Local Stabilization of Linear Systems under Amplitude and Rate Saturating Actuators

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Abstract

This paper addresses the problem of local stabilization of linear systems subject to control amplitude and rate saturation. Considering the actuator represented by a first order system subject to input and state saturation, a condition for the stabilization of an a priori given set of admissible initial states is formulated from certain saturation nonlinearities representation and quadratic stability results. From this condition an algorithm based on the iterative solution of LMI-problems is proposed in order to compute the control law.
Keywords: Amplitude and rate control saturation, local stabilization, LMI.

1 Introduction

Physical and technological constraints do not allow that control actuators provide unlimited amplitude signals neither react unlimited fast. This fact implies that the most of the control systems are susceptible to be submitted to amplitude and rate actuator saturation. The negligence of both amplitude and rate control bounds can be source of limit cycles, parasitic equilibrium points and even instability of the closed-loop system. In particular, rate saturation is responsible by a phase lag that has a high destabilizing effect. Moreover, the problem of actuator amplitude saturation may be more difficult to solve when the actuator is also subject to rate saturation [6], [11], [8]. For these reasons, the study of the problem of control systems subject to amplitude and rate saturation has been received the attention of many researchers in the last years.

More recently, the problem of stabilization of linear systems under actuator amplitude and rate saturation as a more generic problem has been the subject of some papers (see for instance [9], [11], [7], [10]). In these papers, we can identify basically two kinds of saturating actuator modeling. In the first one, the actuator is modeled as an integrator presenting input and output (state) amplitude saturation [9], [10]. The second one can be viewed as a *position-feedback-type model with speed limitation* and considers the actuator represented by a first order system subject to input saturation (corresponding to the amplitude saturation) and output (state) saturation (corresponding to the rate saturation) [11], [7]. It should be noticed that the first model appears as a particular case of the second one. Both models allow to treat the problem only considering amplitude saturations in the control loop. The design approaches can be also classified in two groups: control laws where saturation effectively occurs (i.e. the behavior of the system is nonlinear) [11], [10], and control laws where saturation is avoided (i.e. linear behavior) [7], [9].

In this paper we address the problem of local stabilization of linear systems subject to both actuator amplitude and rate saturation. Our objective is to propose a method for computing state feedback saturating control laws that ensure both asymptotic stability of the closed-loop system with respect to a given set of admissible initial conditions and a certain degree of time-domain performance in a neighborhood of the origin. Considering the actuator represented by a first order system subject to input and state saturation, a condition for the local stabilization is formulated from certain saturation nonlinearities representation and quadratic stability results. From this condition an algorithm based on the iterative solution of LMI-problems is proposed in order to compute the control law. It should be pointed out that, unlike [7] and [9], in our approach effective saturation is allowed and no open stability assumptions are made. On the other hand, our approach differs from [11], [10] and [9] in the sense that the set of admissible initial states to be stabilized is given a priori.

Notations. For any vector $x \in \Re^n$, $x \succeq 0$ means that all the components of x, denoted $x_{(i)}$, are nonnegative. For two vectors x, y of $\Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, $\forall i = 1, \ldots, n$. The elements of a matrix $A \in \Re^{m*n}$ are denoted by $a_{(i,l)}$, $i = 1, \ldots, m$, $l = 1, \ldots, n$. $A_{(i)}$ denotes the ith row of matrix A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A^T denotes the transpose of A. For $x \in \Re^m$,

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$D(x) = diag(x) \in \Re^{m*m}$ denotes a diagonal matrix obtained from vector x, i.e. $d_{(i,i)}(x) = x_{(i)}$. **Co** $\{\cdot\}$ denotes a convex hull. 1_m denotes a m-dimensional vector of ones, i.e. $1_m \triangleq [1 \;\; 1 \ldots 1]^T \in \Re^m$.

2 Problem Statement

Consider the following linear continuous-time system defined as:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \Re^n$ and $u(t) \in \Re^m$ are respectively the state vector and the control vector. Matrices A and B are real constant matrices of appropriate dimensions. The pair (A, B) is supposed to be controllable.

Each control actuator is supposed to be a first order system, that presents both position and rate limitations, that is, $\forall i = 1, \ldots, m$ we have

$$\dot{u}_{(i)} = sat_{r(i)}(\tau_{(i)}(-u_{(i)}(t) + sat_{p(i)}(v_{(i)}(t)))) \tag{2}$$

where $v_{(i)}$ and $u_{(i)}$ are respectively the input and the output (state) of the actuator, $-\tau_{(i)} < 0$, corresponds to the pole of the actuator and

$$\begin{aligned} sat_{r(i)}(\cdot) &\triangleq sign(\cdot)\min(\xi_{(i)}, |\cdot|) \\ sat_{p(i)}(\cdot) &\triangleq sign(\cdot)\min(\rho_{(i)}, |\cdot|) \end{aligned}$$

where $\xi_{(i)}$ and $\rho_{(i)}$ denote respectively the rate and amplitude bounds.

