Discretization of a non-linear, exponentially stabilizing control law using an L_p-gain approach

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Abstract

This paper considers the input-output stability of exponentially stable, non-linear systems with sampled-data output. Results for linear systems are generalized showing that the L_p-gain with respect to the sampled-data output exists and converges to the L_p-gain associated with the continuous-time output when the sampling period approaches $+0$. The results are applied to a non-linear control configuration and compared to a Lyapunov function analysis based approach previously developed for a non-linear sliding-mode like control. In contrast to the Lyapunov function technique, the sampling-time constraint vanishes for a stable plant if no control is used.

1 Introduction

Analysis of discretized continuous-time control systems used for sampled-data control has been of interest for a long period. This is because controllers are often developed in continuous time but need to be applied via sampled-data technology. An extensive frame-work of theory has been developed for linear control of linear systems [1, 2]. Linear time-invariant (LTI) systems are particularly useful for the application of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ techniques [1]. In contrast, non-linear systems often rely on Lyapunov function based analysis techniques [3, 4]. The approach of [3, 5] has been useful for determining reasonably low sampling-frequencies, which are sufficient for stable, robust closed loop control using a discretized non-linear control. The technique has been applied successfully to a sliding-mode based control.

An important result for the discretization of LTI systems is that any stable linear system (A, B, C) followed by a sample-and-hold process has finite L_p-induced norm for constant sampling time $\tau > 0$. Furthermore, the L_p-norm of the discretized output signal approaches the L_p-norm of the continuous output signal of (A, B, C) as $\tau \to +0$ [1]. This result has also been shown for linear time-varying systems [2]. It is of interest to extend this result to non-linear systems, since the L_p-gain has also proved to be useful for non-linear systems. This paper presents an L_p-gain approach for sampled-data outputs of strictly proper, affine, non-linear systems where the continuous outputs are Lipschitz functions of the states $\mathbf{x}$ only. These results are then used to analyse the closed-loop properties of a non-linear sampled-data control. A particular upper bound $\tilde{\tau}$ for the sampling-time τ of the discretized non-linear control is derived so that asymptotic stability of the closed loop system follows for $\tau \leq \tilde{\tau}$. This result is then compared to the Lyapunov function analysis based results of [3, 6] for a sliding mode based non-linear control law. It will be seen that both approaches have their advantages and disadvantages from a theoretical and practical point of view. In the Lyapunov function based approach, an upper bound for the norm difference between the time-sampled state $\mathbf{x}(t_i)$ and the value of the continuous signal $\mathbf{x}(t)$ is found for each particular finite sampling interval $[t_i, t_{i+1}]$:

$$\sup_{t_\alpha \in [t_i, t_{i+1}]} (\|\mathbf{x}(t_\alpha) - \mathbf{x}(t_i)\|) \leq \tau\mathcal{K}\|\mathbf{x}(t)\|\,,\; \mathcal{K} \geq 0,\; t \in [t_i, t_{i+1}]. \tag{1}$$

The upper bound $\tau\mathcal{K}\|\mathbf{x}(t)\|$ decreases with decreasing sampling time τ. A problem of this approach [3, 4, 5] is that the relation (1) can only be derived if an upper bound $\bar{\tau}$ on the sampling-time $\tau < \bar{\tau}$ is assumed. For (1) being valid, this upper bound $\bar{\tau}$ remains finite for any control even if the non-linear system is stable and the applied control is zero. Thus, a minimum sampling frequency is always indirectly imposed. This minimum sampling frequency is not necessary with the L_p-induced norm approach.

2 Preliminaries: L_p-gain of sampled-data LTI systems

Suppose a linear, finite dimensional, strictly proper system G, an input-output map of $\mathbf{u} \to \mathbf{y}$, is given by the triple (A, B, C), where $A \in \mathbb{R}^{n\times n}$ is a stable matrix. A sample-and-hold element may be introduced, measuring the output signal $\mathbf{y}(t)$ at well-defined time instants t_i $(i = 0, 1, 2, 3, ...)$ and holding the output of the element constant over the interval $(t_i, t_{i+1}]$ for constant sampling time $\tau > 0$:

$$\forall\, t \in (t_i, t_{i+1}]:\; \mathbf{y}^\Delta(t, \mathbf{y}) = \mathbf{y}(t_i),\; \mathbf{y}^\Delta(t = 0, \mathbf{y}) = 0,\; t_i = i\tau.$$

To symbolize the change in methodology from continuous to discrete-time, the superscript $^\Delta$(=discrete) has been introduced. It was proved by [1, Theorem 9.3.3] that the input-output map $\mathbf{u} \to (\mathbf{y}^\Delta - \mathbf{y})$ has finite L_p-gain, $(p \in [1, \infty])$. The L_p-gain of the input-output map $\mathbf{u} \to (\mathbf{y}^\Delta - \mathbf{y})$ is bounded from above by:

$$\gamma_{\Delta-I}(A, B, C) \overset{def}{=} \int_0^\infty f_H(t)dt + \tau\phi(0), \tag{2}$$

where the function f_H and the term $\phi(0)$ satisfy:

$$f_H(t) \overset{def}{=} \sup_{a\in(0,\tau)} \left\| Ce^{At}B - Ce^{A(t-a)}B \right\|,\; \phi(0) \overset{def}{=} \sup_{t\in[0,\tau)} \left\| Ce^{At}B \right\|$$

and $\|\cdot\|$ is the induced Euclidean norm. This implies for the L_p-norm on the in/finite horizon $(T \leq \infty)$:

$$\|\mathbf{u}(t)\|_{p,T} = \left(\int_0^T \|\mathbf{u}(t)\|^p \, dt \right)^{\frac{1}{p}}$$

of a Lebesgue-integrable input signal $\mathbf{u}(t) \in L_p[0, T]$:

$$\|\mathbf{y}^\Delta(t) - \mathbf{y}(t)\|_{p,T} \leq \gamma_{\Delta-I}\|\mathbf{u}(t)\|_{p,T} + \beta(A, B, C, \mathbf{x}(0), T), \tag{3}$$

The bias term $\beta(A, B, C, \mathbf{x}(0), T)$ is bounded above by:

[1] G. Herrmann is supported by a grant of the European Commission (TMR-grant, project number: FMBICT983463).

$$\beta(A,B,C,\mathbf{x}(0),T)\leq\left(\int_0^T \sup_{\alpha\in(0,\tau)}\left(\left\|Ce^{At}(I-e^{-A\alpha})\mathbf{x}(0)\right\|^p\right)dt\right)^{\frac{1}{p}}.$$

It can be seen that the upper limit of the L_p-gain $(\int_0^\infty f_H(t)dt + \tau\phi(0))$ converges to 0 for $\tau \to 0$ using the following upper estimates of f_H and the term $\phi(0)$

$$f_H(t)\leq\|B\|\|C\|\left\|e^{tA}\right\|\left|e^{\tau\|A\|}-1\right|,\ \phi(0)\leq\|B\|\|C\|e^{\tau\|A\|}.$$

Note that $\int_0^\infty \left\|e^{tA}\right\| dt$ is finite for stable A. A similar argument also shows that $\lim_{\tau\to 0+}\beta = 0$.

