Robust $\mathcal{H}_\infty$ Control of Uncertain Linear Systems via Parameter-Dependent Lyapunov Functions*

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ABSTRACT

This paper addresses the problems of robust $\mathcal{H}_\infty$ performance analysis and control synthesis for linear systems subject to uncertain real time-varying parameters. The uncertain parameters enter affinely in the matrices of the system state-space model and the admissible values of the parameters and their rates of variation are supposed to belong to a given polytope. New LMI based methods of robust $\mathcal{H}_\infty$ analysis and control design via state feedback are proposed using parameter-dependent Lyapunov functions.

1. INTRODUCTION

The past two decades has witnessed significant advances in the theory of $\mathcal{H}_\infty$ control. The state-space approach has been well investigated and a number of solutions have been proposed based on, for instance, Riccati equations and linear matrix inequalities (LMIs); see, e.g. [4, 7, 8, 11] and the references therein. The $\mathcal{H}_\infty$ control approach is known to provide stability robustness against certain types of modeling uncertainty, however it cannot possibly guarantee a prescribed performance in the presence of large modeling uncertainty. This motivated the development of robust $\mathcal{H}_\infty$ control methods to ensure both stability and a prescribed $\mathcal{H}_\infty$ performance, irrespective of the uncertainty. A time-domain approach of robust $\mathcal{H}_\infty$ control to deal with parameter uncertainty that has attracted considerable interest is the approach based on the notion of quadratic stability (see, e.g. [1, 12, 15] and the references therein). One of the main drawbacks of the quadratic stability approach is that the control design uses a parameter-independent Lyapunov matrix which satisfies the bounded real lemma. As a consequence, the system stability and performance bound hold even when the parameters changes arbitrarily fast. It has long been known that such an approach often yields overly conservative results.

A few attempts have been made to reduce the conservatism of the quadratic stability approach by using parameter-dependent Lyapunov functions. In this context, several methods have been proposed to solve a number of important problems such as robust stability analysis, robust stabilization, robust $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control, and gain-scheduled control of linear parameter-varying systems; see, e.g. [2, 5, 6, 9, 10, 13, 14, 16] and the references therein. Most of the numerically tractable results in this area have been developed for Lyapunov functions with affine or multi-affine parameter-dependence [5, 6, 9, 16]. LMI methods of robust stability analysis and robust $\mathcal{H}_2$ control based on quadratically parameter-dependent Lyapunov functions have also been proposed [2, 13]. In spite of the recent developments on parameter-dependent Lyapunov methods, to-date only few results on robust $\mathcal{H}_\infty$ control design are available and the theory is far from being complete.

In this paper we develop LMI based methodologies of robust $\mathcal{H}_\infty$ performance analysis and control design for linear systems with uncertain time-varying parameters using parameter-dependent Lyapunov functions. The uncertain parameters enter affinely in the matrices of the system state-space model and the admissible values of the parameters and their rates of variation are assumed to belong to a given polytope with known vertices. Attention is focused on the design of static state feedback controllers. The proposed methods incorporate information on the bounds on the rates of change of the parameters and have the feature that the system stability and performance bound are based on parameter-dependent Lyapunov functions with quadratic dependence on the uncertain parameters. Our methods include the quadratic stability based approach as a particular case, and as a result they are at most as conservative as the latter approach.

Notation. $\Re^n$ denotes the n-dimensional Euclidean space, $\Re^{n\times m}$ is the set of $n \times m$ real matrices, I_n is the $n \times n$ identity matrix, the notation $S > 0$, for a real matrix S, means that S is symmetric and positive definite and $\mathcal{L}_2$ denotes the space of square integrable vector function on $[0, \infty)$ with norm $\|\cdot\|_2$. For a symmet-

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*This work was supported in part by 'Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq', Brazil, under grants 300459/93-9/PQ and 301653/ 96-8/PQ, PRONEX grant 'Control of Dynamical Systems', and CAPES.

ric block matrix, the symbol $\star$ denotes the transpose of the block matrices above the main diagonal block.