This model of rate-limiting is known as the classical *rate limiter* and can be viewed as a position-feedback-type model with speed limitation [11]. As pointed in [11], for $\tau_{(i)} \to \infty$, this model corresponds to the ideal rate limiter of SIMULINK.

Consider now a state feedback defined as follows

$$v(t) = K_x x(t) + K_u u(t) \tag{3}$$

Let $T \in \Re^{m*m}$ be a diagonal matrix where $t_{(i,i)} = \tau_{(i)}, \quad i = 1, \ldots, m$. The closed-loop system is depicted in figure 1 and can be described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ \dot{u}(t) &= sat_r(-Tu(t) + Tsat_p(K_x x(t) + K_u u(t))) \end{aligned} \tag{4}$$

Define the regions R_{Lp} and R_{Lr}, respectively as the region where there is no occurrence of amplitude saturation and the region where there is no occurrence of rate saturation:

$$R_{Lp} \triangleq \{x \in \Re^n, u \in \Re^m \;;\; |[K_x \;\; K_u]\begin{bmatrix} x \\ u \end{bmatrix}| \leq \rho\}$$

$$R_{Lr} \triangleq \{x \in \Re^n, u \in \Re^m; |-Tu + Tsat_p(K_x x + K_u u)| \leq \xi\}$$

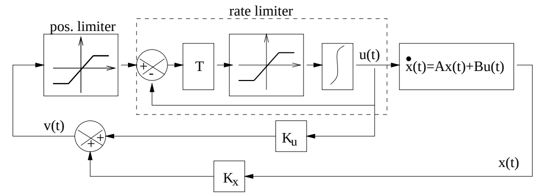

Figure 1: closed-loop system

It follows that $R_L \triangleq R_{Lp} \cap R_{Lr}$ is the region of linear behavior of system (1), i.e. where no saturation occurs.

Define the augmented state vector $z(t) \triangleq \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \Re^{n+m}$. Let now $\mathcal{Z}_0$ be a set of admissible initial states in the state space $\Re^{n+m}$. This set can be viewed as the *zone of operation* of the system (4).

From the definitions and considerations above the problem we intend to solve is stated as follows:

Problem 1 *Find matrices K_x and K_u such that*
1. System (4) is locally asymptotically stable in $\mathcal{Z}_0$, that is, $\forall z(0) = [x(0)^T \;\; u(0)^T]^T \in \mathcal{Z}_0$, the corresponding trajectories converge asymptotically to the origin.
2. When the system operates inside the linearity region R_L, a certain time-domain performance specification is satisfied.

3 System Representation

Consider now the following definitions[1]:

- a vector $\alpha(t) \in \Re^m$ with

$$\alpha_{(i)}(t) \triangleq \min(1, \frac{\rho_{(i)}}{|\mathcal{K}_{(i)} z(t)|})$$

- a vector $\beta(t) \in \Re^m$ with

$$\beta_{(i)}(t) \triangleq \min(1, \frac{\xi_{(i)}}{|\dot{u}_{(i)}(t)|})$$

- the matrices: $\mathcal{A}(\beta(t)) = \begin{bmatrix} A & B \\ 0 & -D(\beta(t))T \end{bmatrix}$; $\mathcal{B}(\beta(t)) = \begin{bmatrix} 0 \\ D(\beta(t))T \end{bmatrix}$; $\mathcal{K} = [K_x \;\; K_u]$

From the definitions above, it follows that the closed-loop system (4) is equivalent to the following one

$$\dot{z}(t) = (\mathcal{A}(\beta(t)) + \mathcal{B}(\beta(t))D(\alpha(t))\mathcal{K})z(t) \tag{5}$$

[1] $D(.)$ denotes a diagonal matrix (see Notations).

