_1. \tag{12}$$

3 Solution to the Lyapunov Functional

The differential equation (11) will be transformed into a two point boundary problem in Theorem 2. To do that, $Q(s)$ and $R(s)$ are defined as follows:

$$\begin{aligned}Q(s) &\triangleq P_{00}F_1(s) + \int_0^s P_{01}(h-s+t)'F_1(t)\,dt \\ R(s) &\triangleq \int_0^s F_1(h-s+t)'P_{01}(t)\,dt.\end{aligned} \tag{13}$$

Theorem 2 *The matrix differential equation (11) can be equivalently described by*

$$\frac{d}{ds}\left[\begin{array}{c}\operatorname{cs}P_{01}(s)\\ \operatorname{cs}Q(s)\\ \operatorname{cs}R(s)\\ \hline \operatorname{cs}P_{01}(h-s)\\ \operatorname{cs}Q(h-s)\\ \operatorname{cs}R(h-s)\end{array}\right] = H\left[\begin{array}{c}\operatorname{cs}P_{01}(s)\\ \operatorname{cs}Q(s)\\ \operatorname{cs}R(s)\\ \hline \operatorname{cs}P_{01}(h-s)\\ \operatorname{cs}Q(h-s)\\ \operatorname{cs}R(h-s)\end{array}\right] \tag{14}$$

where

$$H \triangleq \left[\begin{array}{ccc|ccc} E_1 & E_2 & E_3 & E_4 & 0 & 0\\ 0 & E_5 & 0 & E_6 & 0 & 0\\ E_7 & 0 & E_8 & 0 & 0 & 0\\ \hline -E_4 & 0 & 0 & -E_1 & -E_2 & -E_3\\ -E_6 & 0 & 0 & 0 & -E_5 & 0\\ 0 & 0 & 0 & -E_7 & 0 & -E_8\end{array}\right] \tag{15}$$

$$\begin{gathered}E_1 \triangleq (I\otimes A_0'),\ E_2 \triangleq (F_3'\otimes I),\ E_3 \triangleq I\otimes F_3' \\ E_4 \triangleq (A_1'\otimes I)T,\ E_5 \triangleq (F_2'\otimes I),\ E_6 \triangleq (F_1(0)'\otimes I)T \\ E_7 \triangleq I\otimes F_1(h)',\ E_8 \triangleq -(I\otimes F_2').\end{gathered}$$

Matrix T is defined by

$$T \triangleq [\ T_1 \mid T_2 \mid \ \cdots \ \mid T_{n^2}\ ],\ T_k \in \mathbb{R}^{n^2\times 1}. \tag{16}$$

Row vector $T_k,\ 1\le k\le n^2$ is defined by

$$T_{(i-1)n+j} \triangleq e_{(j-1)n+i},\ 1\le i,j\le n,$$

where $e_k \in \mathbb{R}^{n^2\times 1},\ 1\le k\le n^2$ is a row vector whose k-th element is 1 and all other elements are 0.

Now from (14), relationship between P_{00} and $P_{01}(h)$ satisfying (11) will be derived so that simultaneous equations (10) and (11) can be solved.

Theorem 3 *The P_{00} and $P_{01}(h)$ pair satisfying (10) and (11) can be computed from the following equation:*

$$\left[\begin{array}{cccc}(I\otimes A_0')+(A_0'\otimes I) & I+T & 0 & 0\\ R_1 & R_2 & R_3 & R_4\end{array}\right]\left[\begin{array}{c}\operatorname{cs}P_{00}\\ \operatorname{cs}P_{01}(h)\\ \operatorname{cs}Q(h)\\ \operatorname{cs}R(h)\end{array}\right] = \left[\begin{array}{c}-\operatorname{cs}X\\ 0\end{array}\right] \tag{17}$$

where $R_1 \sim R_4$ can be computed from the following: first compute the singular value decomposition of the following matrix:

$$\begin{aligned}(J-\exp(Hh))\left[\begin{array}{cccc}(A_1'\otimes I) & 0 & 0 & 0\\ (F_1(0)'\otimes I) & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{array}\right] \\ = U\left[\begin{array}{cc}\Sigma_1 & 0\\ 0 & 0\end{array}\right]V'\end{aligned} \tag{18}$$

where U and V are unitary matrices, Σ_1 is a diagonal matrix whose diagonal elements are nonzero, and J is defined by

$$J \triangleq \left[\begin{array}{cc}0 & I\\ I & 0\end{array}\right].$$

$R_1 \sim R_4$ are defined by

$$[\ R_1 \quad R_2 \quad R_3 \quad R_4\ ] \triangleq [\ \Sigma_1\ 0\ ]\,V'. \tag{19}$$

Unfortunately, the solution to (5) alone is not sufficient for stability check because the definiteness condition (4) also need to be checked, which is not easy to check. In the next section, we will derive a stability condition, which does not require the definiteness condition (4).

4 Stability Condition

Theorem 4 *System (1) is stable for $h\in[0,h_{\max}]$ if (i) (1) is stable for $h=0$ (ii) The matrix in (17) is nonsingular for $h\in[0,h_{\max}]$.*

The condition (i) can be checked by investigating eigenvalues of A_0+A_1. The condition (ii) can be checked, for example, by investigating minimum singular values of the matrix in (17) for $h\in[0,h_{\max}]$.

References

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