2. PRELIMINARIES

Consider the uncertain linear system (G)

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B_u(\theta(t))u(t) + B_w(\theta(t))w(t) \\ z(t) &= C(\theta(t))x(t) + D_u(\theta(t))u(t) + D_w(\theta(t))w(t) \end{aligned} \tag{1}$$

with $\theta(t) := (\theta_1(t), \cdots, \theta_p(t)) \in \Re^p$ and

$$A(\theta) = A_0 + \sum_{i=1}^{p} \theta_i A_i, \quad B_u(\theta) = B_{u_0} + \sum_{i=1}^{p} \theta_i B_{u_i}$$

$$B_w(\theta) = B_{w_0} + \sum_{i=1}^{p} \theta_i B_{w_i}, \quad C(\theta) = C_0 + \sum_{i=1}^{p} \theta_i C_i$$

$$D_u(\theta) = D_{u_0} + \sum_{i=1}^{p} \theta_i D_{u_i}, \quad D_w(\theta) = D_{w_0} + \sum_{i=1}^{p} \theta_i D_{w_i}$$

where $x(t) \in \Re^n$ is the state, $u \in \Re^{n_u}$ is the control input, $w(t) \in \Re^{n_w}$ is the disturbance input, $z(t) \in \Re^{n_z}$ is the performance output, A_i, B_{u_i}, B_{w_i}, C_i, D_{u_i} and D_{w_i}, $i = 0, \ldots, p$, are known constant real matrices of appropriate dimensions, and $\theta_i(t)$, $i = 1, \ldots, p$, are uncertain bounded real time-varying parameters with rates of variation $\dot{\theta}_i(t)$. It is assumed that $(\theta(t), \dot{\theta}(t))$, $\forall\, t \geq 0$, lies in a given polytopic domain $\mathcal{D}$ with known ℓ vertices.

This paper addresses the problems of robust $\mathcal{H}_\infty$ performance analysis and control for system (1) using parameter-dependent Lyapunov functions. Inspired by the work of [13] on robust stability analysis via parameter-dependent Lyapunov functions with quadratic dependence on the uncertain parameters, in this paper we shall develop LMI methods of robust $\mathcal{H}_\infty$ performance analysis and controller based on a version of the so-called bounded real lemma for linear time-varying systems, where a quadratically parameter-dependent solution to the underlying Riccati inequality, or the corresponding LMI, is used. Attention is focused on the design of a static state feedback controller.

In order to develop the $\mathcal{H}_\infty$ analysis and synthesis methods, the state equation of system (1) will be rewritten in the form:

$$\begin{aligned} \dot{x}(t) = {} & \left[A_0 + \Theta^T(t)\mathcal{A}\right] x(t) + \left[B_{u_0} + \Theta^T(t)\mathcal{B}_u\right] u(t) \\ & + \left[B_{w_0} + \Theta^T(t)\mathcal{B}_w\right] w(t) \end{aligned} \tag{2}$$

where $\mathcal{A} \in \Re^{q \times n}$, $\mathcal{B}_u \in \Re^{q \times n_u}$ and $\mathcal{B}_w \in \Re^{q \times n_w}$ are known matrices, and $\Theta \in \Re^{q \times n}$ is an uncertain matrix such that each row depends linearly on the uncertain parameters θ_i, and where the value of q depends on the structure of the matrices A_i, B_{u_i} and B_{w_i}, $i = 1, \ldots, p$, and on the structure chosen for Θ.

Note that the representation of (2) always exits and does not impose any loss of generality. For example, a direct, and natural, choice of $\mathcal{A}$, $\mathcal{B}_u$, $\mathcal{B}_w$ and Θ is

$$\mathcal{A} = \begin{bmatrix} A_1 \\ \vdots \\ A_p \end{bmatrix}, \quad \mathcal{B}_u = \begin{bmatrix} B_{u_1} \\ \vdots \\ B_{u_p} \end{bmatrix}, \quad \mathcal{B}_w = \begin{bmatrix} B_{w_1} \\ \vdots \\ B_{w_p} \end{bmatrix}, \quad \Theta = \begin{bmatrix} \theta_1 I_n \\ \vdots \\ \theta_p I_n \end{bmatrix}.$$

It turns out that $q = np$ for the above Θ and this is possibly the choice of Θ of largest dimensions which is not redundant.