Let now $\underline{\alpha} \triangleq [\underline{\alpha}_{(1)} \quad \underline{\alpha}_{(2)} \quad \dots \quad \underline{\alpha}_{(m)}]^T$ and $\underline{\beta} \triangleq [\underline{\beta}_{(1)} \quad \underline{\beta}_{(2)} \quad \dots \quad \underline{\beta}_{(m)}]^T$ be respectively a vector of lower bounds for $\alpha(t)$ and for $\beta(t)$. Consider now all the possible m-order vectors such that the ith entry takes the value 1 or $\underline{\alpha}_{(i)}$ and respectively 1 or $\underline{\beta}_{(i)}$. Hence, there exists a total of 2^m different vectors. By denoting each one of these vectors by $\gamma_j(\underline{\alpha})$ and, respectively, $\gamma_j(\underline{\beta})$ $j = 1, \dots, 2^m$, define the following matrices:

$$D_j(\underline{\alpha}) = diag(\gamma_j(\underline{\alpha})) \quad ; \quad D_j(\underline{\beta}) = diag(\gamma_j(\underline{\beta}))$$

Associated to $\underline{\alpha}$ and $\underline{\beta}$ we can now define the following regions in the space $\Re^{n+m}$:

$$R_{Lp}(\underline{\alpha}) = \{z \in \Re^{n+m} \; ; \; |\mathcal{K}z| \preceq \rho(\underline{\alpha})\} \qquad (6)$$

$$R_{Lr}(\underline{\beta}) = \cap_{j=1}^{2^m} R_{Lr}(\underline{\beta})_j \qquad (7)$$

where $R_{Lr}(\underline{\beta})_j \triangleq \{z \in \Re^{n+m} \; ; \; |[TD_j(\underline{\alpha})K_x \quad (-T + TD_j(\underline{\alpha})K_u)]z| \preceq \xi(\underline{\beta})\}$ with $\rho(\underline{\alpha})_{(i)} \triangleq \frac{\rho_{(i)}}{\underline{\alpha}_{(i)}}$, $\xi_{(i)}(\underline{\beta}) \triangleq \frac{\xi_{(i)}}{\underline{\beta}_{(i)}}$, $\forall i = 1, \dots, m$.

Notice that for all $z(t) \in R_{Lp}(\underline{\alpha}) \cap R_{Lr}(\underline{\beta})$ one has

$$\begin{array}{l} 0 < \underline{\alpha}_{(i)} \leq \alpha_{(i)}(t) \leq 1 \\ 0 < \underline{\beta}_{(i)} \leq \beta_{(i)}(t) \leq 1 \end{array} \qquad \forall i = 1, \dots, m \qquad (8)$$

Consider now the variable change

$$\theta_{(i)}(t) = \alpha_{(i)}(t) - 1 + 0.5(1 - \underline{\alpha}_{(i)}) \qquad (9)$$

then, if $\alpha_{(i)}(t)$ verifies (8) it follows that

$$|\theta_{(i)}(t)| \leq 0.5(1 - \underline{\alpha}_{(i)}) \triangleq \underline{\theta}_{(i)}$$

Defining now the vectors $\theta(t) \triangleq [\theta_{(1)}(t) \quad \dots \quad \theta_{(m)}(t)]^T$ and $\underline{\theta} \triangleq [\underline{\theta}_{(1)} \quad \dots \quad \underline{\theta}_{(m)}]^T$, from (5) and (9) it follows that, $\forall z(t) \in R_{Lp}(\underline{\alpha})$, $\dot{z}(t)$ can be computed by:

$$\dot{z}(t) = (\mathcal{A}(\beta(t)) + \mathcal{B}(\beta(t))D_1(\underline{\alpha}))\mathcal{K})z(t) + \mathcal{B}(\beta(t))D(\theta(t))\mathcal{K}z(t) \qquad (10)$$

with $|\theta_{(i)}(t)| \leq \underline{\theta}_{(i)}$ and where $D_1(\underline{\alpha}) \triangleq D(0.5(1_m + \underline{\alpha}))$.

Define now matrices

$$\mathcal{A}_j(\underline{\beta}) = \begin{bmatrix} A & B \\ 0 & -D_j(\underline{\beta})T \end{bmatrix} \; ; \; \mathcal{B}_j(\underline{\beta}) = \begin{bmatrix} 0 \\ D_j(\underline{\beta})T \end{bmatrix}$$

Note that the matrices $\mathcal{A}_j(\underline{\beta})$ and $\mathcal{B}_j(\underline{\beta})$, are the vertices of convex polytopes of matrices, i.e.,

$$\mathcal{A}(\beta(t)) \in \mathbf{Co}\{\mathcal{A}_1(\underline{\beta}), \mathcal{A}_2(\underline{\beta}), \dots, \mathcal{A}_{2^m}(\underline{\beta})\}$$

$$\mathcal{B}(\beta(t)) \in \mathbf{Co}\{\mathcal{B}_1(\underline{\beta}), \mathcal{B}_2(\underline{\beta}), \dots, \mathcal{B}_{2^m}(\underline{\beta})\}$$

It follows that at instant t, if $z(t) \in R_{Lp}(\underline{\alpha}) \cap R_{Lr}(\underline{\beta})$ there exist nonnegative scalars $\lambda_j(z(t))$, $j = 1, \dots, m$, $\sum_{j=1}^{2^m} \lambda_j(z(t)) = 1$, such that $\dot{z}(t)$ can be computed as follows:

$$\dot{z}(t) = \sum_{j=1}^{2^m} \lambda_j(z(t))[(\mathcal{A}_j(\underline{\beta}) + \mathcal{B}_j(\underline{\beta})D_1(\underline{\alpha})\mathcal{K}) + \mathcal{B}_j(\underline{\beta})D(\theta(t))\mathcal{K}]z(t) \qquad (11)$$

with $|\theta_{(i)}(t)| \leq \underline{\theta}_{(i)}$.