3 L_p-gain of non-linear systems with sampled output

Assume for a system with bounded input gain $\left\|\underline{g}\right\| \leq \tilde{\mathcal{G}}$

$$\dot{\mathbf{x}}(t) = \underline{f}(\mathbf{x})+\underline{g}(\mathbf{x})\mathbf{u},\ \mathbf{x}\in\mathbb{R}^n,\ \underline{f}:\mathbb{R}^n\to\mathbb{R}^n,\ \underline{g}:\mathbb{R}^n\to\mathbb{R}^{n\times\tilde{m}},$$
$$\mathbf{y} = \underline{h}(\mathbf{x}),\ \underline{h}:\mathbb{R}^n\to\mathbb{R}^p,\ \underline{h}(0)=0. \quad (4)$$

that the unperturbed system is exponentially stable. Further, suppose $\underline{f}(\mathbf{x})$ and $\underline{g}(\mathbf{x})$ are continuously differentiable in $\mathbf{x}$. The output measurements $\mathbf{y} = \underline{h}(\mathbf{x})$ and the system function $\underline{f}(\mathbf{x})$ satisfy the following Lipschitz properties:

$$\|\underline{h}(\mathbf{x}_1)-\underline{h}(\mathbf{x}_2)\|\leq\mathcal{K}_{\underline{h}}\|\mathbf{x}_1-\mathbf{x}_2\|,\ \|\underline{f}(\mathbf{x}_1)-\underline{f}(\mathbf{x}_2)\|\leq\mathcal{K}_{\underline{f}}\|\mathbf{x}_1-\mathbf{x}_2\|. \quad (5)$$

Assuming exponential stability for $\mathbf{u}=0$, there must exist by the converse stability theorem [7, Theorem 3.12] a $\mathcal{V}(\mathbf{x})$ such that for $c_1,\ c_2,\ c_3,\ c_4>0$:

$$\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\underline{f}(\mathbf{x})\leq -c_3\|\mathbf{x}\|^2,\ \left\|\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\right\|\leq c_4\|\mathbf{x}\|,\ c_1\|\mathbf{x}\|^2\leq\mathcal{V}(\mathbf{x})\leq c_2\|\mathbf{x}\|^2. \quad (6)$$

Combining now L_p-stability characteristics of exponentially stable systems with the L_p-gain of linear sampled systems, it can be shown that stable affine systems (4) have finite L_p-gain with respect to an output $\underline{h}^\Delta(t,\mathbf{x})-\underline{h}(\mathbf{x})$ and that this gain approaches 0 for $\tau\to+0$:

Theorem 1 *An exponentially stable system (4) with sampled data output* $\underline{h}^\Delta(t,\mathbf{x})$*:*

$$\forall\ t\in(t_i,t_{i+1}]:\ \underline{h}^\Delta(t,\mathbf{x})=\underline{h}(\mathbf{x}(t_i)),\ \underline{h}^\Delta(t_0,\cdot)=0.$$

satisfying (5-6) has finite L_p*-gain* ($p\in[0,\infty]$) *in the space* $L_p[0,T]$ ($0\leq T\leq\infty$)*. The gain converges to the value of the* L_p*-gain with respect to the continuous output* $\underline{h}(\mathbf{x})$ *as* $\tau\to+0$*. Further, it can be shown that for the output signal* $\underline{h}^\Delta(t,\mathbf{x})-\underline{h}(\mathbf{x})$ *the* L_p*-induced norm has the limit* 0 *as* $\tau\to+0$*.*

Proof First consider an auxiliary output $\tilde{\mathbf{y}}$ which enables the L_p-stability result for sampled-data LTI systems to be used. Using a stable matrix $F\in\mathbb{R}^{n\times n}$, the output function $\tilde{\mathbf{y}}$ for the system (4) is:

$$\tilde{\mathbf{y}}(\mathbf{x})=\underline{f}(\mathbf{x})-F\mathbf{x}$$

The auxiliary output $\tilde{\mathbf{y}}$ is globally Lipschitz in $\mathbf{x}$ since $\underline{f}(\mathbf{x})$ is Lipschitz. Further, due to exponential stability, the equality $\tilde{\mathbf{y}}(0)=0$ holds. Thus, the input-output map $\mathbf{u}\to\tilde{\mathbf{y}}$ has finite L_p-gain using [7, Theorem 6.1]:

$$\|\tilde{\mathbf{y}}\|_{p,T}\leq\gamma_{\mathbf{u}\to\tilde{\mathbf{y}}}\|\mathbf{u}(t)\|_{p,T}+\delta_{\mathbf{u}\to\tilde{\mathbf{y}}},\ \gamma_{\mathbf{u}\to\tilde{\mathbf{y}}}\geq0,\ \delta_{\mathbf{u}\to\tilde{\mathbf{y}}}\geq0 \quad (7)$$

Rewriting the system equation (4) using $\tilde{\mathbf{y}}(\mathbf{x})$ yields:

$$\dot{\mathbf{x}}(t)=F\mathbf{x}+(\tilde{\mathbf{y}}(\mathbf{x})+\underline{g}(\mathbf{x})\mathbf{u}).$$

It is now possible to interpret the function $(\tilde{\mathbf{y}}(\mathbf{x})+\underline{g}(\mathbf{x})\mathbf{u})$ as the input signal for a linear system (F,I,I). Since $\underline{h}(\mathbf{x})$ is globally Lipschitz with Lipschitz constant $\mathcal{K}_{\underline{h}}$ and $\left\|\underline{g}\right\|$ is bounded by $\tilde{\mathcal{G}}$, the relation (3) for discretized linear systems implies:

$$\|\underline{h}^\Delta(s,\mathbf{x})-\underline{h}(\mathbf{x})\|_{p,T}\ \leq\ \mathcal{K}_{\underline{h}}\gamma_{\Delta-I}(F,I,I)\|\tilde{\mathbf{y}}\|_{p,T}$$
$$+\mathcal{K}_{\underline{h}}\tilde{\mathcal{G}}\gamma_{\Delta-I}(F,I,I)\|\mathbf{u}\|_{p,T}+\mathcal{K}_{\underline{h}}\beta(F,I,I,\mathbf{x}(0),T) \quad (8)$$

The term $\|\tilde{\mathbf{y}}\|_{p,T}$ in (8) can now be replaced by the expression in (7) implying that $(\underline{h}^\Delta(t,\mathbf{x})-\underline{h}(\mathbf{x}))$ has bounded L_p-norm for L_p-norm bounded input $\mathbf{u}$. The norm vanishes for $\tau\to+0$ since $\gamma_{\Delta-I}$ and β converge to 0 in this case. The latter and the existence of a finite L_p-gain with respect to the map $\mathbf{u}\to\underline{h}(\mathbf{x})$ [7, Theorem 6.1] implies the remaining part of the claim using the triangle inequality. ■

