Optimal Control of Linear Nonstandard Singularly Perturbed Discrete Systems

Myo-Taeg Lim and Beom-Soo Kim

School of Electrical Engineering, Korea University
Seoul, Korea 136-701, Email: mlim@elec.korea.ac.kr

Abstract

This paper introduces the definition of the nonstandard linear discrete-time singularly perturbed system and shows how to solve the corresponding linear-quadratic optimal control problem since the methodology that exists in the literature for the solution of the standard singularly perturbed discrete linear-quadratic optimal control problem can not be extended to the corresponding nonstandard counterpart. The solution of the optimal control problem is obtained in terms of the pure-slow and pure-fast, reduced-order, *continuous-time*, algebraic Riccati equations.

1 Introduction

Standard discrete-time linear singularly perturbed systems have been studied in the past by several researchers, see for example [3, 4, 5, 11, 12, 13]. However, the nonstandard singularly perturbed *discrete* linear systems have not been yet defined in the control literature. Motivated by the existence of results for continuous-time nonstandard singularly perturbed linear systems, [7, 8, 9, 16], in this paper, we formally define the nonstandard singularly perturbed linear systems in discrete time, and show how to solve the corresponding linear-quadratic optimal control problem.

In this paper we use the formulation of singularly perturbed linear discrete-time control systems of [12, 13] given by

$$\begin{aligned} x_1(k+1) &= (I_{n_1}+\epsilon A_1)x_1(k)+\epsilon A_2 x_2(k)+\epsilon B_1 u(k) \\ x_2(k+1) &= A_3 x_1(k)+A_4 x_2(k)+B_2 u(k) \\ & \qquad x_1(0)=x_{10}, \quad x_2(0)=x_{20} \end{aligned} \tag{1}$$

with slow variables $x_1 \in R^{n_1}$, fast state variables $x_2 \in R^{n_2}$, and control inputs $u \in R^m$. ϵ is a small positive parameter.

The system (1) is said to be the standard form [12, 13] when $(I_{n_2}-A_4)$ is nonsingular, otherwise it is said to be in the nonstandard form. This system has been studied in the control literature under the assumption imposed by [12, 13] that

$$det(I_{n_2}-A_4) \neq 0 \tag{2}$$

This assumption is used to define the quasi-steady state for the fast subsystem as

$$x_{2s}(k) = A_3 x_{1s}(k)+A_4 x_{2s}(k)+B_2 u(k) \tag{3}$$

which in fact gives the slow portion of the fast variable

$$x_{2s}(k) = (I_{n_2}-A_4)^{-1}(A_3 x_{1s}(k)+B_2 u(k)) \tag{4}$$

so that the approximate slow subsystems is obtained by replacing (4) in the first equation of (1).

In analogy with the corresponding continuous-time problem [9] defined by

$$\begin{aligned} \dot{x}_1(t) &= A_1^c x_1(t)+A_2^c x_2(t)+B_1^c u(t) \\ \epsilon \dot{x}_2(t) &= A_3^c x_1(t)+A_4^c x_2(t)+B_2^c u(t) \end{aligned} \tag{5}$$

the quasi-steady state for the fast system is

$$0 = A_3^c x_{1s}(t)+A_4^c x_{2s}(t)+B_2^c u(t) \tag{6}$$

or

$$x_{2s}(t) = -{A_4^c}^{-1}(A_3^c x_{1s}(t)+B_2^c u(t)) \tag{7}$$

Hence, the discrete-time condition (2) corresponds to the continuous-time condition

$$det(A_4^c) \neq 0 \tag{8}$$

Such kinds of continuous-time singularly perturbed linear systems are called standard singularly perturbed linear systems. However, it has been noticed that in some real physical applications like a flexible space structure the fast subsystem matrix A_4^c is singular. Such systems with singular matrix A_4^c are called nonstandard continuous-time singularly perturbed systems. They have been studied by several researchers [7, 8, 9, 16]. It is interesting to point out that nonstandard singularly perturbed linear systems have not been studied yet in the discrete-time.

In this paper we introduce a formal definition of discrete-time nonstandard singularly perturbed control system and show that the results of [11] can be used for optimal control of both standard and nonstandard singularly perturbed discrete systems.

Definition: Nonstandard linear singularly perturbed system in discrete-time is defined by (1) with $det(I_{n_2} - A_4) = 0$.