4 Main Results

In order to solve Problem 1 we consider that the following data are a priori given:

• A set of initial conditions $\mathcal{Z}_0$ defined by a polyhedral set in $\Re^{n+m}$ described by its vertices:

$$\mathcal{Z}_0 \triangleq \mathbf{Co}\{v_1, \dots, v_{n_v}\}, \quad v_s \in \Re^{n+m}, \quad \forall s = 1, \dots, n_{nv} \qquad (12)$$

• A region $\mathcal{D}_p$, contained in the left half complex plane

$$\mathcal{D}_p \triangleq \{s \in \mathcal{C} \quad ; \quad (H + sG + \bar{s}G^T) < 0\} \qquad (13)$$

where H and G are $l*l$ symmetric real matrices and s is a complex number with its conjugate $\bar{s}$. This is an LMI region as defined in [2]. We assume that if the poles of $\begin{bmatrix} A & B \\ TK_x & -T + TK_u \end{bmatrix}$ are located in the region $\mathcal{D}_p$ the time-domain performance specifications in the zone of linear behavior of the system (4) are satisfied.

Proposition 1 *If there exist a matrix $W = W^T > 0$, $W \in \Re^{(n+m)*(n+m)}$, a diagonal matrix $S > 0$, $S \in \Re^{m*m}$, matrix $Y \in \Re^{m*(n+m)}$ and vectors $\underline{\alpha} \in \Re^m$ and $\underline{\beta} \in \Re^m$, satisfying the following matrix inequalities:*

$$(i) \quad \begin{bmatrix} X_j - \mathcal{B}_j(\underline{\beta})S\mathcal{B}_j^T(\underline{\beta}) & \star \\ D_2(\underline{\alpha})Y & S \end{bmatrix} > 0, \; j = 1, \dots, 2^m$$

$$(ii) \quad \begin{bmatrix} W & \star \\ \underline{\alpha}_{(i)}Y_{(i)} & \rho_{(i)}^2 \end{bmatrix} \geq 0 \quad , \quad i = 1, \dots, m$$

$$(iii) \quad \begin{bmatrix} 1 & v_s^T \\ v_s & W \end{bmatrix} \geq 0 \quad , \quad s = 1, \dots, n_v$$

$$(iv) \quad \begin{bmatrix} W & \star \\ (-N_{(i)} + \underline{\alpha}_{(i)}M_{(i)}) & (\frac{\xi_{(i)}}{\underline{\beta}_{(i)}})^2 \end{bmatrix} \geq 0 \quad i = 1, \dots, m$$

$$(v) \quad \begin{bmatrix} W & \star \\ (-N_{(i)} + M_{(i)}) & (\frac{\xi_{(i)}}{\underline{\beta}_{(i)}})^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m$$

$$(vi) \quad \begin{cases} 0 < \underline{\alpha}_{(i)} \leq 1 \\ 0 < \underline{\beta}_{(i)} \leq 1 \end{cases} \quad i = 1, \dots, m$$

$$(vii) \quad h_{(i,j)}W + g_{(i,j)}(\mathcal{A}W + \mathcal{B}Y) + g_{(i,j)}(\mathcal{A}W + \mathcal{B}Y)^T < 0 \quad 1 \leq i, j \leq l$$

where
$D_2(\underline{\alpha}) = D(0.5(1_m - \underline{\alpha}))$
$X_j = -W\mathcal{A}_j^T(\underline{\beta}) - \mathcal{B}_j(\underline{\beta})D_1(\underline{\alpha})Y - \mathcal{A}_j(\underline{\beta})W - Y^T D_1(\underline{\alpha})\mathcal{B}_j(\underline{\beta})^T$
$N_{(i)} = [0 \;\; T_{(i)}]W$
$M_{(i)} = T_{(i)}Y$
Then $\mathcal{K} \stackrel{\triangle}{=} YW^{-1}$ *solves Problem 1.*

Proof:

For notational simplicity consider: $\mathcal{A}_j(\underline{\beta}) = \mathcal{A}_j$, $\mathcal{B}_j(\underline{\beta}) = \mathcal{B}_j$ and $\lambda_j(z(t)) = \lambda_j$.

