Estimation of the region of attraction by first order approximation

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Abstract

Conditions are given that guarantee the convergence of arbitrary solutions of an autonomous dynamical system towards some equilibrium point of the system. The conditions are formulated in terms of matrix inequalities involving the variational equation. A connection with analytic estimates of the Hausdorff dimension of invariant compact sets is given.

1 Introduction

Stability analysis of closed loop nonlinear systems in control theory is usually based on Lyapunov functions (nonlocal methods) or first order approximation (local methods). Control design which is based on linearization around a desired position is far simpler in general than nonlocal design. However, local methods do not allow to estimate the region of attraction of the target position. With this in mind it would be nice to have a tool to estimate the region of attraction of an equilibrium point of a system of differential equations in terms of the variational equation for the closed loop system.

It can be shown that if all solutions of an autonomous system enter a domain diffeomorphic to a ball and this domain does not contain periodic solutions then any solution converges to an equilibrium point in this domain, provided the equilibria are isolated (recall that if there is a positively invariant domain homeomorphic to a ball then this domain contains at least one equilibrium point). Thus the question of estimation of the region of attraction leads to a question under which conditions a positively invariant domain does not contain a periodic solution. In this paper we investigate this question by a method which allows to estimate the Hausdorff dimension of invariant compact sets [10].

In the paper we use the following notations. The Euclidean norm in $\mathbb{R}^n$ is denoted as $|\cdot|$, $|x|^2 = x^\top x$, where $^\top$ stands for the transpose. For matrices the notation $||P||$ stands for the spectral norm of P, i.e. $||P||^2$ is the largest eigenvalue of the matrix $P^\top P$. Eigenvalues of the matrix $P^\top P$ are called singular values of P. If a quadratic form $x^\top Px$ with a symmetric matrix $P = P^\top$ is positive definite then the matrix P is called positive definite. For positive definite matrices we use the notation $P > 0$. $\mathbb{R}_+$ stands for the set of nonnegative reals. The solution of the system of differential equations $\dot{x} = f(t,x)$, with $x \in \mathbb{R}^n$ starting at t_0 in x_0 and calculated at t will be denoted as $x(t, t_0, x_0)$, i.e. $x(t_0, t_0, x_0) = x_0$. Sometimes, if no confusion arises, we will omit the dependence of some arguments.

2 Hausdorff dimension and Hausdorff measure

Consider a compact subset K of a compact metric space X. Given $d \geq 0$, $\varepsilon > 0$, consider a covering of K by open spheres B_i with radii $r_i \leq \varepsilon$. Denote by

$$\mu(K, d, \varepsilon) = \inf \sum_i r_i^d \tag{1}$$

the d-measured volume of covering of the set K. Here the infimum is calculated over all ε-coverings of K. There exists a limit, which may be infinite,

$$\mu_d(K) := \sup_{\varepsilon > 0} \mu(K, d, \varepsilon).$$

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It can be proved that μ_d is an outer measure on X (see, e.g. Proposition 5.3.1 in [7]).

Definition 1 *The measure μ_d is called the* Hausdorff d-measure.

The properties of the measure μ_d can be summarized as follows. There exists a single value $d = d_*$, such that for all $d < d_*$, $\mu_d(K) = +\infty$ and for all $d > d_*$, $\mu_d(K) = 0$. Here

$$d_* = \inf\{d \ : \ \mu_d(K) = 0\} = \sup\{d \ : \ \mu_d(K) = +\infty\}.$$

Definition 2 *The value d_* is called the* Hausdorff dimension *of the set K.*

In the sequel, we will use the notation $\dim_H K$ for the Hausdorff dimension of the set K.