Due to the fact that $det(I_{n_2} - A_4) = 0$ we are not able to define the quasi-steady state of the fast variable by (4), hence we can not study the corresponding linear-quadratic optimal control problem along the lines of [12, 13]. Also, the recursive fixed point iteration schemes of [5] can not be used since the corresponding slow-fast decompositions of the algebraic Lyapunov and Riccati equations are not valid any more since they explicitly require $det(I_{n_2} - A_4) \neq 0$. Note that also the celebrated Chang transformation [2] requires in the discrete-time domain nonsingularity assumption on $(I_{n_2} - A_4)$. (See [1, 5]).

2 Linear-Quadratic Optimal Control Problem

Consider a nonstandard singularly perturbed linear discrete system in (1) where $(I_{n_2} - A_4)$ may be singular. With (1) a quadratic performance criterion to be optimized subject to (8) is associated

$$J = \frac{\epsilon}{2} \sum_{k=0}^{\infty} \left[x(k)^T Q x(k) + u(k)^T R u(k) \right] \tag{9}$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \geq 0, \quad R > 0 \tag{10}$$

The solution of the above optimal control problem is given by

$$\begin{aligned} u(k) &= -R^{-1} B^T \lambda(k+1) \\ &= -\left(R + B^T P B\right)^{-1} B^T P A x(k) \end{aligned} \tag{11}$$

where $\lambda(k)$ is a costate variable and P is the positive semi-definite stabilizing solution of the discrete algebraic Riccati equation [10] given by

$$\begin{aligned} P &= Q + A^T P \left[I + SP\right]^{-1} A \\ &= Q + A^T P A - A^T P B \left[R + B^T P B\right]^{-1} B^T P A \end{aligned} \tag{12}$$

For the singularly perturbed discrete systems, corresponding matrices in (11)-(13) are given by

$$\begin{gathered} A = \begin{bmatrix} I_{n_1} + \epsilon A_1 & \epsilon A_2 \\ A_3 & A_4 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{1}{\epsilon} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \\ B = \begin{bmatrix} \epsilon B_1 \\ B_2, \end{bmatrix}, \; S = B R^{-1} B^T = \begin{bmatrix} \epsilon^2 S_1 & \epsilon Z \\ \epsilon Z^T & S_2 \end{bmatrix} \\ S_1 = B_1 R^{-1} B_1^T, \; S_2 = B_2 R^{-1} B_2^T, \; Z = B_1 R^{-1} B_2^T \end{gathered} \tag{13}$$

The Hamiltonian form of (1) and (9) can be written as the forward recursion [10]

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = \mathbf{H} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \tag{14}$$

with

$$\mathbf{H} = \begin{bmatrix} A + BR^{-1}B^T A^{-T} Q & -BR^{-1}B^T A^{-T} \\ -A^{-T} Q & A^{-T} \end{bmatrix} \tag{15}$$

where $\mathbf{H}$ is the symplectic matrix, which has the property that the eigenvalues of $\mathbf{H}$ are grouped into two disjoint subsets Γ_1 and Γ_2, such that for every $\lambda_c \in \Gamma_1$ there exists $\lambda_d \in \Gamma_2$, which satisfies $\lambda_c \times \lambda_d = 1$, and we can choose either Γ_1 or Γ_2 to contain only the stable eigenvalues [15].

In (15) the assumption that the matrix A is invertible is used, which requires the invertibility of the matrix A_4. In that case

$$A^{-1} = \begin{bmatrix} I_{n_1} & 0 \\ -A_4^{-1} A_3 & A_4^{-1} \end{bmatrix} + O(\epsilon) \tag{16}$$

Hence, the presentation requires the following assumption.

Assumption 1: The fast subsystem matrix A_4 is nonsingular.

In the following, we show how to obtain exactly the solution of the discrete-time algebraic Riccati equation of singularly perturbed systems, (13), in terms of solutions of two reduced-order *continuous-time*, pure-slow and pure-fast, algebraic Riccati equations.