Consider the set $\mathcal{E} \stackrel{\triangle}{=} \{z \in \Re^{n+m} \; ; \; z^T W^{-1} z \leq 1\}$ with $P \stackrel{\triangle}{=} W^{-1}$ and the regions $R_{Lp}(\underline{\alpha})$ and $R_{Lr}(\underline{\beta})$ defined respectively in (6) and (7). Consider also $\mathcal{K} = YW^{-1}$.

Satisfaction of conditions (i) to (vii) leads to the following facts:

1. Pre and post-multiplying (i) by $\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}$, with $P = W^{-1}$, considering $\mathcal{K} = YW^{-1}$ and applying Schur's complement it follows that (i) is equivalent to
$$\begin{bmatrix} L_j - \mathcal{K}^T D_2(\underline{\alpha})S^{-1}D_2(\underline{\alpha})\mathcal{K} & P\mathcal{B}_j \\ \mathcal{B}_j^T P & S^{-1} \end{bmatrix} > 0 \quad (14)$$
where $L_j = -(\mathcal{A}_j + \mathcal{B}_j D_1(\underline{\alpha})\mathcal{K})^T P - P(\mathcal{A}_j + \mathcal{B}_j D_1(\underline{\alpha})\mathcal{K})$.
Since (i) holds for $j = 1, \ldots, 2^m$ by convexity one obtains
$$\sum_{j=1}^{2^m}\lambda_j(z^T(t)L_j z(t) + z^T(t)P\mathcal{B}_j q(t) + q^T(t)\mathcal{B}_j^T P z(t) - \sum_{i=1}^{m}\sigma_{(i)}[z^T(t)\underline{\theta}_{(i)}\mathcal{K}_{(i)}^T\mathcal{K}_{(i)}\underline{\theta}_{(i)}z(t) - q_{(i)}^2(t)]) > 0 \quad (15)$$
$\forall z(t), q(t)$ and $\lambda_j \geq 0$, $j = 1, \ldots, m$, such that $\sum_{j=1}^{2^m}\lambda_j = 1$, with $\sigma_{(i)}$ denoting the i^{th} diagonal component of S^{-1}.
2. (ii) and (vi) ensure that $\mathcal{E} \subset R_{Lp}(\underline{\alpha}) \cap R_{Lr}(\underline{\beta})$ [4],[1].
3. (iii) ensures that $\mathcal{Z}_0 \subset \mathcal{E}$
4. (vii) ensures that the poles of $(\mathcal{A} + \mathcal{B}\mathcal{K})$ are placed in the region $\mathcal{D}_p$ defined in (13) [2].

Using the variable change (9), if (8) is verified, that is $\forall z(t) \in (R_{Lr}(\underline{\beta}) \cap R_{Lp}(\underline{\alpha}))$, $\dot{z}(t)$ can be computed by:
$$\dot{z}(t) = \sum_{j=1}^{2^m}\lambda_j(\mathcal{A}_j + \mathcal{B}_j D_1(\underline{\alpha})\mathcal{K})z(t) + \sum_{j=1}^{2^m}\lambda_j\mathcal{B}_j D(\theta(t))\mathcal{K}z(t) \quad (16)$$
with $|\theta_{(i)}(t)| \leq \underline{\theta}_{(i)}$.

Suppose now that $z(t) \in \mathcal{Z}_0$. Since (ii),(iii),(iv) and (v) are verified we have that $z(t) \in \mathcal{E} \subset (R_{Lr}(\underline{\beta}) \cap R_{Lp}(\underline{\alpha}))$, $\forall\theta(t)$ such that $|\theta_{(i)}(t)| \leq \underline{\theta}_{(i)}$, $i = 1, \ldots, m$ and therefore $\dot{z}(t)$ can be computed by (16). Consider now that $q(t) = -D(\theta(t))\mathcal{K}z(t)$. Since, $|\theta_{(i)}(t)| \leq \underline{\theta}_{(i)}$ it follows that
$$z^T(t)\underline{\theta}_{(i)}\mathcal{K}_{(i)}^T\mathcal{K}_{(i)}\underline{\theta}_{(i)}z(t) \geq q_{(i)}^2(t) \quad (17)$$