Observe that the representation of (2) is not unique. Further, the choice of the dimension q of Θ should be based on the tradeoff between the conservatism and the computational effort required by both the performance analysis and control design methods. Indeed, an increase of q is likely to reduce the conservatism of those methods as it increases the number of decision variables in the underlying optimization problem, however it will increase the required computational effort.

We end this section by recalling the notion of $\mathcal{H}_\infty$ norm and a version of the bounded real lemma for linear time-varying systems which will be used in the paper. Let the linear system $(\tilde{G})$:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)w(t) \\ z(t) &= C(t)x(t) + D(t)x(t) \end{aligned}$$

where $x(t) \in \Re^n$ is the state, $w(t) \in \Re^{n_w}$ is the input, $z(t) \in \Re^{n_z}$ is output, and $A(t)$, $B(t)$, $C(t)$ and $D(t)$ are real valued piecewise-continuous matrix functions with appropriate dimensions.

Definition 2.1. ([8]) *Suppose that the system $\tilde{G}$ is exponentially stable. Then the $\mathcal{H}_\infty$ norm of $\tilde{G}$ is defined as the $\mathcal{L}_2$-induced norm of the input-output operator $\tilde{G} : w \to z$, namely*

$$\|\tilde{G}\|_\infty = \sup_w \left\{ \frac{\|z\|_2}{\|w\|_2} \;:\; w \in \mathcal{L}_2,\, w \neq 0;\; x(0) = 0 \right\}$$

Lemma 1. ([3]) *Consider the system $\tilde{G}$ and let $\gamma > 0$ be a given scalar. Then the following statements are equivalent*

(1) The System $\tilde{G}$ is exponentially stable and $\|\tilde{G}\|_\infty < \gamma$.

(2) There exists a bounded positive definite matrix function $P(t)$ on $[0, \infty)$ such that the following inequality holds for all $t \in [0, \infty)$

$$\begin{bmatrix} -\dot{P}(t) + A(t)P(t) + P(t)A^T(t) & P(t)C^T(t) & B(t) \\ C(t)P(t) & -\gamma I & D(t) \\ B^T(t) & D^T(t) & -\gamma I \end{bmatrix} < 0 \tag{3}$$

3. ROBUST $\mathcal{H}_\infty$ PERFORMANCE ANALYSIS

This section is concerned with the problem of robust $\mathcal{H}_\infty$ performance analysis for the unforced system of (1) with $u(t) \equiv 0$, namely the system (G_0):

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B_w(\theta(t))w(t) \\ z(t) &= C(\theta(t))x(t) + D_w(\theta(t))w(t) \end{aligned} \tag{4}$$

The goal is to ascertain exponential stability and ensure an upper-bound for the $\mathcal{H}_\infty$ norm of system (4) for all $(\theta, \dot{\theta}) \in \mathcal{D}$. We shall develop a method of $\mathcal{H}_\infty$ performance analysis based on the bounded real lemma inequality of (3) with a quadratically parameter-dependent matrix $P(t)$. More specifically, the aim is to find conditions which ensure the existence of a real symmetric matrix function $\mathcal{P}(\theta)$ which is quadratic in θ and satisfies the following inequalities for all $(\theta, \dot{\theta}) \in \mathcal{D}$:

$$\mathcal{P}(\theta) > 0 \tag{5}$$

$$\begin{bmatrix} -\dot{\mathcal{P}}(\theta) + A(\theta)\mathcal{P}(\theta) + \mathcal{P}(\theta)A^T(\theta) & \star & \star \\ C(\theta)\mathcal{P}(\theta) & -\gamma I & \star \\ B_w^T(\theta) & D_w^T(\theta) & -\gamma I \end{bmatrix} < 0 \tag{6}$$

Note that as the inequalities of (5) and (6) are nonconvex in θ, the problem of testing if these inequalities are feasible is not tractable, in general, neither analytically nor numerically. In the sequel we shall develop sufficient LMI conditions for checking the feasibility of (5) and (6) .