4 Discretizing a non-linear, continuous-time control

The procedure for controller discretization is introduced using a generic example. Certain assumptions are made on the system and an exponentially stabilizing control:
Assumption 1: Assume for a system with bounded input gain $\|\mathsf{g}\|\leq\mathcal{G},\mathcal{G}>0$

$$\dot{\mathbf{x}}(t)=\mathsf{f}(\mathbf{x})+\mathsf{g}(\mathbf{x})\mathbf{u},\ \mathbf{x}\in\mathbb{R}^n,\ \mathsf{f}:\mathbb{R}^n\to\mathbb{R}^n,\ \mathsf{g}:\mathbb{R}^n\to\mathbb{R}^{n\times m} \quad (9)$$

an exponentially stabilizing continuous-time control $\mathbf{u}=\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t))$ exists. Further, suppose $\mathsf{f}(\mathbf{x})$, $\mathsf{g}(\mathbf{x})$ and $\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t))$ are continuously differentiable in $\mathbf{x}$.
Assumption 2: The controlled system shall have certain Lipschitz properties:

$$\|\mathsf{f}(\mathbf{x}_1)-\mathsf{f}(\mathbf{x}_2)\|\leq\mathcal{K}_{\mathsf{f}}\|\mathbf{x}_1-\mathbf{x}_2\|,\ \left\|\mathbf{u}^{\mathcal{C}}(\mathbf{x}_1)-\mathbf{u}^{\mathcal{C}}(\mathbf{x}_2)\right\|\leq\mathcal{K}_{\mathbf{u}}\|\mathbf{x}_1-\mathbf{x}_2\| \quad (10)$$

Assumption 3: Suppose values of the continuous-time control are available via sampled measurements at well defined time instants $\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_i))=\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t))|_{t=t_i},\ t_i=i\tau,\ i=0,1,2,3,\ldots$. It is then possible to apply to the system a discretized control:

$$\forall\ t\in(t_i,t_{i+1}]:\ \mathbf{u}(t)=\mathbf{u}^\Delta(t,\mathbf{x})=\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_i)),\ \mathbf{u}(t_0)=0. \quad (11)$$

The following theorem ensures stability of the discretized control using an L_2-gain approach:

Theorem 2 *There exists a* $\hat{\tau}$ *small enough such that for any* $\tau<\hat{\tau}$*, the discretized control for the system (9) is asymptotically stable. The constraints on the sampling time* τ *vanishes,* $\tilde{\tau}\to+\infty$*, for a control with Lipschitz constant* $\mathcal{K}_{\mathbf{u}}\to+0$ *and exponentially stable system (9).*

Note the contrast to [3] where relations similar to (1) need to hold and therefore the upper bound on the sampling time τ remains finite for any control even if the control is zero and the system is open-loop stable.

Proof
Step I: It can be verified using the principle of complete induction that a continuous solution to the discrete controlled system exists [3].
Step II: Due to the exponential stability of $\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t))$, the equilibrium point is $\mathsf{f}(0)+\mathsf{g}(0)\mathbf{u}^{\mathcal{C}}(0)=0$. Thus, there exists a constant $\mathcal{K}_{\mathcal{C}}>0$ using the Lipschitz property of $\mathsf{f}(\cdot)$, the assumption on $\|\mathsf{g}\|$ in (9) and on $\mathbf{u}^{\mathcal{C}}(\cdot)$ in (10) such that

$$\|\mathsf{f}(\mathbf{x}_1)+\mathsf{g}(\mathbf{x}_1)\mathbf{u}^{\mathcal{C}}(\mathbf{x}_1)-\mathsf{f}(\mathbf{x}_2)-\mathsf{g}(\mathbf{x}_2)\mathbf{u}^{\mathcal{C}}(\mathbf{x}_2)\|\le\mathcal{K}_{\mathcal{C}}\|\mathbf{x}_1-\mathbf{x}_2\|$$
$$\|\mathsf{f}(\mathbf{x})+\mathsf{g}(\mathbf{x})\mathbf{u}^{\mathcal{C}}(\mathbf{x})\|\le\mathcal{K}_{\mathcal{C}}\|\mathbf{x}\|.$$

Further, there must exist a Lyapunov function $\mathcal{V}$ [7, Theorem 3.12] such that:

$$\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\left(\mathsf{f}(\mathbf{x}(t))+\mathsf{g}(\mathbf{x}(t))\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t))\right)\le -c_3\|\mathbf{x}(t)\|^2,\ c_3>0$$
$$\left\|\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\right\|\le c_4\|\mathbf{x}\|,\ c_1\|\mathbf{x}\|^2\le\mathcal{V}(\mathbf{x})\le c_2\|\mathbf{x}\|^2. \tag{12}$$

Step III: Investigate a system with state $\hat{\mathbf{x}}$ and input perturbation vector $\hat{\mathbf{u}}$

$$\dot{\hat{\mathbf{x}}}=\underbrace{\left(\mathsf{f}(\hat{\mathbf{x}}(t))+\mathsf{g}(\hat{\mathbf{x}}(t))\mathbf{u}^{\mathcal{C}}(\hat{\mathbf{x}}(t))\right)}_{\underline{f}}+\underbrace{\mathsf{g}(\hat{\mathbf{x}}(t))}_{\underline{g}}\hat{\mathbf{u}}, \tag{13}$$

For $\hat{\mathbf{x}}(t=0)=\mathbf{x}(t=0)$ and a particular choice of $\hat{\mathbf{u}}$, it will be seen that system (13) is equivalent to the closed loop with discretized control. Using Lemma 1 of the Appendix, it follows that the system (13) with input perturbation $\hat{\mathbf{u}}$, output $\tilde{\mathbf{y}}$

$$\tilde{\mathbf{y}}=\underbrace{\mathsf{f}(\hat{\mathbf{x}}(t))+\mathsf{g}(\hat{\mathbf{x}}(t))\mathbf{u}^{\mathcal{C}}(\hat{\mathbf{x}}(t))-F\hat{\mathbf{x}}}_{\underline{h}}, \tag{14}$$

and arbitrary but fixed and stable matrix $F\in\mathbb{R}^{n\times n}$ has finite L_2-induced norm with upper bound $\gamma_{subopt}=\frac{c_4(\mathcal{K}_{\mathcal{C}}+\|F\|)\mathcal{G}}{c_3}$. Choosing a scalar κ, $0<\kappa\le 1$, it follows for $\epsilon=\frac{\mathcal{K}_{\mathcal{C}}+\|F\|}{\sqrt{c_3\kappa}}$ using Lemma 1:

$$\frac{\delta\mathcal{V}}{\delta\hat{\mathbf{x}}}\left(\mathsf{f}(\hat{\mathbf{x}}(t))+\mathsf{g}(\hat{\mathbf{x}}(t))\mathbf{u}^{\mathcal{C}}(\hat{\mathbf{x}}(t))\right)+\frac{\delta\mathcal{V}}{\delta\hat{\mathbf{x}}}\mathsf{g}(\hat{\mathbf{x}})\hat{\mathbf{u}}$$
$$\le\frac{1}{2}\frac{\gamma_{subopt}^2}{\kappa^2}\frac{1}{\epsilon^2}\|\hat{\mathbf{u}}\|^2-\frac{1}{2}\frac{1}{\epsilon^2}\|\tilde{\mathbf{y}}\|^2-(1-\kappa)c_3\|\hat{\mathbf{x}}\|^2. \tag{15}$$

Thus, integrating inequality (15) and using $\mathcal{V}(\hat{\mathbf{x}})\le c_2\|\hat{\mathbf{x}}\|^2$ from (12) it follows that:

$$\mathcal{V}(\mathbf{x}(t=T))-\mathcal{V}(\mathbf{x}(t=0))\ \le\ -(1-\kappa)\frac{c_3}{c_2}\int_0^T\mathcal{V}(\mathbf{x}(s))ds$$
$$+\frac{1}{2\epsilon^2}\left(\frac{\gamma_{subopt}^2}{\kappa^2}\int_0^T\|\hat{\mathbf{u}}(s)\|^2ds-\int_0^T\|\tilde{\mathbf{y}}(s)\|^2ds\right) \tag{16}$$

