Robustness of Stability under Delay Perturbations in Linear Time Delay Systems

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Abstract: We consider the robustness of stability under delay perturbations in a class of linear delay systems and obtain an equation dependent norm bound on allowable perturbations. We apply this bound to get estimates on sampling intervals in the stabilization of hybrid systems with feedback delays.
Key words: delay equations, robust stability, delay perturbations

1. Introduction

In this paper we obtain system dependent norm bounds on delay perturbations which guarantee the preservation of asymptotic stability of the unperturbed system. The technique uses the fact that the asymptotic stability of the unperturbed system implies that the components of its fundamental solution go to zero at infinity, which in turn imply that it is possible to get accurate numerical estimates of these components using finite time intervals, and consequently derive norm bounds onthe allowable perturbations in a relatively straightforward fashion.

In Section 2 we derive a condition for the preservation of stability for a large class of equations assuming: i) smallness of perturbations only for sufficiently large times, (hence we allow perturbation which are not "small" initially), ii) the knowledge of the integral of the absolute value of the fundamental solution over $[0, \infty)$. Note that for asymptotically stable systems, it is relatively easy to obtain good estimates for the above integral, and therefore our condition may provide a useful tool for applications. Furthermore, in the special case, when the fundamental solution is positive, the condition for preservation of stability can be formulated in terms of the coefficient matrices of the given system. In Section 3 we consider numerical examples. Example 3.2 demonstrates how our results can be used to obtain an estimate on the maximum allowable sampling interval while preserving stability of a hybrid system with feedback delay. (Note that this problem was studied in [5] in the case when the plant is described by an ordinary differential equation.)

Investigating stability properties of perturbed delay equations, uncertain delay equations, or robust stability of delay equations is a reasonably active research area. Without claiming completeness we refer the reader to [1], [4], [6]–[7], [11]–[14], [15] and the references therein for related articles on these topics.

2. Preservation of Stability

Consider the delay differential equation

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - r_i - \eta_i(t)), \qquad t \geq 0 \tag{2.1}$$

with initial condition

$$x(t) = \varphi(t), \qquad -r \leq t \leq 0, \tag{2.2}$$

where A_i $(i = 0, \ldots, m)$ denote constant $n \times n$ matrices, $0 = r_0 \leq r_1 \leq \ldots \leq r_m$, $\varphi : [-r, 0] \to \mathbb{R}^n$ is a continuous function, and we shall assume that the piecewise continuous delay perturbations, $\eta_i(\cdot)$ $(i = 0, \ldots, m)$, satisfy

$$t - r \leq t - r_i - \eta_i(t) \leq t \qquad \text{for} \quad t \geq 0 \quad (i = 0, \ldots, m). \tag{2.3}$$

The solution of initial value problem (2.1)-(2.2) is an absolutely continuous function, which satisfies (2.2) for all $t \in [-r, 0]$, and satisfies (2.1) a.e. $t \geq 0$. Under our assumptions initial value problem (2.1)-(2.2) is a delay differential equation and has a unique solution, which is continuously differentiable at the points where $r_i(t)$ $(i = 0, \ldots, m)$ are continuous.

We consider the corresponding unperturbed system with constant delays, i.e.,

$$\dot{y}(t) = \sum_{i=0}^{m} A_i y(t - r_i), \qquad t \geq 0, \tag{2.4}$$

and we assume that

(H) the trivial solution of (2.4) is asymptotically stable.

The fundamental solution of (2.4), $V(t)$, is defined as the solution of the following system

$$\dot{V}(t) = \sum_{i=0}^{m} A_i V(t - r_i), \qquad t \geq T \tag{2.5}$$

and

$$V(t) = \begin{cases} I, & t = T, \\ 0, & t < T, \end{cases} \tag{2.6}$$

where $I, 0 \in \mathbb{R}^{n\times n}$ are the identity and the zero matrix, respectively.

Remark 2.1 *To emphasize the dependence of $V(\cdot)$ on T we use the notation $V(t;T)$. Note that $V(t;T) = V(t-T;0)$ for $t \geq T$ because* (2.4) *is autonomous (see e.g. [10]), hence*

$$\int_0^\infty V(t;T)\,dt = \int_0^\infty V(t;0)\,dt.$$

We can rewrite (2.1) in the form

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t-r_i) + f(t), \tag{2.7}$$

where

$$f(t) \equiv \sum_{i=0}^{m} A_i \Big(x(t-r_i-\eta_i(t)) - x(t-r_i) \Big). \tag{2.8}$$

In this setting (2.4) can be considered as the homogeneous equation corresponding to (2.7). The variation-of-constants formula (see e.g. [10]) gives the following expression for the solution of initial value problem (2.1)-(2.2):

$$x(t) = y(t) + \int_T^t V(t-s)f(s)\,ds, \qquad t \geq T, \tag{2.9}$$

where $T > 0$, and y is the solution of (2.4) with initial function $y(t) = x(t)$ for $T - r \leq t \leq T$ and $V(\cdot) = V(\cdot;T)$ is the fundamental solution of (2.4).