Partitioning $\lambda(k)$ as $\lambda(k) = \left[\lambda_1^T(k) \; \lambda_2^T(k)\right]^T$ with $\lambda_1(k) \in \Re^{n_1}$ and $\lambda_2(k) \in \Re^{n_2}$, we get

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \lambda_1(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \mathbf{H} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \tag{17}$$

It can been shown after some algebra that the Hamiltonian matrix (15) has the following form [11]

$$\mathbf{H} = \begin{bmatrix} I_{n_1} + \epsilon \overline{A_1} & \epsilon \overline{A_2} & \epsilon^2 \overline{S_1} & \epsilon \overline{S_2} \\ \overline{A_3} & \overline{A_4} & \epsilon \overline{S_3} & \overline{S_4} \\ \overline{Q_1} & \overline{Q_2} & I_{n_1} + \epsilon \overline{A_{11}^T} & \overline{A_{21}^T} \\ \overline{Q_3} & \overline{Q_4} & \epsilon \overline{A_{12}^T} & \overline{A_{22}^T} \end{bmatrix} \tag{18}$$

Note that in the remaining part of this section there is no need for analytical expressions for the bared matrices. Those matrices have to be formed by the computer in the process of calculations, which can be done easily, for example, by using MATLAB.

Interchanging second and third rows in (17) and using the following partitioning and scaling

$[p_1(k) \;\; p_2(k)]^T = [\epsilon\lambda_1(k) \;\; \lambda_2(k)]^T$ in (17) yield

$$\begin{bmatrix} x_1(k+1) \\ p_1(k+1) \\ x_2(k+1) \\ p_2(k+1) \end{bmatrix} = \overline{\mathbf{H}} \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} = \begin{bmatrix} I_{2n_1} + \epsilon T_1 & \epsilon T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} \tag{19}$$

where

$$\overline{\mathbf{H}} = \begin{bmatrix} I_{n_1} + \epsilon\overline{A_1} & \epsilon\overline{S_1} & \epsilon\overline{A_2} & \epsilon\overline{S_2} \\ \epsilon\overline{Q_1} & \overline{I_{n_1} + \epsilon A_{11}^T} & \epsilon\overline{Q_2} & \epsilon\overline{A_{21}^T} \\ \overline{A_3} & \overline{S_3} & \overline{A_4} & \overline{S_4} \\ \overline{Q_3} & \overline{A_{12}^T} & \overline{Q_4} & \overline{A_{22}^T} \end{bmatrix}$$

$$T_1 = \begin{bmatrix} \overline{A_1} & \overline{S_1} \\ \overline{Q_1} & \overline{A_{11}^T} \end{bmatrix}, \quad T_2 = \begin{bmatrix} \overline{A_2} & \overline{S_2} \\ \overline{Q_2} & \overline{A_{21}^T} \end{bmatrix}$$
$$T_3 = \begin{bmatrix} \overline{A_3} & \overline{S_3} \\ \overline{Q_3} & \overline{A_{12}^T} \end{bmatrix}, \quad T_4 = \begin{bmatrix} \overline{A_4} & \overline{S_4} \\ \overline{Q_4} & \overline{A_{22}^T} \end{bmatrix} \tag{20}$$

Introducing the notation

$$U(k) = \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix}, \quad V(k) = \begin{bmatrix} x_2(k) \\ p_2(k) \end{bmatrix} \tag{21}$$

we have the singularly perturbed discrete-time linear system

$$\begin{aligned} U(k+1) &= (I_{2n_1} + \epsilon T_1) U(k) + \epsilon T_2 V(k) \\ V(k+1) &= T_3 U(k) + T_4 V(k) \end{aligned} \tag{22}$$

Applying to (22) the discrete-time version of the Chang transformation [2, 5] defined by

$$\mathbf{T} = \begin{bmatrix} I_{2n_1} - \epsilon HL & -\epsilon H \\ L & I_{2n_2} \end{bmatrix}$$
$$\mathbf{T}^{-1} = \begin{bmatrix} I_{2n_1} & \epsilon H \\ -L & I_{2n_2} - \epsilon LH \end{bmatrix}$$
$$\begin{bmatrix} \eta(k) \\ \xi(k) \end{bmatrix} = \mathbf{T} \begin{bmatrix} U(k) \\ V(k) \end{bmatrix} \tag{23}$$

produces in the new coordinates two completely decoupled subsystems

$$\begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} = \eta(k+1) = (I_{2n_1} + \epsilon T_1 - \epsilon T_2 L)\,\eta(k) \tag{24}$$

$$\begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} = \xi(k+1) = (T_4 + \epsilon L T_2)\,\xi(k) \tag{25}$$