Step IV: The system dynamics (13) may be rewritten in terms of the signal $\tilde{\mathbf{y}}$ in (14) and an auxiliary output $\tilde{\mathbf{u}}$:

$$\begin{aligned}\dot{\hat{\mathbf{x}}}&=F\hat{\mathbf{x}}+(\tilde{\mathbf{y}}+\mathsf{g}(\hat{\mathbf{x}}(t))\hat{\mathbf{u}})\\ \tilde{\mathbf{u}}&=\mathbf{u}^{\Delta}(t,\hat{\mathbf{x}})-\mathbf{u}^{\mathcal{C}}(\hat{\mathbf{x}}(t)).\end{aligned} \tag{17}$$

The signal $\mathbf{u}^{\Delta}(t,\hat{\mathbf{x}}(t))$ results as in (11) from a sample and hold process operating on the non-linear control $\mathbf{u}^{\mathcal{C}}(\cdot)$, which is now a function of $\hat{\mathbf{x}}(t)$. Considering $(\tilde{\mathbf{y}}+\mathsf{g}(\hat{\mathbf{x}}(t))\hat{\mathbf{u}})$ as the input signal, the dynamics in (17) have, as discussed in Section 2, a finite L_2-gain associated with the linear system (F,I,I). Using the bounds on $\|\mathsf{g}\|$ in (9), the Lipschitz constant for the continuous control $\mathbf{u}^{\mathcal{C}}$ in (10) ($\|\tilde{\mathbf{u}}\|\le K_{\mathbf{u}}\left\|\hat{\mathbf{x}}^{\Delta}(t,\hat{\mathbf{x}})-\hat{\mathbf{x}}(t)\right\|$) and the relation $|a|+|b|+|c|\le\sqrt{3}\sqrt{|a|^2+|b|^2+|c|^2}$, it follows that:

$$\|\tilde{\mathbf{u}}(t)\|_{p,T}\le\sqrt{3}\mathcal{K}_{\mathbf{u}}\Big[(\gamma_{\Delta-I}(F,I,I))^2\left(\|\tilde{\mathbf{y}}\|_{p,T}^2+\mathcal{G}^2\|\hat{\mathbf{u}}\|_{p,T}^2\right)$$
$$+(\beta(F,I,I,\mathbf{x}(0),T))^2\Big]^{\frac{1}{2}} \tag{18}$$

Step V: Consider now inequality (18). Both sides may be squared and multiplied by the constant:

$$\frac{\gamma_{subopt}^2}{2\epsilon^2\kappa^2(1-2\mathcal{K}_{\mathbf{u}}^2(\gamma_{\Delta-I}(F,I,I))^2\mathcal{G}^2)},$$

which has to be positive. This is satisfied if the sampling time τ has been chosen small enough so that $2\mathcal{K}_{\mathbf{u}}^2(\gamma_{\Delta-I}(F,I,I))^2\mathcal{G}^2<1$. The resulting inequality can now be added to (16) and equivalently rewritten so that:

$$\mathcal{V}(\mathbf{x}(t=T))-\mathcal{V}(\mathbf{x}(t=0))\le-(1-\kappa)\frac{c_3}{c_2}\int_0^T\mathcal{V}(\mathbf{x}(s))ds$$
$$+\frac{1}{2\epsilon^2}\frac{\gamma_{subopt}^2\left(\|\hat{\mathbf{u}}(s)\|_{p,T}^2-\|\tilde{\mathbf{u}}(s)\|_{p,T}^2\right)}{\kappa^2(1-2\mathcal{K}_{\mathbf{u}}^2(\gamma_{\Delta-I}(F,I,I))^2\mathcal{G}^2)}$$
$$+\frac{1}{2\epsilon^2}\left(\frac{2\gamma_{subopt}^2\mathcal{K}_{\mathbf{u}}^2\gamma_{\Delta-I}(F,I,I))^2}{\kappa^2(1-2\mathcal{K}_{\mathbf{u}}^2(\gamma_{\Delta-I}(F,I,I))^2\mathcal{G}^2)}-1\right)\|\tilde{\mathbf{y}}(s)\|_{p,T}^2$$
$$+\frac{1}{2\epsilon^2}\frac{\gamma_{subopt}^2 2\mathcal{K}_{\mathbf{u}}^2\beta^2(F,I,I,\tilde{\mathbf{x}}(0),T)}{\kappa^2(1-2\mathcal{K}_{\mathbf{u}}^2(\gamma_{\Delta-I}(F,I,I))^2\mathcal{G}^2)} \tag{19}$$

The perturbation $\hat{\mathbf{u}}$ may now be chosen to be $\hat{\mathbf{u}}=\tilde{\mathbf{u}}$, which makes system (13) equivalent to the discrete controlled system (9). Hence, if the sampling frequency has been chosen small enough so that

$$\begin{aligned}2\mathcal{K}_{\mathbf{u}}^2(\gamma_{\Delta-I}(F,I,I))^2\mathcal{G}^2&<1,\\ \frac{\gamma_{subopt}}{\kappa}\frac{\sqrt{2}\mathcal{K}_{\mathbf{u}}\gamma_{\Delta-I}(F,I,I)}{\sqrt{1-2\mathcal{K}_{\mathbf{u}}^2(\gamma_{\Delta-I}(F,I,I))^2\mathcal{G}^2}}&\le 1\end{aligned} \tag{20}$$

then it follows from (19) for $\epsilon=\frac{\mathcal{K}_{\mathcal{C}}+\|F\|}{\sqrt{c_3\kappa}}$:

$$\mathcal{V}(\mathbf{x}(t=0))+\frac{1}{\epsilon^2}\left(\frac{\gamma_{subopt}^2}{\kappa^2}\frac{\mathcal{K}_{\mathbf{u}}^2\beta^2(F,I,I,\mathbf{x}(0),\infty)}{(1-2\mathcal{K}_{\mathbf{u}}^2(\gamma_{\Delta-I}(F,I,I))^2\mathcal{G}^2)}\right)$$
$$\ge\mathcal{V}(\mathbf{x}(t=T))+(1-\kappa)\frac{c_3}{c_2}\int_0^T\mathcal{V}(\mathbf{x}(s))ds \tag{21}$$

From the boundedness of $\mathcal{V}(\mathbf{x}(t=T))$ and $\int_0^T\mathcal{V}(\mathbf{x}(s))ds$, it follows that the system is asymptotically stable [8, Theorem 3.10.4]. Observe that the right hand side of (18) has the value 0 and the constraint on τ in (20) vanishes if the Lipschitz constant $\mathcal{K}_{\mathbf{u}}$ is zero. This would be satisfied if there is no control necessary to stabilize the system. ■