Now, following Douady and Oesterlé [2], we define the elliptic Hausdorff d-measure of a compact set $K \subset \mathbb{R}^n$. Let E be an open ellipsoid in $\mathbb{R}^n$. Let $a_1(E) \geq a_2(E) \geq \ldots \geq a_n(E)$ be the lengths of semiaxis of E numbered in decreasing order. Represent an arbitrary number d, $0 \leq d \leq n$ in the form $d = d_0 + s$, where $d_0 \in \mathbb{Z}_+$ and $s \in [0, 1)$ and introduce the following

$$\omega_d(E) = \prod_{i=1}^{d_0} a_i(E)(a_{d_0+1}(E))^s. \tag{2}$$

Fix a certain d and $\varepsilon > 0$ and consider all possible finite coverings of the compact K by ellipsoids E_i for which

$$[\omega_d(E_i)]^{1/d} \leq \varepsilon$$

(if $d = 0$ we put $[\omega_d(E_i)]^{1/d} = a_1(E_i)$). Similar to the definition of Hausdorff d-measure we denote

$$\tilde{\mu}_d(K, d, \varepsilon) = \inf \sum_i \omega_d(E_i),$$

where the infimum is calculated over all coverings.

Definition 3 *The value*

$$\tilde{\mu}_d(K) = \sup_{\varepsilon > 0} \tilde{\mu}(K, d, \varepsilon)$$

is called the Hausdorff elliptical d-measure *of the compact set K.*

It was proven in [2, 13] that the elliptical and spherical Hausdorff d-measures are equivalent and therefore, using extremal properties of μ_d, the values of the Hausdorff dimensions determined by means of spherical and elliptic coverings are equal.

Let $\{\varphi^t\}$ be a one-parameter semigroup of diffeomorphisms $\Omega \to \Omega$, $\Omega \subset \mathbb{R}^n$, $t \in \mathbb{I}$. We will only consider the case $\mathbb{I} = \mathbb{R}_+$, the case $\mathbb{I} = \mathbb{Z}_+$ can be treated in the same fashion. A subset $\gamma(x_0)$ of $\mathbb{R}^n$ of the form $\gamma(x_0) = \{x \ : \ \exists t \in \mathbb{R}_+ \ x = \varphi^t(x_0)\}$ is called the *trajectory, or orbit,* of the point x_0.

By $T_x\varphi^t$ we denote the derivative of φ^t with respect to x (Jacobian) at the point $x \in \mathbb{R}^n$, that is, $T_x\varphi^t$ is a linear operator $\mathbb{R}^n \to \mathbb{R}^n$. For a linear operator $L \ : \ \mathbb{R}^n \to \mathbb{R}^n$ denote by $a_1(L) \geq a_2(L) \geq \ldots \geq a_n(L)$ its singular values. For arbitrary $k \in \mathbb{Z}_+$, $0 \leq k \leq n$ we denote

$$\omega_k(L) = \begin{cases} \prod_{i=1}^{k} a_i(L), \ k > 0 \\ 1, \ k = 0 \end{cases}$$

For arbitrary $d \in [0, n]$ we put $d = d_0 + s$, where $d_0 \in \mathbb{Z}_+$ and $s \in [0, 1)$ and introduce the following definition

$$\omega_d(L) = \omega_{d_0}^{1-s}(L)\omega_{d_0+1}^s(L).$$

Consider a compact set $\tilde{K}$ such that $K \subset \tilde{K}$, $\varphi^t(K) \subset \tilde{K}$.

Theorem 1 *Assume that there exists $d \in [0, n]$ such that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $t \geq t_\varepsilon$*

$$\sup_{x \in \tilde{K}} \omega_d(T_x\varphi^t) \leq \varepsilon. \tag{3}$$

Then

$$\mu_d(K) < \infty \Longrightarrow \lim_{t \to \infty} \mu_d(\varphi^t(K)) = 0$$

Additionally, if the compact set K is invariant (i.e. $\varphi^t(K) = K$, $\forall t \in \mathbb{R}_+$) then $\dim_H K \leq d$.