where the matrices L and H satisfy

$$-L + T_4 L - T_3 - \epsilon L (T_1 - T_2 L) = 0 \tag{26}$$

$$H + T_2 - H T_4 + \epsilon (T_1 - T_2 L) H - \epsilon H L T_2 = 0 \tag{27}$$

The unique solutions of algebraic equations (26) and (27) exist, by the implicit function theorem [14], under the condition that the matrix $T_4 - I_{2n_2}$ is nonsingular. It can be shown from (18)-(20) that the matrix T_4 is given by

$$T_4 = T_4^{(0)} + O(\epsilon) = \begin{bmatrix} A_4 + S_2 A_4^{-T} Q_3 & -S_2 A_4^{-T} \\ -A_4^{-T} Q_3 & A_4^{-T} \end{bmatrix} + O(\epsilon) \tag{28}$$

From (15) we see that $T_4^{(0)}$ represents the Hamiltonian matrix of the fast subsystem. The nonsingularity of $T_4^{(0)} - I_{2n_2}$ requires the following assumption.

Assumption 2: The triple $\left(A_4, B_2, \sqrt{Q_3}\right)$ is stabilizable-detectable.

It follows that under Assumption 2, the matrix $T_4 - I_{2n_2}$ is nonsingular for sufficiently small values of ϵ.

The algebraic equations (26) and (27) can be solved using the Newton method, similarly to the solution of the corresponding continuous-time algebraic equations [6]. The Newton method converges quadratically in the neighborhood of the sought solution, that is, its rate of convergence is $O\left(\epsilon^{2^i}\right)$. The initial guess required for the Newton method is easily obtained with the accuracy of $O(\epsilon)$, by setting $\epsilon = 0$ in equation (26), that is

$$L^{(0)} = (T_4 - I_{2n_2})^{-1} T_3 = L + O(\epsilon) \tag{29}$$

The Newton algorithm can be constructed by setting $L^{(i+1)} = L^{(i)} + \Delta L^{(i)}$ and neglecting $O(\Delta L)^2$ terms. This leads to a Lyapunov-type equation of the form

$$D_1^{(i)} L^{(i+1)} + L^{(i+1)} D_2^{(i)} = Q^{(i)} \tag{30}$$

with

$$D_1^{(i)} = T_4 - I_{2n_1} + \epsilon L^{(i)} T_2, \quad i = 0, 1, 2, \ldots \tag{31}$$
$$D_2^{(i)} = -\epsilon\left(T_1 - T_2 L^{(i)}\right), \quad Q^{(i)} = T_3 + \epsilon L^{(i)} T_2 L^{(i)}$$

where the initial condition is obtained from (29). The Newton sequence will be $O\left(\epsilon^2\right)$, $O\left(\epsilon^4\right)$, $O\left(\epsilon^8\right)$, ..., $O\left(\epsilon^{2^i}\right)$ close to the exact solution, respectively, in each iteration. Having found the solution of (26) up to the desired degree of accuracy, one can get the solution of (27) by solving directly the algebraic Lyapunov-like (Sylvester) equation of the form

$$H^{(i)} D_1^{(i)} + D_2^{(i)} H^{(i)} = T_2 \tag{32}$$

which implies $H^{(i)} = H + O\left(\epsilon^{2^i}\right)$.

The rearrangement and modification of variables in (19) is done by using the permutation matrix E_1 of the form

$$\begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & \epsilon I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix}$$

$$= E_1 \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \tag{33}$$

From (21), (23)-(25), and (33), we obtain the relationship between the original coordinates and the new ones

$$\begin{aligned} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \\ \eta_2(k) \\ \xi_2(k) \end{bmatrix} &= E_2^T \mathbf{T} E_1 \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} = \Pi \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \\ &= \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \end{aligned} \tag{34}$$

where E_2 is a permutation matrix of the form

$$E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{35}$$

Since $\lambda(k) = Px(k)$, where P satisfies the discrete algebraic Riccati equation (13), it follows from (34) that

$$\begin{aligned} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} &= (\Pi_1 + \Pi_2 P)\, x(k) \\ \begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} &= (\Pi_3 + \Pi_4 P)\, x(k) \end{aligned} \tag{36}$$

In the original coordinates, the required optimal solution has a closed-loop nature. We have the same characteristic for the new systems (24) and (25), that is,

$$\begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} \tag{37}$$

Then (36) and (37) yield

$$\begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1} \tag{38}$$

Following the same logic, we can find P reversely by introducing

$$E_1^{-1} \mathbf{T}^{-1} E_2 = \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \Pi^{-1} \tag{39}$$

and it yields

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \right) \left(\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix} \right)^{-1} \tag{40}$$

It can be shown, by estimating the order of quantity for the entries Π_1, Π_2, Ω_1, Ω_2, that required matrix in (38) and (40) is invertible [11].