5 A sliding-mode like control law

Linear, uncertain systems shall be considered

$$\dot{\mathbf{x}}=A\mathbf{x}+B\mathbf{u}+\mathbf{F}(t,\mathbf{x}) \tag{22}$$

where $\mathbf{x}\in\mathbb{R}^n$, $\mathbf{u}\in\mathbb{R}^m$ and the known matrix pair (A,B) defines the nominal linear system, which is assumed to be controllable with B of full rank. The unknown function $\mathbf{F}(.,.):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ models parametric uncertainties and non-linearities of the system lying in the range space of B. Under these conditions, it is possible to define a linear transformation $\tilde{T}$ such that the system (22) becomes:

$$\dot{\mathbf{z}}_1=\Sigma\mathbf{z}_1+A_{12}\phi, \tag{23}$$
$$\dot{\phi}=(\Theta+\Delta\Theta)\mathbf{z}_1+(\Omega+\Delta\Omega)\phi+B_2\mathbf{u}, \tag{24}$$
$$\text{where}\quad\tilde{\mathbf{z}}=\tilde{T}\mathbf{x}=\left[\mathbf{z}_1^T\ \phi^T\right]^T,\ \mathbf{z}_1\in\mathbb{R}^{n-m},\ \phi\in\mathbb{R}^m \tag{25}$$

and Σ is a stable design matrix. The sub-system (23) defines *the null-space dynamics* and the sub-system (24) represents the *range-space dynamics*. The matrices $\Delta\Theta$ and $\Delta\Omega$ stemming from the transformation of $\mathbf{F}(.,.)$ are bounded

$$\Delta\Theta(t,\tilde{\mathbf{z}}): \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^{m\times(n-m)},\ \|\Delta\Theta\|\le K_\Theta,$$
$$\Delta\Omega(t,\tilde{\mathbf{z}}): \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^{m\times m},\ \|\Delta\Omega\|\le K_\Omega$$

The associated non-negative bounds K_Θ and K_Ω are assumed known and finite. It is assumed that $\mathbf{F}(.,.)$ is Caratheodory to ensure existence of solution (see [3, 9]). It can be shown similarly to [6], that the sliding-mode like control law outlined below forces the system states $\tilde{\mathbf{z}}$ to satisfy $\phi(t) \approx 0$ and the control is robust to matched uncertainty. This control has two parts:

$$\mathbf{u}(t) = \mathbf{u}_L^{\mathcal{C}}(\mathbf{z}_1(t),\phi(t)) + \mathbf{u}_{NL}^{\mathcal{C}}(\mathbf{z}_1(t),\phi(t)) \qquad (26)$$

where $\mathbf{u}_L^{\mathcal{C}}(\cdot)$ and $\mathbf{u}_{NL}^{\mathcal{C}}(\cdot)$ are the linear and the non-linear control components. The linear controller part is

$$\mathbf{u}_L^{\mathcal{C}}(\mathbf{z}_1(t),\phi(t)) \stackrel{def}{=} -B_2^{-1}\left(\Theta\mathbf{z}_1(t) + (\Omega-\Omega^*)\phi(t)\right), \qquad (27)$$

where Ω^* is a stable design matrix and P_2 satisfies $P_2\Omega^* + {\Omega^*}^T P_2 = -I_m$, where I_m is the $(m\times m)$ identity matrix. A Lyapunov function $V_2(t)$ for the analysis of the range-space dynamics is

$$V_2(t) \stackrel{def}{=} \frac{1}{2}\phi^T(t)P_2\phi(t), \qquad (28)$$

and

$$V_1(t) \stackrel{def}{=} \frac{1}{2}{\mathbf{z}_1}^T(t)P_1\mathbf{z}_1(t) \qquad (29)$$

with $P_1\Sigma + \Sigma^T P_1 = -I_{(n-m)}$ is appropriate for the null-space dynamics. The non-linear control component

$$\mathbf{u}_{NL}^{\mathcal{C}} \stackrel{def}{=} \begin{cases} \frac{-\rho(\mathbf{z}_1(t),\phi(t))B_2^{-1}P_2\phi(t)}{\|P_2\phi(t)\|+\delta(\|\mathbf{z}_1(t)\|+\|\phi(t)\|)} & for\ [\mathbf{z}_1^T,\phi^T]^T\neq 0 \\ 0 & for\ [\mathbf{z}_1^T,\phi^T]^T=0 \end{cases},\ \delta\in\mathbb{R}^+ \qquad (30)$$

achieves robustness by counteracting the matched uncertainties and ensuring reachability of a cone shaped layer around the sliding mode. The expression $\delta(\|\mathbf{z_1}(t)\| + \|\phi(t)\|)$ results in a cone shaped layer around $\phi = 0$, which is defined by the relation $V_2 = \omega^2 V_1$, where $\omega \in \mathbb{R}^+$

$$\omega \stackrel{def}{=} \frac{\lambda_{max}(P_2^{\frac{1}{2}})\delta/\lambda_{min}(P_1^{\frac{1}{2}})}{\left(\lambda_{max}(P_2^{\frac{1}{2}})\lambda_{min}(P_2^{\frac{1}{2}})(\gamma_1-1)-\delta\right)}$$

has to be positive, which can be ensured by imposing the constraint $\lambda_{min}(P_2^{\frac{1}{2}})\lambda_{max}(P_2^{\frac{1}{2}}) > \frac{\delta}{\gamma_1-1}$. The function $\rho(\mathbf{z}_1(t),\phi(t))$ is defined as:

$$\rho(\mathbf{z}_1(t),\phi(t)) \stackrel{def}{=} \gamma_1(\eta_1\|P_2\phi(t)\| + \eta_3\|\mathbf{z}_1(t)\|). \qquad (31)$$

where $\gamma_1 > 1$. Note, that the non-linear control given in (30) and (31) is continuous and globally Lipschitz. The gains η_1 and η_3 in (31) have been defined so that they ensure robustness with respect to the matched disturbances and reachability of the cone shaped sliding mode layer:

$$\eta_1 \stackrel{def}{=} \max\left(\sup_{\Delta\Omega}\left(\frac{1}{2}\lambda_{max}\{P_2^{-1}\Upsilon^T + \Upsilon P_2^{-1}\}\right),0\right), \qquad (32)$$

$$\eta_3 \stackrel{def}{=} \sup_{\Delta\Theta}(\|\Delta\Theta\|) + \omega^2\|P_1A_{12}P_2^{-1}\| + \frac{\omega}{2}\|P_1^{-\frac{1}{2}}\|\|P_2^{-\frac{1}{2}}\|, \qquad (33)$$

$$\Upsilon \stackrel{def}{=} ((1-\gamma_3)\Omega^* + \Delta\Omega),\ 1\ge\gamma_3>0. \qquad (34)$$

However, it is also possible to adjust the parameter $1 \ge \gamma_3 > 0$ for a compromise of using linear control for performance ($\gamma_3 = 1$) or robustness ($\gamma_3 \to 0$). A proof of stability [3, 6] makes it necessary to show that the null space Lyapunov function $V_1(t)$ in (29) is decreasing as soon as the states have entered the cone shaped layer, i.e. by imposing a quadratic stability constraint which will limit the choice of ω with an upper bound. An implicit bound on ω will be given in Sect. 5.1. A theorem showing asymptotic stability of the control can be easily derived from results in [3, 6].

5.1 Discretizing the sliding-mode like control

Since the sliding-mode like control $\mathbf{u}_L^{\mathcal{C}}(\mathbf{z}_1,\phi) + \mathbf{u}_{NL}^{\mathcal{C}}(\mathbf{z}_1,\phi)$ of (26-27) and (30) is globally Lipschitz, a time-discretized control $\mathbf{u}_L^{\Delta}(t,\mathbf{z}_1,\phi) + \mathbf{u}_{NL}^{\Delta}(t,\mathbf{z}_1,\phi)$ may be applied to the system (22) instead of the continuous control (11). The proof of stability for the discretized control using the L_2-gain approach may be sketched by following the different steps of the proof for Theorem 2.