Without loss of generality, the matrix $\mathcal{P}(\theta)$ is assumed to be of the following form

$$\begin{aligned} \mathcal{P}(\Theta) &= P_0 + P_1\Theta + \Theta^T P_1^T + \Theta^T P_2 \Theta \\ &= \tilde{\Theta}^T P \tilde{\Theta} \end{aligned} \tag{7}$$

where

$$\tilde{\Theta} := \begin{bmatrix} I_n \\ \Theta \end{bmatrix}, \quad P := \begin{bmatrix} P_0 & P_1 \\ P_1^T & P_2 \end{bmatrix} \in \Re^{(n+q)\times(n+q)} \tag{8}$$

The robust $\mathcal{H}_\infty$ performance analysis result is presented below. Due to space limitation, all proofs will be omitted. First, introduce the notation:

$$\Lambda(\theta, \dot{\theta}) = \begin{bmatrix} A_0 & -\dot{\Theta}^T + A_0\Theta^T \\ \mathcal{A} & \mathcal{A}\Theta^T \end{bmatrix}, \quad \tilde{\mathcal{B}}_w = \begin{bmatrix} B_{w_0} \\ \mathcal{B}_w \end{bmatrix} \tag{9}$$

$$H_x(\theta) = \begin{bmatrix} \Theta & -I_q \end{bmatrix}. \tag{10}$$

Theorem 3.1. *Consider the system* G_0 *of (4) and let* $\mathcal{D}$ *be a polytopic domain of admissible* $(\theta, \dot{\theta})$ *and* $\gamma > 0$ *a given scalar. Suppose there exist a symmetric matrix* P *and matrices* L *and* M *such that the following LMIs are satisfied at all the vertices of* $\mathcal{D}$*:*

$$P + LH_x(\theta) + H_x^T(\theta)L^T > 0 \tag{11}$$

$$\begin{bmatrix} W(\theta,\dot{\theta}) & \star & \star & \star \\ \tilde{\Theta}^T P & 0 & \star & \star \\ 0 & 0 & -\gamma I & \star \\ \tilde{\mathcal{B}}_w^T & 0 & D_w^T(\theta) & -\gamma I \end{bmatrix} + MH(\theta) + H^T(\theta)M^T < 0 \tag{12}$$

where

$$W(\theta, \dot{\theta}) = \Lambda(\theta, \dot{\theta})P + P\Lambda^T(\theta, \dot{\theta}) \tag{13}$$

$$H(\theta) = \begin{bmatrix} H_x(\theta) & 0 & 0 & 0 \\ 0 & -I & C^T(\theta) & 0 \end{bmatrix}. \tag{14}$$

Then the system G_0 *is exponentially stable and* $\|G_0\|_\infty < \gamma$ *for all* $(\theta, \dot{\theta}) \in \mathcal{D}$*. Moreover,* $V(x, \theta) = x^T \mathcal{P}^{-1}(\theta) x$*, where* $\mathcal{P}(\theta) = \tilde{\Theta}^T P \tilde{\Theta}$*, is a parameter-dependent Lyapunov function for the system* G_0*.*

Theorem 3.1 provides an LMI based method for computing an upper-bound for the $\mathcal{H}_\infty$ norm of system (4). The proposed method incorporates information on the bounds on the rates of variation of the parameters and is based on a parameter-dependent Lyapunov function. Further, finding the minimum γ is a convex optimization problem in terms of an eigenvalue minimization problem with LMIs constraints (see, e.g. [1]).