Partitioning (24) and (25) as

$$\begin{aligned} \begin{bmatrix} \eta_1(k+1) \\ \eta_2(k+1) \end{bmatrix} &= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} \\ &= (I_{2n_1} + \epsilon T_1 - \epsilon T_2 L) \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} \end{aligned} \tag{41}$$

$$\begin{aligned} \begin{bmatrix} \xi_1(k+1) \\ \xi_2(k+1) \end{bmatrix} &= \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} \\ &= (T_4 + \epsilon L T_2) \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} \end{aligned} \tag{42}$$

and using (38) yield two reduced-order nonsymmetric algebraic Riccati equations

$$P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s = 0 \tag{43}$$

$$P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f = 0 \tag{44}$$

It is very interesting that the algebraic Riccati equation of singularly perturbed discrete-time control systems is completely and exactly decomposed into two reduced-order nonsymmetric *continuous-time* algebraic Riccati equations (43)-(44).

The pure-fast algebraic Riccati equation (44) is nonsymmetric, but its $O(\epsilon)$ perturbation is symmetric one. This can be observed from the fact that

$$\begin{aligned} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} &= (T_4 + \epsilon L T_2) = \begin{bmatrix} b_1^{(0)} & b_2^{(0)} \\ b_3^{(0)} & b_4^{(0)} \end{bmatrix} + O(\epsilon) \\ &= T_4^{(0)} + O(\epsilon) \end{aligned} \tag{45}$$

with $T_4^{(0)}$ given in (28). The coefficients of the Hamiltonian matrix $T_4^{(0)}$ imply the following approximate, fast subsystem, discrete-time algebraic Riccati equation

$$\begin{aligned} P_f^{(0)} &= A_4^T P_f^{(0)} A_4 + Q_3 \\ &\quad - A_4^T P_f^{(0)} B_2 \left(R + B_2^T P_f^{(0)} B_2 \right)^{-1} B_2^T P_f^{(0)} A_4 \end{aligned} \tag{46}$$

such that $P_f = P_f^{(0)} + O(\epsilon)$. Note that the positive semidefinite stabilizing solution of (46) exists under Assumption 2. Equation (46) is identical to the approximate fast discrete-time algebraic Riccati equation of [12, 13].

In order to establish that an $O(\epsilon)$ approximation of the pure-slow algebraic Riccati equation (43) is symmetric, we use the following arguments. It follows from (29) and (41) that

$$\begin{aligned} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} &= I_{2n_1} + \epsilon (T_1 - T_2 L) = I_{2n_1} + \epsilon T_s \\ &= I_{2n_1} + \epsilon \left(T_1 - T_2 L^{(0)} \right) + O(\epsilon) \\ &= I_{2n_1} + \epsilon T_s^{(0)} + O(\epsilon) = \begin{bmatrix} a_1^{(0)} & a_2^{(0)} \\ a_3^{(0)} & a_4^{(0)} \end{bmatrix} + O(\epsilon) \\ &= I_{2n_1} + \epsilon \left(T_1 - T_2 (T_4 - I_{2n_2})^{-1} T_3 \right) + O(\epsilon) \end{aligned} \tag{47}$$

On the other hand, the approximate slow continuous-time algebraic Riccati equation can be obtained from

$$\begin{bmatrix} x_1(k+1) \\ p_1(k+1) \\ x_2(k+1) \\ p_2(k+1) \end{bmatrix} = \begin{bmatrix} I_{2n_1} + \epsilon T_1 & \epsilon T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} \tag{48}$$

by using the methodology of [12, 13] and assuming that the fast variables $x_2(k)$ and $p_2(k)$ are at the steady state. Using the fact that at the steady state $x_2(k+1) = x_2(k)$ and $p_2(k+1) = p_2(k)$ we get from (48)

$$\begin{bmatrix} x_2(k) \\ p_2(k) \end{bmatrix} = (I_{2n_2} - T_4)^{-1} \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} \tag{49}$$

and

$$\begin{bmatrix} x_1(k+1) \\ p_1(k+1) \end{bmatrix} = \left\{ I_{2n_1} + \epsilon \left(T_1 - T_2 (T_4 - I_{2n_2})^{-1} T_3 \right) \right\} \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} = \left(I_{2n_1} + \epsilon T_s^{(0)} + O(\epsilon^2) \right) \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} \tag{50}$$