Basically, this theorem is a reformulation of the well known Douady-Oesterlé theorem [2], an analog of this statement for arbitrary Hilbert spaces is proved in [13]. Leonov [4] (see also Theorem 5.4.1 in [7] and Theorem 8.1.2 in [6]) proved a generalization of the Douady-Oesterlé theorem: instead of (3) it is sufficient to require that

$$\sup_{x \in \tilde{K}} \left[\frac{p(\varphi^t(x))}{p(x)} \omega_d(T_x\varphi^t) \right] \leq \varepsilon \tag{4}$$

where $p \ : \ \tilde{K} \to (0, \infty)$ is a scalar positive continuous function. This approach turns out to be useful for estimates of the Hausdorff dimension in terms of auxillary (Lyapunov) functions satisfying certain partial differential inequalities. We will apply this idea in the sequel.

Consider the system

$$\dot{x} = f(x), \tag{5}$$

where $x \in \Omega$ and $f : \Omega \to \Omega$, is a smooth vector-function on $\Omega \subset \mathbb{R}^n$. Let $\varphi^t(x_0) : x_0 \mapsto x(t, x_0)$ and $K \subset \tilde{K} \subset \Omega$.

Along with the system (5) consider the first order approximation

$$\dot{y} = J(x(t, x_0))y, \tag{6}$$

where $y \in \mathbb{R}^n$ and

$$J(x(t, x_0)) = \frac{\partial f}{\partial x}(x(t, x_0)).$$

Consider a matrix function $G : \tilde{K} \to \mathbb{R}^{n \times n}$ which is smooth and invertible in $\tilde{K}$. For any $t \in \mathbb{R}_+$ and $x \in \tilde{K}$, $G(\varphi^t(x))$ defines a linear operator $\mathbb{R}^n \to \mathbb{R}^n$. For any $x \in \tilde{K}$ the singular values of $G(\varphi^t(x))$ are bounded from above and below. Given an arbitrary nonsingular matrix $S(t)$ which is bounded from below and above for all $t \in \mathbb{R}_+$, we have for the singular values $\sigma_i'(t)$ of the matrix $X(t)S(t)$ the following simple estimate $\xi_{\min}\sigma_i(t) \le \sigma_i'(t) \le \xi_{\max}\sigma_i(t)$ where $\xi_{\min}$ and $\xi_{\max}$ are the lower and upper bounds for the singular values of the matrix $S(t)$ and $\sigma_i(t)$ are the singular values of the matrix $X(t)$. Therefore, using Theorem 1, we arrive at the following result.

Theorem 2 *Assume that there exists $d \in [0, n]$ such that*

$$\sup_{x \in \tilde{K}} \omega_d \left[G(\varphi^t(x))T_x\varphi^t\right] \to 0 \quad \text{as} \quad t \to \infty. \tag{7}$$

Then

$$\mu_d(K) < \infty \Longrightarrow \lim_{t \to \infty} \mu_d(\varphi^t(K)) = 0,$$

if $\varphi^t(K) = K, \quad \forall t \in \mathbb{R}_+$ then $\dim_H K \le d$,

Proof: Since

$$\omega_d \left[G(\varphi^t(x))T_x\varphi^t\right] \ge \xi^d \omega_d(T_x\varphi^t),$$

where ξ is the minimal singular value of G on $\tilde{K}$, the result follows from Theorem 1. ∎

Consider some symmetric positive definite matrix $P(x)$ which is continuously differentiable in $\tilde{K}$, and which therefore is bounded from above and below in $\tilde{K}$ and the symmetric matrix $Q(x(t, x_0))$ defined by

$$Q = \dot{P} + PJ + J^\top P.$$

Here $\dot{P}(x(t, x_0)) = \frac{d}{dt}P(x(t, x_0))$ stands for the matrix with entries equal to

$$\left(\frac{\partial p_{ij}(x(t, x_0))}{\partial x} f(x(t, x_0))\right)_{ij}.$$