Step I: The existence of a solution of (22) has been proved by [3] and the respective system (13-14) has also a Caratheodory solution if $\hat{\mathbf{u}}$ is of Caratheodory type.
Step II: A global Lyapunov function can be provided by non-smooth theory using a max-Lyapunov function as described in [3, 5]:

$$V(\mathbf{z}_1(t),\phi(t)) = \begin{cases} V_1(\mathbf{z}_1(t)) + kV_2(\phi(t)) & for\ V_2\ge\omega^2V_1 \\ (k\omega^2+1)V_1(\mathbf{z}_1(t)) & for\ V_2\le\omega^2V_1 \end{cases} \qquad (35)$$

where $k > 0$ has to be determined [3]. It has been shown in [3] that this function is differentiable for almost all t since the max-Lyapunov function is Lipschitz in $\mathbf{z}_1$ and ϕ and the solution $\mathbf{z}_1(t)$ and $\phi(t)$ is absolutely continuous in t [10]. Thus, the equality $\int^t \dot{V}(\mathbf{z}_1(s),\phi(s))ds = (V(\mathbf{z}_1(t),\phi(t)) + const.)$ applies. This integrability feature of $\dot{V}(\mathbf{z}_1(t),\phi(t))$ allows the non-smooth max-Lyapunov function to be used as $\mathcal{V}$ within the proof for Theorem 2. Further, it can be shown similar to [3, 10] that the relation $\dot{V} \le -\tilde{\vartheta}V$, $(\tilde{\vartheta} > 0)$ holds for almost all t. This implies exponentially fast decay of $V(\mathbf{z}_1(t),\phi(t))$ and exponential stability of the continuous-time controlled system.
Step III: The auxiliary output for (13-14) may be introduced with respect to the sliding-mode-like control system:

$$\tilde{\mathbf{y}} = \underbrace{\left(\begin{bmatrix}\Sigma & A_{12}\\ 0 & \Omega^*\end{bmatrix} + \begin{bmatrix}0 & 0\\ \Delta\Theta & \Delta\Omega\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \psi\end{bmatrix} - F\right)}_{\tilde{C}}\begin{bmatrix}\mathbf{z}_1\\ \phi\end{bmatrix}, \qquad (36)$$

$$\psi \stackrel{def}{=} \begin{cases} \frac{-\rho(\mathbf{z}_1(t),\phi(t))P_2}{\|P_2\phi(t)\|+\delta(\|\mathbf{z}_1(t)\|+\|\phi(t)\|)} & for\ [\mathbf{z}_1^T\phi^T]^T \neq 0 \\ 0 & for\ [\mathbf{z}_1^T\phi^T]^T = 0 \end{cases},$$

where the scalar ψ derived from (30-31) is bounded: $0 \le \psi \le \tilde{\rho}_{max}$. Thus, $\|\tilde{C}\|$ is bounded and it is permissible to define

$$F = \begin{bmatrix}\Sigma & A_{12}\\ 0 & \Omega^* - \frac{\tilde{\rho}_{max}P_2}{2}\end{bmatrix}.$$

The output matrix $\tilde{C}$ is partitioned as follows:

$$\tilde{C} = \begin{bmatrix}\tilde{C}_1 & \tilde{C}_2\end{bmatrix},\ \tilde{C}_1 \in \mathbb{R}^{n\times(n-m)},\ \tilde{C}_2 \in \mathbb{R}^{n\times m}.$$

The definition of $\underline{f}$ and $\underline{g}$ for (13) are then implicit using the transformed system dynamics (23-24) and the respective continuous-time control (26). This also determines the sub-states $\hat{\mathbf{z}}_1$ and $\hat{\phi}$ and the input perturbation $\hat{\mathbf{u}} \in \mathbb{R}^m$ as for (13). In the following, the max-Lyapunov function

$V(\hat{\mathbf{z}}_1(t), \hat{\phi}(t))$ from (35) and the respective sub-functions V_1 in (28) and $V_1 + kV_2$ in (28-29) shall be investigated with respect to the newly introduced states $\hat{\mathbf{z}}_1$ and $\hat{\phi}$. As a result, the L_2-gain of the system with input $\hat{\mathbf{u}}$ and output $\tilde{\mathbf{y}}$ will be determined. Similar to [6], it can be shown for $V_2 \geq \omega^2 V_1$

$$\hat{\phi}^T P_2 (B_2 \mathbf{u}_{NL}^{\mathcal{C}}(\hat{\mathbf{z}}_1(t), \hat{\phi}(t)) + \frac{\rho(\hat{\mathbf{z}}_1(t), \hat{\phi}(t))}{\gamma_1}) \leq 0.$$

Thus, the uncertainty is compensated for $V_2 \geq \omega^2 V_1$ [6]. In this case, the derivative $\dot{V}_1(\hat{\mathbf{z}}_1(t)) + k\dot{V}_2(\hat{\phi}(t))$ satisfies for almost all t (and considering the perturbation $\hat{\mathbf{u}}$):

$$\begin{aligned}\dot{V}_1(\hat{\mathbf{z}}_1(t)) + k\dot{V}_2(\hat{\phi}(t)) \quad &\leq \quad \begin{bmatrix} \hat{\mathbf{z}}_1(t) \\ \hat{\phi}(t) \\ \frac{1}{\epsilon}\hat{\mathbf{u}} \end{bmatrix}^T \Psi_1 \begin{bmatrix} \hat{\mathbf{z}}_1(t) \\ \hat{\phi}(t) \\ \frac{1}{\epsilon}\hat{\mathbf{u}} \end{bmatrix} \\ &- \hat{\vartheta}(V_1(\hat{\mathbf{z}}_1(t)) + kV_2(\hat{\phi}(t)) + \frac{\gamma^2}{\epsilon^2}\|\hat{\mathbf{u}}\|^2 - \frac{1}{\epsilon^2}\|\tilde{\mathbf{y}}\|^2, \end{aligned} \quad (37)$$

$$\Psi_1 = \begin{bmatrix} \frac{-I+\hat{\vartheta}P_1}{2} + \frac{\tilde{C}_1^T\tilde{C}_1}{\epsilon^2} & \frac{P_1A_{12}}{2} + \frac{\tilde{C}_1^T\tilde{C}_2}{\epsilon^2} & 0 \\ \frac{A_{12}^T P_1}{2} + \frac{\tilde{C}_2^T\tilde{C}_1}{\epsilon^2} & -k\frac{\gamma_3 I - \hat{\vartheta}P_2}{2} + \frac{\tilde{C}_2^T\tilde{C}_2}{\epsilon^2} & \frac{kP_2B_2}{2}\epsilon \\ 0 & \frac{kB_2^T P_2}{2}\epsilon & -\gamma^2 I_m \end{bmatrix} \quad (38)$$

The scalars $\epsilon > 0$, $\hat{\vartheta} > 0$ and $\gamma \geq 0$ have been introduced to ensure that the symmetric matrix $\Psi_1 \leq 0$. For that reason, $\frac{1}{\epsilon}$ and $\hat{\vartheta}$ should not be chosen too large to ensure that the principal minors of Ψ_1 are negative semi-definite for any value of $\tilde{C}$. Similar arguments need to be made for $k > 0$. Further, the value of γ should be chosen as small as possible so that $\Psi_1 \leq 0$ for any value of $\tilde{C}$. The scalar γ will act as an upper bound on the L_2-gain from $\hat{\mathbf{u}}$ to $\tilde{\mathbf{y}}$.