Remark 3.1. Note that Theorem 3.1 specializes to $\mathcal{H}_\infty$ performance analysis methods based on quadratic stability and affine quadratic stability by imposing certain constraints to the matrix P as partitioned below

$$P = \begin{bmatrix} P_0 & P_1 \\ P_1^T & P_2 \end{bmatrix}; \quad P_0 \in \Re^{n\times n}, \quad P_2 \in \Re^{q\times q}.$$

Indeed, when $P_1 = 0$ and $P_2 = 0$, it turns out that $\mathcal{P}(\theta)$ becomes parameter-independent, namely $\mathcal{P}(\theta) = P_0$, and thus Theorem 3.1 provides a quadratic stability based approach. On the other hand, by imposing the restriction $P_2 = 0$ renders the matrix $\mathcal{P}(\theta)$ affine in the parameters θ_i. □

It should be observed that the conditions of Theorem 3.1 with $P_1 = 0$ and P_2=0 are necessary and sufficient for the bounded real lemma inequalities of (5) and (6) to hold with a parameter-independent matrix $\mathcal{P}(\theta) = P_q$. Indeed, the sufficiency follows immediately as pointed out in Remark 3.1, whereas the converse result is provided by the next lemma.

Lemma 2. *Suppose that the bounded real lemma inequalities of (5) and (6) are satisfied with* $\mathcal{P}(\theta) = P_q$*. Then there exists a scalar* $\varepsilon > 0$ *such that the conditions of Theorem 3.1 are satisfied with* $P_0 = P_q$*,* $P_1 = 0$*,* $P_2 = 0$ *and*

$$L = \begin{bmatrix} 0 \\ -\varepsilon I \end{bmatrix}, \quad M = \begin{bmatrix} P_q \mathcal{A}^T & P_q \\ \varepsilon I & 0 \\ 0 & \varepsilon I \\ 0 & 0 \\ \mathcal{B}_w^T & 0 \end{bmatrix}.$$

In the light of Remark 3.1 and Lemma 2, the robust $\mathcal{H}_\infty$ performance analysis method of Theorem 3.1 is at most as conservative as the quadratic stability approach and is likely to provides less conservative results than the methods based on affine quadratic stability.

We end this section by noting that in the case where the matrix $C(\theta)$ is parameter-independent, namely $C(\theta) = C_0$, Theorem 3.1 specializes to the following result.

Corollary 3.1. *Consider the system G_0 of (4) with $C(\theta) = C_0$ and let $\mathcal{D}$ be a polytope of admissible $(\theta, \dot{\theta})$ and $\gamma > 0$ a given scalar. Suppose there exist a symmetric matrix P and matrices L and M such that the following LMIs are satisfied at all the vertices of $\mathcal{D}$:*

$$P + LH_x(\theta) + H_x^T(\theta)L^T > 0$$

$$\begin{bmatrix} W(\theta,\dot{\theta}) & \star & \star \\ C_0\tilde{\Theta}^T P & -\gamma I & \star \\ \tilde{\mathcal{B}}_w^T & D_w^T(\theta) & -\gamma I \end{bmatrix} + M\hat{H}(\theta) + \hat{H}^T(\theta)M^T < 0$$

where $\tilde{\mathcal{B}}_w$, $H_x(\theta)$ and $W(\theta,\dot{\theta})$ are given in (9), (10) and (13), respectively, and

$$\hat{H}(\theta) = \begin{bmatrix} H_x(\theta) & 0 & 0 \end{bmatrix}.$$

Then the system G_0 is exponentially stable and $\|G_0\|_\infty < \gamma$ for all $(\theta, \dot{\theta}) \in \mathcal{D}$. Moreover, $V(x,\theta) = x^T \mathcal{P}^{-1}(\theta)x$, where $\mathcal{P}(\theta) = \tilde{\Theta}^T P \tilde{\Theta}$, is a parameter dependent Lyapunov function for the system G_0.