Consider the equation

$$\det[Q(x) - \lambda(x)P(x)] = 0. \tag{8}$$

For any $x \in \tilde{K}$ the equation (8) has n *real* solutions $\lambda_i(x)$ since the matrix Q is symmetric and P is positive definite. Indeed, (8) can be rewritten as

$$\det[G(x)^\top(G(x)^{-\top}Q(x)G(x)^{-1} - \lambda(x)I_n)G(x)] = 0$$

or, equivalently,

$$\det[G(x)^{-\top}Q(x)G(x)^{-1} - \lambda(x)I_n] = 0,$$

where

$$P(x) = G(x)^\top G(x)$$

and the matrix $G(x)^{-\top}Q(x)G(x)^{-1}$ is symmetric. Order the solutions of (8) in the decreasing order for all x: $\lambda_1(x) \ge \lambda_2(x) \ge \ldots \ge \lambda_n(x)$.

Theorem 3 *Suppose that for some $P(x)$ satisfying the above assumptions there exist numbers $d_0 \in \mathbb{Z}_+$, $s \in [0, 1)$ such that*

$$\limsup_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau [\lambda_1(x(t, x_0)) + \ldots + \lambda_{d_0}(x(t, x_0)) + s\lambda_{d_0+1}(x(t, x_0))]dt < 0 \tag{9}$$

for any $x_0 \in \tilde{K}$. Then $\mu_d(K) < \infty \Longrightarrow \lim_{t\to\infty} \mu_d(\varphi^t(K)) = 0$, additionally, if K is invariant then $\dim_H K \le d_0 + s$

Before we prove this theorem we formulate the following result due to Smith.

Theorem 4 (Smith, [12]) *Let X be a fundamental matrix on $[0, \tau]$ of the linear system*

$$\dot{x} = A(t)x,$$

with singular values $\sigma_i(t), i = 1, \ldots, n$ ordered in decreasing order for all $0 \le t \le \tau$. For all $i, 1 \le i \le n$ the following relations are true: $\sigma_1(\tau)\sigma_2(\tau)\cdots\sigma_i(\tau) \le \exp\frac{1}{2}\left(\int_0^\tau(\lambda_1(t) + \lambda_2(t) + \ldots + \lambda_i(t))dt\right)$, and $\sigma_n(\tau)\sigma_{n-1}(\tau)\cdots\sigma_{n-i+1}(\tau) \ge \exp\frac{1}{2}\left(\int_0^\tau(\lambda_n(t) + \lambda_{n-1}(t) + \ldots + \lambda_{n-i+1}(t))dt\right)$, where $\lambda_i(t)$ are the eigenvalues of the symmetrized matrix $(A(t) + A(t)^\top)$ ordered in decreasing order for all $0 \le t \le \tau$.

Proof of Theorem 3: Since the matrix $P(x) = P(x)^\top$ is positive definite it can be represented in the form $P(x) = G(x)^\top G(x)$ where $G(x)$ is continuously differentiable and bounded together with its inverse in $\tilde{K}$. For the system (6) consider the time-varying coordinate change

$$z = G(x(t, x_0))y. \tag{10}$$

From now for brevity we will omit the dependence of $x(t, x_0)$ and t. In the new coordinates the system (6) can be rewritten as

$$\dot{z} = \dot{G}G^{-1}z + GJG^{-1}z. \tag{11}$$

In view of (10) the fundamental matrix of (11) has the form GH, where H is the fundamental matrix of (6). In order to use Theorem 2 we should estimate the singular values of the matrix GH.

Denote $A = \dot{G}G^{-1} + GJG^{-1}$ and consider the equation

$$\det((A + A^\top) - \lambda I_n) = 0.$$

This equation is equivalent to

$$\det(G^\top GA + A^\top G^\top G + G^\top \dot{G} + \dot{G}^\top G - \lambda G^\top G) = 0,$$

which is the same as (8) since $P = G^\top G$.