For $V_2 \leq \omega^2 V_1$, the derivative $\dot{V}_1$ is investigated; for almost all t:

$$\begin{aligned}\dot{V}_1 = &\frac{1}{2}\begin{bmatrix} \hat{\mathbf{z}}_1^T(t) & \hat{\phi}^T(t) \end{bmatrix} \Psi_2 \begin{bmatrix} \hat{\mathbf{z}}_1^T(t) & \hat{\phi}^T(t) \end{bmatrix}^T \\ &+\xi\left(-V_1 + \frac{1}{\omega^2}V_2\right) - \bar{\vartheta}V_1 - \frac{1}{(k\omega^2+1)\epsilon^2}\|\tilde{\mathbf{y}}\|^2, \end{aligned}$$

$$\Psi_2 = \begin{bmatrix} -I + \bar{\vartheta}P_1 + \frac{2\tilde{C}_1^T\tilde{C}_1}{(k\omega^2+1)\epsilon^2} + \xi P_1 & P_1A_{12} + \frac{2\tilde{C}_1^T\tilde{C}_2}{(k\omega^2+1)\epsilon^2} \\ A_{12}^T P_1 + \frac{2\tilde{C}_2^T\tilde{C}_1}{(k\omega^2+1)\epsilon^2} & -\frac{\xi P_2}{\omega^2} + \frac{2\tilde{C}_2^T\tilde{C}_2}{(k\omega^2+1)\epsilon^2} \end{bmatrix}. \quad (39)$$

where $\xi \in \mathbb{R}$. It can be shown that if $\bar{\vartheta} > 0$, $\omega > 0$ and $\frac{2}{(k\omega^2+1)\epsilon^2}$ are chosen small enough then there always exists a $\xi \geq 0$ so that $\Psi_2 \leq 0$. Assuming $\Psi_2 \leq 0$, $V_2(\hat{\phi}(t)) - \omega^2 V_1(\hat{\mathbf{z}}_1(t)) \leq 0$, it follows for almost all t:

$$\dot{V}_1 \leq -\bar{\vartheta}V_1 - \frac{1}{(k\omega^2+1)\epsilon^2}\|\tilde{\mathbf{y}}\|^2 \quad (40)$$

Thus, the sub-functions V_1 and $V_1 + kV_2$ of the max-Lyapunov functions $V(\hat{\mathbf{z}}_1(t), \hat{\phi}(t))$ (35) have been now investigated with respect to their time derivative (37),(40) at those points where they define the max-Lyapunov function (35). It is then possible using non-smooth analysis to show that for almost all t the derivative $\dot{V}(\hat{\mathbf{z}}_1(t), \hat{\phi}(t))$ satisfies:

$$\begin{aligned}\dot{V}(\hat{\mathbf{z}}_1(t), \hat{\phi}(t)) \quad \leq \quad &-\min(\hat{\vartheta}, (k\omega^2+1)\bar{\vartheta})V(\hat{\mathbf{z}}_1(t), \hat{\phi}(t)) \\ &+\frac{\gamma^2}{\epsilon^2}\|\hat{\mathbf{u}}\|^2 - \frac{1}{\epsilon^2}\|\tilde{\mathbf{y}}\|^2 \end{aligned} \quad (41)$$

The latter inequality actually is equivalent to (15). Therefore, γ is an upper bound of the L_2-gain from $\hat{\mathbf{u}}$ to $\tilde{\mathbf{y}}$, while a quadratic stability margin $(-\min(\hat{\vartheta}, (k\omega^2+1)\bar{\vartheta})V(\hat{\mathbf{z}}_1, \hat{\phi}))$ is considered.

Step IV & V: The next steps follow the proof of Theorem 2. The Lipschitz constant $\mathcal{K}_{\mathbf{u}}$ of the sliding-mode based control has been calculated in [3], which can be used here. In particular, an inequality similar to (21) must follow for any $T \geq 0$ and therefore asymptotic stability can be implied for small enough sampling time τ.

5.2 Examples of discretized sliding-mode based controls

The L_2-gain technique will be demonstrated and compared to the Lyapunov function analysis based discretization methodology of [5] using two examples. First, consider a simple model of the inverted pendulum, which is formed by a light rod and a heavy mass attached to one end of the rod with the pivot of rotation at the other end. The system may be expressed in the form of (22) with

$$A = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \Delta\Theta = 0.5(\mathrm{sinc}(\theta(t)) - 1)$$

$$\Delta\Omega = -b + 0.1, \ \mathrm{sinc}(t) = \begin{cases} \frac{\sin(t)}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases} \quad (42)$$

where θ is the angle of rotation, $\mathbf{x}^T = [\theta \ \dot{\theta}]$, b ($0.09 \leq b \leq 0.11$) the damping coefficient and $\mathbf{u}(t)$ the control torque. Furthermore, it is of interest to investigate the L_2-gain approach on a stable model in the nominal case when the uncertainty is set to zero and $\eta_3 \neq 0$, the non-linear control gain, is only used for reachability of the sliding-mode. Thus, the system satisfies in this case:

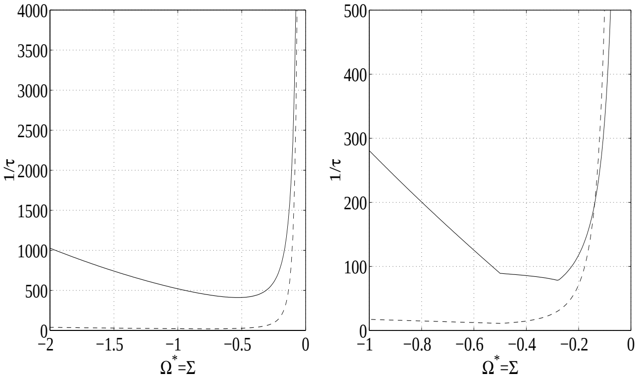
$$A = \begin{bmatrix} -0.5 & 1 \\ 0 & -0.5 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ [\Delta\Theta \ \ \Delta\Omega] = 0. \quad (43)$$

The non-linear controllers for both systems are chosen to satisfy $\gamma_1 = 1.6$, $\eta_1 = 0$. By demanding $\eta_1 = 0$ (32), the linear controller is used to compensate for the parametric uncertainty $(-b + 0.1)$ of Ω. Therefore assuming $\eta_1 = 0$, the value of γ_3 ($0 < \gamma_3 \leq 1$) must be chosen as large as possible, so that the linear control is also used for reachability. Provided the matrix Σ (25) determining the sliding-mode performance, the linear control matrix Ω^* (27) and the respective Lyapunov matrices P_1 (29) and P_2 (28) have been set, then the features of the sliding-mode cone and the parameter ω can be determined using nominal system and controller parameters only:

$$\underbrace{-\lambda_{max}\left(P_1\Sigma + \Sigma^T P_1\right)}_{=1} - 2|P_1A_{12}P_2^{-\frac{1}{2}}|\omega|P_1^{\frac{1}{2}}| = 0.85$$

For the single input, second order systems considered, this constraint is necessary and sufficient so that for any state satisfying $V_2 \leq \omega^2 V_1$, $\dot{V}_1 \leq -0.85|z_1|^2$ for almost all t. This ensures consistency with the results in [5], comparability of the control of the nominal and the uncertain systems and implies stable performance for both examples (39). The value of κ, $0 < \kappa < 1$, is set to 0.5 as a trade-off between controller performance and low sampling frequency as derived in the proof of Theorem 2. The constant k from the max-Lyapunov function (35) has been adjusted via numerical methods so that the sampling frequency bound $1/\hat{\tau}$ for each controller is minimal.