Substituting $q(t)$ in (15) and take into account (17) and the fact that $\sigma_{(i)} \geq 0$ one obtains
$$\sum_{j=1}^{2^m}\lambda_j z^T(t)L_j z(t) - z^T(t)P\sum_{j=1}^{2^m}\lambda_j\mathcal{B}_j D(\theta(t))\mathcal{K}z(t) - z^T(t)\mathcal{K}^T D(\theta(t))\sum_{j=1}^{2^m}\lambda_j\mathcal{B}_j^T P z(t)) > 0$$
$$\Leftrightarrow \quad -z^T(t)[\sum_{j=1}^{2^m}\lambda_j(\mathcal{A}_j + \mathcal{B}_j D_1(\underline{\alpha})\mathcal{K} + \mathcal{B}_j D(\theta(t))\mathcal{K})^T P + P\sum_{j=1}^{2^m}\lambda_j(\mathcal{A}_j + \mathcal{B}_j D_1(\underline{\alpha})\mathcal{K} + \mathcal{B}_j D(\theta(t))\mathcal{K})]z(t) > 0$$

And therefore, from (16) one has
$$\dot{z}^T(t)Pz(t) + z(t)P\dot{z}(t) < 0$$

Since this reasoning is valid $\forall z(t) \in \mathcal{E}$, we can conclude that $\mathcal{E}$ is a positively invariant and contractive set w.r.t the system (5) (or, equivalently (4)) and $V(z(t)) = z^T(t)Pz(t)$ is a strictly decreasing Lyapunov function for system (5) (or, equivalently (4)). Hence, we can conclude that if conditions (i) to (vi) are satisfied with $\mathcal{K} = YW^{-1}$, it follows that $\forall z(0) \in \mathcal{Z}_0$ the corresponding trajectory does not leave $\mathcal{E}$ and converges asymptotically to the origin. Furthermore the poles of $(\mathcal{A} + \mathcal{B}\mathcal{K})$ are placed in $\mathcal{D}_p$ guaranteeing the performance specification in the linearity region R_L. Problem 1 is then solved if the conditions $(i) - (vii)$ are fullfiled. $\diamond$

5 Computation of the control law

We discuss now how to compute the matrix gain $\mathcal{K}$ in order to solve Problem 1 from the conditions stated in Proposition 1.

The variables to be found by applying Proposition 1 are W, Y, $\underline{\alpha}$ and $\underline{\beta}$. Due to terms involving the product between these variables, inequalities $(i), (ii), (iv)$ and (v) of Proposition 1 are nonlinear whereas relations $(iii), (vi)$ and (vii) are linear (i.e. they are LMIs). This fact implies that the attempt to compute $\mathcal{K}$ by solving constraints (i)-(vii) directly as a feasibility problem in the variables W, Y, $\underline{\alpha}$ and $\underline{\beta}$ is very difficult (it is an NP-hard problem) or even impossible.

An way to overcome this problem is to fix, a priori, the value of the components of $\underline{\alpha}$ ([4]) and $\underline{\beta}$. In this case, all inequalities become LMIs and, given $(\mathcal{Z}_0, \mathcal{D}_p)$, it is possible to solve constraints $(i)-(vii)$ of Proposition 1, as a feasibility problem, with efficient numerical algorithms [1]. Of course, considering the fixed vectors $\underline{\alpha}$ and $\underline{\beta}$ and the given data, it may actually be impossible to find a feasible solution. In fact, considering a scaling factor δ, $\delta > 0$, the maximum homothetic set to $\mathcal{Z}_0$, denoted $\delta^\star \mathcal{Z}_0$, that can be stabilized using the proposed approach considering the fixed $\underline{\alpha}$ and $\underline{\beta}$, can be obtained by solving the following convex optimization problem with LMI constraints:

$$\begin{array}{c} \max\limits_{\delta, W, Y} \ \delta \\ \text{subject to} \\ \left\{ \begin{array}{l} \begin{bmatrix} 1 & \delta v_i^T \\ \delta v_i & W \end{bmatrix} > 0 \quad , \quad \forall i = 1, \ldots, n_v \\ \\ \text{LMIs } (i), (ii), (iv), (v) \text{ and } (vii) \text{ of Proposition 1} \end{array} \right. \end{array} \tag{18}$$

Hence, if the optimal value of δ, $\delta^\star$, is greater or equal to 1, it means that it is possible to find a solution considering the fixed $\underline{\alpha}$ and $\underline{\beta}$ for the given data $(\mathcal{Z}_0, \mathcal{D}_p)$. We conjecture that smaller the components of vectors $\underline{\alpha}$ and $\underline{\beta}$, greater can be the optimal value of the scalar δ, that is, it is possible to stabilize larger domains of admissible initial states (see the numerical example). Note that the idea is to render the problem less conservative by allowing more control saturation. In fact smaller are the components of $\underline{\alpha}$ and $\underline{\beta}$, larger is the region $(R_{Lp}(\underline{\alpha}) \cap R_{Lp}(\underline{\beta}))$ where the invariant ellipsoid containing $\mathcal{Z}_0$ can be included. In this case two issues arise: how to choose the initial vectors $\underline{\alpha}$ and $\underline{\beta}$ and how exactly to decrease the components of $\underline{\alpha}$ and $\underline{\beta}$ (if $\delta^\star < 1$). One simple way of handling them is to apply trial and error procedures. Another way is to use an iterative algorithm, where in each step, two or three variables are fixed and a convex optimization problem with LMI constraints is solved:

Algorithm 1 *:*

- **Step 0:** *Initialize $\underline{\alpha}$ and $\underline{\beta}$.*

- **Step 1:** *Fix $\underline{\alpha}$ and $\underline{\beta}$, solve (18) for W, Y and δ.*

- **Step 2:** *Fix $\underline{\beta}$ and Y, solve (18) for W, $\underline{\alpha}$ and δ.*

- **Step 3:** *Fix W, Y, S and $\underline{\alpha}$ solve the following optimization problem*

$$\begin{array}{c} \min\limits_{\underline{\beta}} \ \sum\limits_{i=1}^{m} \underline{\beta}_{(i)} \\ \textit{subject to} \\ \textit{LMIs } (i), (iv), (v) \textit{ and } (vi) \textit{ of Proposition 1} \end{array} \tag{19}$$

- **Step 4:** *Go to Step 1 until no significant change in the optimal value of δ occurs.*

The iteration between these four steps stops when a desired precision for δ is achieved. If $\delta^\star \geq 1$ it means that it is possible to stabilize the system (4) for all initial conditions in $\mathcal{Z}_0$ by considering the pole placement of $(\mathcal{A} + \mathcal{BK})$ inside $\mathcal{D}_p$. In particular, all intermediate solutions with $\delta > 1$ are solutions to Problem 1. In this case, another optimization iterative scheme can be used in order to select a gain under a performance criteria (see for example [5]). Hence, this kind of approach solves, in part, the problem of the choice of vectors $\underline{\alpha}$ and $\underline{\beta}$ by using robust and available packages to solve LMIs [3].

Remark 1 *It is worth to be noticed that if we start the algorithm with $\underline{\alpha} = \underline{\beta} = 1_m$, the convergence to a solution $(W^\star, Y^\star, \underline{\alpha}^\star, \underline{\beta}^\star, \delta^\star)$, is ensured provided that the pair (A, B) is controllable. This follows from the fact that an optimal solution for one step is also a feasible solution for the next step. Of course, taking different initial vectors $\underline{\alpha}$ and $\underline{\beta}$ the proposed algorithm can converge to different values of $(W^\star, Y^\star, \underline{\alpha}^\star, \underline{\beta}^\star, \delta^\star)$.*

6 Numerical Example

Consider the matrices of the system given by [10]:

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.7070 & 1.3230 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.0447 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix} \quad ; \quad T = \begin{bmatrix} 20 & 0 \\ 0 & 15 \end{bmatrix}$$

Notice that matrix A is unstable (the eigenvalues of A are: $-1.9809, -0.3340, 0.2807 \pm 0.0952i$). Consider that the bounds on the control are given by $\rho = \begin{bmatrix} 5 & 2 \end{bmatrix}^T$ and $\xi = \begin{bmatrix} 2 & 5 \end{bmatrix}^T$ and the set of admissible initial conditions is an hypercube in $\Re^4$: $\mathcal{Z}_0 = \{x \in \Re^4, u = 0; -1 \leq x_{(i)} \leq 1, \forall i = 1, \ldots, 4\}$. Given these data the objective is to solve Problem 1. We consider that the performance specification in the linearity region was translated as the placement of the poles of $(\mathcal{A} + \mathcal{BK})$ in the region $\mathcal{D}_p = \{s \in \mathcal{C} \ ; \ Re\{s\} \leq -\mu, \quad \mu > 0\}$.

Notice that greater is μ, farther from the origin are the poles of $(\mathcal{A}+\mathcal{BK})$ and greater tends to be the speed of the convergence of the trajectories to the origin inside the region of linearity.

Considering the data above, Table 1 shows the final values[2] of $\underline{\alpha}$, $\underline{\beta}$, δ obtained from the iterative algorithm proposed in Section 5 from different initial vectors $\underline{\alpha}$ and $\underline{\beta}$ and different values of μ. The number of iterations needed in each case is denoted by n_{iter}. $\delta_{initial}$ and δ_{final} denote the optimal value of δ obtained by the iterative algorithm from respectively $(\underline{\alpha}_{initial}, \underline{\beta}_{initial})$ and $(\underline{\alpha}_{final}, \underline{\beta}_{final})$.