The matrix $T_s^{(0)}$ determines the coefficients for the approximate slow continuous-time algebraic equation of [12]. It can be observed from (47) and (50) that

$$\begin{aligned} \begin{bmatrix} a_1^{(0)} & a_2^{(0)} \\ a_3^{(0)} & a_4^{(0)} \end{bmatrix} &= I_{2n_1} + \epsilon T_s^{(0)} \\ &= I_{2n_1} + \epsilon \left(T_1 - T_2 (T_4 - I_{2n_2})^{-1} T_3 \right) \\ &= \begin{bmatrix} I_{n_1} + \epsilon A_s & -\epsilon B_s R_s^{-1} B_s \\ -\epsilon Q_s & I_{n_1} - \epsilon A_s^T \end{bmatrix} \\ &= \begin{bmatrix} I_{n_1} + \epsilon A_s & -\epsilon S_s \\ -\epsilon Q_s & I_{n_1} - \epsilon A_s^T \end{bmatrix} \end{aligned} \tag{51}$$

The corresponding approximate continuous-time algebraic Riccati equation is given by

$$P_s^{(0)} A_s + A_s^T P_s^{(0)} + Q_s - P_s^{(0)} S_s P_s^{(0)} = 0 \tag{52}$$

such that $P_s = P_s^{(0)} + O(\epsilon)$. The matrices defined in (51) can be also found in [12]. The unique positive semidefinite stabilizing solution of the slow approximate continuous-time algebraic Riccati equation exists under the assumption that the approximate slow subsystem is stabilizable-detectable.

Assumption 3: The approximate slow subsystem determined by $T_s^{(0)}$ is stabilizable-detectable, that is, the triple $\left(A_s, \sqrt{S_s}, \sqrt{Q_s}\right)$ is stabilizable-detectable.

We have established that $O(\epsilon)$ perturbations of (43) and (44) lead to the symmetric reduced-order approximate slow and fast algebraic Riccati equations obtained in [12]. The solutions of these equations can be used as very good initial guesses for the Newton method for solving the obtained nonsymmetric Riccati equations (43) and (44).

The Newton algorithm for (43) is given by

$$P_s^{(i+1)} \left(a_1 + a_2 P_s^{(i)} \right) - \left(a_4 - P_s^{(i)} a_2 \right) P_s^{(i+1)} = a_3 + P_s^{(i)} a_2 P_s^{(i)}, \quad i = 0, 1, 2, ... \tag{53}$$

with the initial guess $P_s^{(0)}$ obtained from the continuous-time approximate slow algebraic Riccati equation (52). The Newton algorithm for (44) is similarly obtained as

$$P_f^{(i+1)} \left(b_1 + b_2 P_f^{(i)} \right) - \left(b_4 - P_f^{(i)} b_2 \right) P_f^{(i+1)} = b_3 + P_f^{(i)} b_2 P_f^{(i)}, \quad i = 0, 1, 2, ... \tag{54}$$

with the initial guess $P_f^{(0)}$ found by solving the discrete-time approximate fast algebraic Riccati equation (46).

Using solutions of both pure-slow and pure-fast Riccati equations and formulas (37), (41)-(42), the completely decoupled pure-slow and pure-fast subsystems in the new coordinates are, respectively, given by

$$\eta_1(k+1) = (a_1 + a_2 P_s)\, \eta_1(k) \tag{55}$$

$$\xi_1(k+1) = (b_1 + b_2 P_f)\, \xi_1(k) \tag{56}$$

The proposed method is very suitable for parallel computations since it allows complete parallelism. In addition, due to complete and exact decomposition of the discrete algebraic Riccati equation, the optimal control at steady state can be performed independently and in parallel at the local levels (slow and fast subsystems).

3 Conclusion

In this paper we have introduced the definition of discrete-time nonstandard singularly perturbed linear systems and shown how to solve the corresponding numerically ill-conditioned linear-quadratic optimal control problem in terms of well-conditioned reduced-order pure-slow and pure-fast continuous-time algebraic Riccati equations. The presented method produces the exact solution for both standard and nonstandard singularly perturbed linear discrete systems.

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