Note that the results (Figure 1) for the L_2-norm based technique are more conservative than those calculated for the Lyapunov function analysis based approach. The sampling frequency which is sufficient to stabilize robustly the unstable inverted pendulum (42) is generally for the L_2-norm based methodology more than nine times higher than for the Lyapunov function based technique. A similar result applies to the stable system with no uncertainties (43) where



a) unstable pendulum (42) b) stable system (43)

Figure 1: Values of sufficient sampling frequencies against changing controller poles $\Sigma = \Omega^*$ using the L_2-norm based approach (**'-'**) and the Lyapunov function analysis based approach (**'--'**)

the non-linear control is only used for reachability of the sliding-mode region. The reason for the more conservative results is connected to the conservative estimate of the L_2-gain γ with respect to the output $\tilde{\mathbf{y}}$ (13-14), (36). Using the Matlab©-LMI toolbox, the matrix $\tilde{C}^T\tilde{C}$ was approximated by a polytopic set. Furthermore in the Lyapunov function based approach, the very large value of the Lipschitz constant of the non-linear control $\mathbf{u}_{NL}^{\mathcal{C}}$ (30) could be compensated by the upper estimate of the sampling error $\sup_{t_\alpha \in [t_i, t_{i+1}]} (\|\mathbf{x}(t_\alpha) - \mathbf{x}(t_i)\|)$ (1) [3], which is not straightforward for the L_2-norm technique.

6 Conclusion

This work has presented a theoretical approach to L_2-stability of discretized non-linear continuous-time control. In particular an approach was presented which shows that a sampled-data output of a strictly proper, exponentially stable, affine, non-linear system has finite L_p-norm approaching the L_p-norm of the continuous-time output for $\tau \to +0$. This result has been used to show asymptotic stability of a discretized, continuous-time, non-linear control for small enough sampling time τ. An example of a sliding-mode based control has shown that the methodology is theoretically viable but practically disadvantageous compared to a Lyapunov function analysis based technique.

Appendix

Lemma 1 *The exponentially stable system (4) satisfying (5-6) has finite L_2-induced gain and*

$$\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\underline{f}(\mathbf{x}) + \frac{\delta\mathcal{V}}{\delta\mathbf{x}}\underline{g}(\mathbf{x})\mathbf{u} \le \frac{1}{2}\gamma_{subopt}^2 \frac{1}{\epsilon^2}\|\mathbf{u}\|^2 - \frac{1}{2}\frac{1}{\epsilon^2}\|\mathbf{y}\|^2,$$

$$\gamma_{subopt} = \frac{\mathcal{K}_{\underline{h}} c_4 \tilde{\mathcal{G}}}{c_3}, \; \epsilon = \frac{\mathcal{K}_{\underline{h}}}{\sqrt{c_3}} \tag{44}$$

Hence, γ_{subopt} is an upper bound on the L_2-gain. Furthermore, a quadratic stability margin $-(1-\kappa)c_3\|\mathbf{x}\|^2$ is part of the upper bound for $\dot{\mathcal{V}}$ if the gain γ is increased to $\gamma = \frac{\gamma_{subopt}}{\kappa}$ using a scalar κ, $0 < \kappa \le 1$ and

$$\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\underline{f}(\mathbf{x}) + \frac{\delta\mathcal{V}}{\delta\mathbf{x}}\underline{g}(\mathbf{x})\mathbf{u} \le \frac{\gamma_{subopt}^2}{\kappa^2}\frac{\|\mathbf{u}\|^2}{2\epsilon^2} - \frac{\|\mathbf{y}\|^2}{2\epsilon^2} - (1-\kappa)c_3\|\mathbf{x}\|^2. \tag{45}$$

holds for $\epsilon = \frac{\mathcal{K}_{\underline{h}}}{\sqrt{c_3\kappa}}$.

Proof Introduce the auxiliary system input gain $\underline{\tilde{g}} = \epsilon \underline{g}$ and the auxiliary output $\underline{\tilde{h}} = \frac{1}{\epsilon}\underline{h}$ using the positive scalar $\epsilon > 0$ from (44). Upper bounds for the left hand side of the Hamilton-Jacobian inequality of the input-to-output map $\frac{1}{\epsilon}\mathbf{u} \to \frac{1}{\epsilon}\mathbf{y}$ can be derived using the inequalities (5) and (6):

$$\begin{aligned}
&\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\underline{f}(\mathbf{x}) + \frac{1}{2\gamma^2}\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\underline{\tilde{g}}(\mathbf{x})\underline{\tilde{g}}^T(\mathbf{x})\frac{\delta\mathcal{V}}{\delta\mathbf{x}}^T + \frac{1}{2}\underline{\tilde{h}}^T(\mathbf{x})\underline{\tilde{h}}(\mathbf{x}) \\
&= \frac{\delta\mathcal{V}}{\delta\mathbf{x}}\underline{f}(\mathbf{x}) + \frac{\epsilon^2}{2\gamma^2}\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\underline{g}(\mathbf{x})\underline{g}^T(\mathbf{x})\frac{\delta\mathcal{V}}{\delta\mathbf{x}}^T + \frac{1}{2\epsilon^2}\underline{h}^T(\mathbf{x})\underline{h}(\mathbf{x}) \\
&\le -c_3\|\mathbf{x}\|^2 + \frac{\epsilon^2}{2\gamma^2}c_4^2\tilde{\mathcal{G}}^2\|\mathbf{x}\|^2 + \frac{1}{2\epsilon^2}\mathcal{K}_{\underline{h}}^2\|\mathbf{x}\|^2 \\
&\le \frac{1}{2\gamma^2\epsilon^2}c_4^2\tilde{\mathcal{G}}^2 \underbrace{\left((\epsilon^2)^2 - (\epsilon^2)2\gamma^2\frac{c_3}{c_4^2\tilde{\mathcal{G}}^2} + \gamma^2\frac{\mathcal{K}_{\underline{h}}^2}{c_4^2\tilde{\mathcal{G}}^2}\right)}_{\psi}\|\mathbf{x}\|^2
\end{aligned}$$

The term $\psi = \epsilon^4(1 - \frac{\gamma^2}{\gamma_{subopt}^2})$ is non-positive for $\gamma \ge \gamma_{subopt}$. This implies using [7, Theorem 6.5], that the L_2-induced norm of the map $\frac{1}{\epsilon}\mathbf{u} \to \frac{1}{\epsilon}\mathbf{y}$ is smaller or equal γ_{subopt} and the inequality (44) holds. Thus, the map $\mathbf{u} \to \mathbf{y}$ must have the same upper bound on the L_2 induced norm. The second claim in (45) follows by carrying out the same steps including the quadratic stability margin. ■

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