4. ROBUST $\mathcal{H}_\infty$ CONTROL SYNTHESIS

This section presents a robust state feedback $\mathcal{H}_\infty$ control design method for system (1) via LMIs and based on a parameter-dependent Lyapunov function. Attention is focused on a control law of the form $u(t) = Kx(t)$, where $K \in \Re^{n_u \times n}$ is a constant matrix gain to be found, such that the closed-loop system (G_c):

$$\begin{aligned} \dot{x}(t) &= \big[A(\theta) + B_u(\theta)K\big]x(t) + B_w(\theta)w(t) \\ z(t) &= \big[C(\theta) + D_u(\theta)K\big]x(t) + D_w(\theta)w(t) \end{aligned} \qquad (15)$$

satisfies the bounded real conditions of (5) and (6) with a quadratically parameter-dependent matrix $\mathcal{P}(\theta)$. The robust $\mathcal{H}_\infty$ controller design is presented in the next theorem.

Theorem 4.1. *Consider the system (1) and let $\mathcal{D}$ be a polytopic domain of admissible $(\theta, \dot{\theta})$ and $\gamma > 0$ a given scalar. Suppose there exist a symmetric matrix P and matrices L, M, N, R and S such that the following conditions hold at all the vertices of $\mathcal{D}$:*

$$P + LH_x(\theta) + H_x^T(\theta)L^T > 0 \qquad (16)$$

$$\Psi(\theta,\dot{\theta}) + MH_c(\theta) + H_c^T(\theta)M^T < 0 \qquad (17)$$

$$R\tilde{\Theta}^T = \tilde{\Theta}^T P \qquad (18)$$

where

$$\Psi(\theta,\dot{\theta}) = \begin{bmatrix} W(\theta,\dot{\theta}) & \star & \star & \star & \star \\ \tilde{\Theta}^T P & 0 & \star & \star & \star \\ S^T \tilde{\mathcal{B}}_u^T & 0 & 0 & \star & \star \\ 0 & 0 & D_u(\theta)S & -\gamma I & \star \\ \tilde{\mathcal{B}}_w^T & 0 & 0 & D_w^T(\theta) & -\gamma I \end{bmatrix}$$

$$H_c(\theta) = \begin{bmatrix} H_x(\theta) & 0 & 0 & 0 & 0 \\ 0 & -I_n & 0 & C^T(\theta) & 0 \\ \tilde{\Theta}^T & 0 & -I_n & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathcal{B}}_u = \begin{bmatrix} B_{u_0} \\ \mathcal{B}_u \end{bmatrix}$$

and $\tilde{\mathcal{B}}_w$, $H_x(\theta)$ and $W(\theta,\dot{\theta})$ are given in (9), (10) and (13), respectively.

Then the control law $u(t) = Kx(t)$, where $K = SR^{-1}$, ensures that the resulting closed-loop system G_c is exponentially stable and $\|G_c\|_\infty < \gamma$ for all $(\theta, \dot{\theta}) \in \mathcal{D}$. Moreover, $V(x,\theta) = x^T \mathcal{P}^{-1}(\theta)x$, where $\mathcal{P}(\theta) = \tilde{\Theta}^T P \tilde{\Theta}$, is a parameter-dependent Lyapunov function for the closed-loop system.

Remark 4.1. Theorem 4.1 provides an LMI method for the design of a robust state feedback $\mathcal{H}_\infty$ controller for system (1). The developed $\mathcal{H}_\infty$ control method has the feature that the control law is based on a quadratically parameter-dependent Lyapunov function for the closed-loop system. Indeed, it turns out that the control law of Theorem 4.1 ensures the exponential stability of the adjoint system corresponding to the closed-loop system (15) via a Lyapunov function which depends quadratically on the uncertain parameters. ☐

Note that the equality constraint $R\tilde{\Theta}^T = \tilde{\Theta}^T P$ of (18) may render the synthesis method of Theorem 4.1 somehow conservative as this equality implies hard constraints on the structure of the matrix P. To alleviate this undesirable feature, the condition of (18) can be replaced by

$$R\Omega^T(\theta) = \tilde{\Theta}^T P \qquad (19)$$

where $\Omega(\theta)$ is a given affine matrix function of θ with the same dimensions of $\tilde{\Theta}$ and having full column-rank. Indeed, it can be readily verified from the proof of Theorem 4.1 that this theorem still holds when the equality constraint of (19) is used *in lieu* of (18) and $\tilde{\Theta}$ in the matrix $H_c(\theta)$ is replaced by $\Omega(\theta)$.