Estimates of the singular values $\sigma_1, \sigma_2, \ldots$ of the fundamental matrix for the system (11) can be obtained using Smith's theorem formulated above.

From the equality

$$\begin{aligned} &\sigma_1(\tau)\sigma_2(\tau)\cdots\sigma_{d_0+1}^s(\tau) = \\ &(\sigma_1(\tau)\cdots\sigma_{d_0}(\tau))^{1-s}\,(\sigma_1(\tau)\cdots\sigma_{d_0+1}(\tau))^s \end{aligned} \tag{12}$$

for the singular values of GH it follows from Theorem 4 that

$$\begin{aligned} &\sigma_1(\tau)\cdots\sigma_{d_0}(\tau)\sigma_{d_0+1}^s(\tau) \\ \leq\;& \exp\frac{1}{2}\left(\int_0^\tau(\lambda_1(t) + \ldots + \lambda_{d_0}(t) + s\lambda_{d_0+1}(t))dt\right) \end{aligned}$$

where λ_i are the solutions of (8) and hence for all $\tau > 0$ and arbitrary positive ε there is $N = N(\varepsilon)$ such that

$$\sup_{x_0\in K}\sigma_1(\tau)\sigma_2(\tau)\cdots\sigma_{d_0}(\tau)\sigma_{d_0+1}^s(\tau) \leq Ne^{(-\alpha/2+\varepsilon)\tau}$$

where

$$-\alpha = \sup_{x_0\in K}\limsup_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau(\lambda_1(t) + \ldots + s\lambda_{d_0+1}(t))dt$$

and therefore the conditions of Theorem 2 are satisfied. ■

Corollary 1 (Leonov [4, 7]) *Let $\lambda_i(x)$, $i = 1, \ldots, n$ be the eigenvalues of the matrix $(J(x) + J(x)^\top)/2$ ordered in decreasing order. Suppose there exist numbers $d_0 \in \mathbb{Z}_+$, $s \in [0, 1)$, and a continuously differentiable in $\tilde{K}$ function $v : \tilde{K} \to \mathbb{R}$ such that*

$$\lambda_1(x) + \ldots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x) + \frac{\partial v}{\partial x}f(x) < 0 \tag{13}$$

for any $x_0 \in \tilde{K}$. Then

$$\mu_d(K) < \infty \Longrightarrow \lim_{t\to\infty}\mu_d(\varphi^t(K)) = 0.$$

Additionally, if K is invariant then $\dim_H K \leq d_0 + s$.

Proof: The result directly follows from Theorem 3 if one takes $P(x) = p^2(x)I_n$ where $p(x) > 0$ is a scalar differentiable function bounded from below and above in $\tilde{K}$ and denote $v(x) = (\log p(x))/(d_0 + s)$. In this case (8) is equivalent to the equation

$$\det[J(x)^\top + J(x) + \frac{2\dot{p}}{p}I_n - \lambda I_n] = 0$$

Since

$$2\frac{\dot{p}}{p(d_0 + s)} = 2\frac{dv}{dt} = 2\frac{\partial v}{\partial x}f(x)$$

the result follows from Theorem 3. ■

3 Estimation of the region of attraction by means of first order approximation

The results obtained in the previous section allow to find conditions based on the first order approximation ensuring convergence of any solution to an equilibrium point.

Consider again system (5). We assume that in Ω there exists a bounded open simply connected positively invariant set D, the boundary ∂D of which transversally intersects any trajectory originating in ∂D. This assumption means that positive invariance of D is preserved under small perturbations of the vector field f. Moreover assume that the set D is diffeomorphic to an open ball B^n. The existence of such a set can be established by the direct Lyapunov method. Suppose finally that D contains a finite number of equilibria.

Let $P(x)$ be a continuously differentiable positive definite matrix function defined in cl$D$. As before, let λ_i be the roots of the equation (8) ordered in the decreasing order for all $x \in$ cl$D$.