μ	$\underline{\alpha}_{initial}$	$\underline{\beta}_{initial}$	$\delta_{initial}$
0.3	$[1\ \ 1]^T$	$[1\ \ 1]^T$	2.928
0.3	$[1\ \ 0.9]^T$	$[0.9\ \ 0.2]^T$	3.550
0.5	$[1\ \ 1]^T$	$[1\ \ 1]^T$	1.267
0.5	$[1\ \ 0.9]^T$	$[0.9\ \ 0.2]^T$	1.573
1	$[1\ \ 1]^T$	$[1\ \ 1]$	0.343
1	$[1\ \ 0.9]^T$	$[0.9\ \ 0.2]^T$	0.406

μ	$\underline{\alpha}_{final}$	$\underline{\beta}_{final}$	δ_{final}	n_{iter}
0.3	$[1\ \ 0.952]^T$	$[0.847\ \ 0.208]^T$	3.596	35
0.3	$[1\ \ 0.869]^T$	$[0.867\ \ 0.201]^T$	3.648	28
0.5	$[1\ \ 0.935]^T$	$[0.831\ \ 0.323]^T$	1.535	35
0.5	$[1\ \ 0.884]^T$	$[0.899\ \ 0.200]^T$	1.579	6
1	$[1\ \ 0.954]^T$	$[0.763\ \ 0.425]^T$	0.397	17
1	$[1\ \ 0.885]^T$	$[0.899\ \ 0.200]^T$	0.409	4

Table 1: algorithm performance

Regarding Table 1 we can notice the following:
- Smaller are the components of $\underline{\alpha}$ and $\underline{\beta}$, greater tends to be the δ obtained from (18). This illustrates the fact that by allowing saturation we can stabilize the system for a larger set of initial conditions.
- More stringent is the performance requirement (greater μ, in this case) smaller is the region of admissible initial states for which we can find a solution.

7 Concluding Remarks

A sufficient condition for the existence of a control law that guarantees the asymptotic stability of a given region of initial states in presence of control amplitude and rate saturation was proposed. The condition applies to both stable and unstable systems. Furthermore, the effectively saturation is considered. It is shown, by means of an example, that this fact can allow the stabilization of larger sets of initial conditions. The proposed stabilization condition leads to an LMI-based framework to compute the control law.

[2] The numerical results presented in this section were obtained from the MATLAB LMI Control Toolbox [3] by using a precision of 0.001 and feasibility radius of 10^4.

References

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM Studies in Applied Mathematics, 1994.

[2] M. Chilali and P. Gahinet. H_∞ design with pole placement constraints: an lmi approach. *IEEE Transactions on Automatic Control*, 41(3):358–367, 1996.

[3] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali. *LMI Control Toolbox User's Guide*. The Math Works Inc., 1995.

[4] J.M. Gomes da Silva Jr., A. Fischman, S. Tarbouriech, J. M. Dion, and L. Dugard. Synthesis od state feedback for linear systems subject to control saturation by an LMI-based approach. In *Proceedings of the 2nd IFAC Symposium on Robust Control (ROCOND'97)*, pages 229–234, Budapest, Hungary, 1997.

[5] J.M. Gomes da Silva Jr and S. Tarbouriech. Local stabilization of discrete-time linear systems with saturating controls: an LMI-based approach. In *Proceedings of the 17th American Control Conference (ACC'98) (to appear also in IEEE-Transactions on Automatic Control)*, pages 92–96, Philadelphia, USA, June 1998.

[6] P. Kapasouris and M. Athans. Control systems with rate and magnitude saturation for neutrally stable open-loop systems. In *Proc. of the 29th IEEE Conference on Decision and Control (CDC'90)*, pages 3404–3409, Honolulu (Hawaii), 1990.

[7] Z. Lin. Semi-global stabilization of linear systems with position and rate-limited actuators. *System & Control Letters*, 30:1–11, 1997.

[8] T. Nguyen and F. Jabbary. Output feedback controllers for disturbance attenuation with actuator amplitude and rate saturation. In *Proceedings of 1999 american Control Conference (ACC'99)*, pages 1997–2001, San Diego (USA), 1999.

[9] J.M. Shewchun and E. Feron. High performance bounded control. In *Proceedings of the 16th American Control Conference (ACC'97)*, volume 5, pages 3250–3254, Albuquerque, USA, June 1997.

[10] S. Tarbouriech, G. Garcia, and D. Henrion. Local stabilization of linear systems with position and rate bounded actuators. In *Proceedings of IFAC World Congress 1999*, Beijing, China, 1999.

[11] F. Tyan and D. Bernstein. Dynamic output feedback compensation for linear systems with independent amplitude and rate saturation. *International Journal of Control*, 67(1):89–116, 1997.