Observe that by setting $\Omega(\theta) = \begin{bmatrix} I_n & 0 \end{bmatrix}^T$, it turns out that the block partitions P_1 and P_2 of the matrix P in (8) must satisfy $P_1 = 0$ and $P_2 = 0$. This implies that the matrix $\mathcal{P}(\theta)$ reduces to $\mathcal{P}(\theta) = P_0$, i.e. it becomes parameter-independent, and thus the method of Theorem 4.1 reduces to a quadratic stability approach for robust $\mathcal{H}_\infty$ control. Other choices of $\Omega(\theta)$ may lead to suitable parameter-dependent Lyapunov functions. The best choice of $\Omega(\theta)$ which renders the constraint of (19) somehow nonconservative is an important issue that we are currently investigating.

It should be noted that when some of the system matrices are parameter-independent, the condition (17) of Theorem 4.1 can be simplified as follows.

(i) When $B_u(\theta) = B_{u_0}$, $C(\theta) = C_0$ and $D_u(\theta) = D_{u_0}$, the inequality of (17) can be replaced by the following:

$$\tilde{\Psi}(\theta,\dot{\theta}) + M\hat{H}_c(\theta) + \hat{H}_c^T(\theta)M^T < 0$$

where

$$\tilde{\Psi}(\theta,\dot{\theta}) = \begin{bmatrix} W(\theta,\dot{\theta}) + \tilde{W}(\theta) & \star & \star \\ C_0 P\tilde{\Theta}^T P + D_{u_0} S\Omega^T(\theta) & -\gamma I & \star \\ \tilde{\mathcal{B}}_w^T & D_w^T(\theta) & -\gamma I \end{bmatrix}$$

$$\tilde{W}(\theta) = \tilde{\mathcal{B}}_u S\Omega^T(\theta) + \Omega(\theta)S^T\tilde{\mathcal{B}}_u^T$$

$$\hat{H}_c(\theta) = \begin{bmatrix} H_x(\theta) & 0 & 0 \end{bmatrix}$$

and $W(\theta,\dot{\theta})$ is as in (13). Note that now $\tilde{\mathcal{B}}_u$ reduces to $\tilde{\mathcal{B}}_u = \begin{bmatrix} B_{u_0}^T & 0 \end{bmatrix}^T$.

(ii) When $C(\theta) = C_0$, the inequality of (17) can be replaced by the following:

$$\begin{bmatrix} W(\theta,\dot{\theta}) & \star & \star & \star \\ S^T\tilde{\mathcal{B}}_u^T & 0 & \star & \star \\ C_0\tilde{\Theta}^T P & D_u(\theta)S & -\gamma I & \star \\ \tilde{\mathcal{B}}_w^T & 0 & D_w^T(\theta) & -\gamma I \end{bmatrix} + M\tilde{H}_c + \tilde{H}_c^T M^T < 0$$

where $W(\theta,\dot{\theta})$ is as in (13) and

$$\tilde{H}_c(\theta) = \begin{bmatrix} H_x(\theta) & 0 & 0 & 0 \\ \Omega^T(\theta) & -I_n & 0 & 0 \end{bmatrix}.$$

5. EXAMPLES

Example 1. This example deals with the problem of robust $\mathcal{H}_\infty$ performance analysis for the unforced system of (4) with

$$A(\theta) = \begin{bmatrix} -1 - 1.3\theta & -0.5 - 20\theta \\ -1 + 2\theta & -2 - 10\theta \end{bmatrix},$$