The following theorem can be treated as a generalization of theorems due to Hartman-Olech [3] and Leonov [5].

Theorem 5 *Suppose that for some $P(x)$ satisfying the above assumptions*

$$\lambda_1(x) + \lambda_2(x) < 0 \tag{14}$$

for any $x \in$ cl$D$. Then any solution originating in D tends to some equilibrium point.

In the proof of Theorem 5 we need the well-known closing lemma due to Pugh [11]. Let us formulate this lemma in a form convenient for us.

Lemma 1 (Closing lemma, [9]) *Let $\tilde{x}$ be an ω-limit point of a bounded solution $x(t, x_0)$ of (5), $t \geq 0$, lying* *in D, that is not an equilibrium. For any $\delta > 0$ there is a vector field $g \in \mathcal{C}^1$ such that*

$$\max_{\mathrm{cl}D} |f(x) - g(x)| + \max_{\mathrm{cl}D} \left\| \frac{\partial f}{\partial x}(x) - \frac{\partial g}{\partial x}(x) \right\| < \delta$$

and a closed trajectory γ of the system $\dot{x} = g(x)$ passes through $\tilde{x}$.

The proof of Theorem 5 follows the same lines as the proof of the Leonov theorem (Theorem 8.3.1 in [6]), the difference is that the condition (14) is weaker than that imposed in [5, 6] and we added the assumption that D is diffeomorphic to B^n not explicitely stated in [5, 6].

Proof of Theorem 5: Let $x_0 \in \mathrm{cl}D$. Since D is bounded and positively invariant, the ω-limit set Ω_{x_0} of $x(t, x_0)$ is nonempty. Let $\tilde{x} \in \Omega_{x_0}$. If a closed trajectory $\gamma \subset \mathrm{cl}D$ passes through a point $\tilde{x}$ then γ is an invariant set. We put on γ some smooth two-dimensional surface $K \subset \mathbb{R}^n$ having finite area. The existence of such a surface for a smooth curve is shown, for example, in [1]. Moreover, since D is diffeomorphic to a ball we can suppose that $K \subset \mathrm{cl}D$ (any closed smooth curve in $\mathrm{cl}D$ is diffeomorphic to some closed smooth curve in $\mathrm{cl}B^n$). As before, we denote by φ^t the shift operator along trajectories of system (5). Let $\mu(S)$ be the Hausdorff 2-measure of a smooth 2-dimensional surface S. Since γ is invariant under φ^t and $K \subset \mathrm{cl}D$ for any $t \geq 0$ we have

$$\inf_{t \geq 0} \mu\left(\varphi^t(K)\right) > 0. \tag{15}$$

At the same time, using (14) from Theorem 3, it follows that

$$\lim_{t \to \infty} \mu\left(\varphi^t(K)\right) = 0, \tag{16}$$

which contradicts (15).

When $\tilde{x} \subset \Omega_{x_0}$ is not an equilibrium point and the trajectory passing through this point is not closed, we use Pugh's lemma. We slightly perturb f such that the conditions of the Pugh lemma are satisfied and the theorem hypothesis holds for the perturbed f with the same $P(x)$. Then there is a closed trajectory of the perturbed system passing through $\tilde{x}$. Using the above arguments we again arrive at a contradiction.

Thus $\tilde{x}$ is an equilibrium. By assumption, the equilibria are isolated in D and the result follows. ∎

Remark 1 If we take $P(x) = \alpha I_n, \alpha > 0$, or, equivalently, $v(x) = \text{const}$, condition (14) is equivalent to the condition imposed in the Hartman-Olech theorem [3].

4 Conclusion

In the paper we presented sufficient conditions guaranteeing that any solution entering a set diffeomorphic to a ball tends to an equilibrium point. The conditions are based on the properties of some matrix pencil associated with the first order approximation system. An interesting connection of the approach used in this paper with the problem of estimation of the Hausdorff dimension of invariant compact sets is emphasized.

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