$$B_w(\theta) = \begin{bmatrix} 1 + 2.2\theta & -4 + 0.5\theta \\ -1 - 6\theta & -1 - 5\theta \end{bmatrix}, \quad C(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where θ is an uncertain real time-varying scalar. We shall consider the problem of finding the smallest γ attenuation level obtainable from Theorem 3.1 for different rates of variation of θ. Using Theorem 3.1, and assuming that $\theta \in [0,1]$ (which ensures quadratic stability of the above system), three types of Lyapunov matrices $\mathcal{P}(\theta)$ will be considered to illustrate the dependence of the minimum γ on both $\mathcal{P}(\theta)$ and the parameter rate of variation. We shall refer to the cases where $\mathcal{P}(\theta)$ depends quadratically and affinely on θ as *bi-quadratic* and *affine-quadratic* approaches, respectively, whereas the *quadratic approach* corresponds to $\mathcal{P}(\theta)$ independent of θ. Note that an affine-quadratic approach can be obtained from Theorem 3.1 by zeroing the block partition P_2 of (8), whereas the quadratic approach corresponds to the case where $P_1 = 0$ and $P_2 = 0$. Fig. 1 displays the minimum bound γ for the system $\mathcal{H}_\infty$ norm in function of $\dot{\theta}$ obtained from Theorem 3.1 for the three cases as above. Clearly we conclude from Fig. 1, that the bi-quadratic approach provides smaller $\mathcal{H}_\infty$ performance bounds.

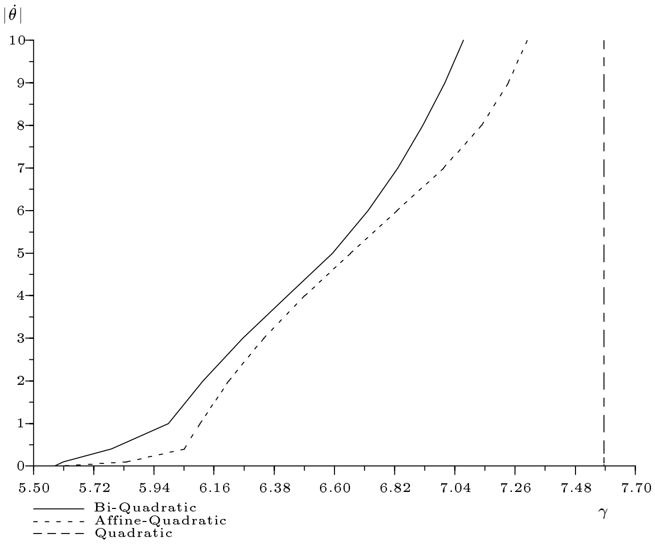

Figure 1. Minimum γ as function of $\dot{\theta}$ for $\theta \in [0,1]$

Example 2. Consider the system of (1) with

$$A(\theta) = \begin{bmatrix} -4.1 - 3\theta & 1 \\ -2\theta & 2 - 3.2\theta \end{bmatrix},$$

$$B_w(\theta) = \begin{bmatrix} -0.03 \\ -0.47 \end{bmatrix}, \quad B_u = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

$$C(\theta) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_u(\theta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_w(\theta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It can be easily verified that the above system is unstable for $\theta < 0.56414$. We shall consider the problem of designing a state-feedback controller which ensures exponential stability and a guaranteed $\mathcal{H}_\infty$ performance for any θ satisfying $|\theta| \leq 4$ and $|\dot{\theta}| \leq 5$. Using the method of Theorem 4.1, the control gain that provides the smallest bound γ for the system $\mathcal{H}_\infty$ norm is $K = \begin{bmatrix} 1.15437 & -34.23263 \end{bmatrix}$ and the corresponding value of γ is 2.65248.

6. CONCLUSIONS

This paper have presented LMI methods for solving the problems of robust $\mathcal{H}_\infty$ performance analysis and control design for linear systems with real time-varying uncertain parameters in the system state-space model. The uncertain parameters enter affinely in the matrices of the state-space model and the admissible values of the parameters and their rates of variation are supposed to belong to a given polytope. The developed methods incorporate information on the bounds on the rates of variation of the parameters and have the feature that the Lyapunov function as well as the upper-bound on the system $\mathcal{H}_\infty$ norm are parameter-dependent. A method for designing a robust $\mathcal{H}_\infty$ controller via static state feedback has been provided.

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