For future convenience, we introduce the $\tilde{}$ operation on vectors and on matrices, which means taking the absolute value of the vector or matrix componentwise, i.e., if $x = (x_1, x_2, \ldots, x_n)^T$, then by definition $\tilde{x} \equiv (|x_1|, |x_2|, \ldots, |x_n|)^T$, and similarly if $A = (a_{ij})_{n\times n}$, then $\tilde{A} \equiv (|a_{ij}|)_{n\times n}$. The relation $\leq$ between vectors means a componentwise comparison, i.e., $(x_1, x_2, \ldots, x_n)^T \leq (y_1, y_2, \ldots, y_n)^T$ if for all the components $x_i \leq y_i$.

Remark 2.2 *Hypothesis (H4) implies (see e.g. [10]) that the trivial solution of* (2.1) *is exponentially stable, and there exist constants $K > 0$ and $\alpha > 0$, such that $\|V(t)\| \leq Ke^{-\alpha t}$ for $t \geq 0$, (where $\|\cdot\|$ is the matrix norm induced by the vector norm $\|(x_1, x_2, \ldots, x_n)\| \equiv max\{|x_1|, |x_2|, \ldots, |x_n|\}$), and then every element of the matrix*

$$\int_0^\infty \tilde{V}(s)\,ds$$

is finite.

The next theorem shows that if the perturbations of the delays in (2.1) are small enough for large t, then the equation remains asymptotically stable.

Theorem 2.3 *Assume (H) and that the matrix*

$$M \equiv \int_0^\infty \tilde{V}(s)\,ds \left(\sum_{i=0}^{m} \overline{\lim_{t\to\infty}} |\eta_i(t)| \cdot \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \tag{2.10}$$

has spectral radius less than 1, *i.e.,* $\rho(M) < 1$. *Then the trivial solution of* (2.1) *is asymptotically stable.*

Proof: We prove the theorem in three steps. First we give an estimate of $\tilde{f}(t)$ for large t. Next we show that $\tilde{x}(t)$ is bounded, i.e., $\overline{\lim}_{t\to\infty} \tilde{x}(t)$ is finite, and then we show that $\overline{\lim}_{t\to\infty} \tilde{x}(t) = 0$, which proves the theorem.

(i) We will need an estimate of $f(t)$ for large t. Fix a constant $T > r$, then (2.3) implies that

$$t - r_i - \eta_i(t) \geq 0 \qquad \text{for} \qquad t > T, \quad i = 0, \ldots, m. \tag{2.11}$$

It is easy to see that for $t > r$ the solution of (2.1) is piecewise continuously differentiable and we can write

$$f(t) = \sum_{i=0}^{m} A_i \int_{t-r_i}^{t-r_i-\eta_i(t)} \dot{x}(s)\,ds.$$

Using (2.1) we get

$$f(t) = \sum_{i=0}^{m} A_i \int_{t-r_i}^{t-r_i-\eta_i(t)} \sum_{j=0}^{m} A_j x(s - r_j - \eta_j(s))\,ds. \tag{2.12}$$

This relation and the definition of the $\tilde{}$ operation imply the inequality

$$\tilde{f}(t) \leq \sum_{i=0}^{m} \tilde{A}_i \left| \int_{t-r_i}^{t-r_i-\eta_i(t)} \sum_{j=0}^{m} \tilde{A}_j \tilde{x}(s - r_j - \eta_j(s))\,ds \right|. \tag{2.13}$$

Introduce the simplifying notation

$$\max_{0\leq s\leq t} \tilde{x}(s) \equiv \left(\max_{0\leq s\leq t} |x_1(s)|, \ldots, \max_{0\leq s\leq t} |x_n(s)| \right)^T.$$

In addition to (2.11), we choose T large enough that all the arguments of $\tilde{x}(\cdot)$ in the integrals in (2.13) are positive. Then we can estimate all $\tilde{x}(\cdot)$ by $\max_{0\leq s\leq t} \tilde{x}(s)$, therefore we obtain from (2.13)

$$\tilde{f}(t) \leq \left(\sum_{i=0}^{m} |\eta_i(t)| \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \max_{0\leq s\leq t} \tilde{x}(s), \qquad t \geq T. \tag{2.14}$$

Define the matrix

$$M_0 \equiv \int_0^\infty \tilde{V}(s)\,ds \left(\sum_{i=0}^{m} \tilde{A}_i \right)^2. \tag{2.15}$$

(We note that according to Remark 2.1, the matrices M and M_0 are independent of the choice of T.) It is easy to see that $\rho(M) < 1$ implies that there exists $\delta > 0$ such that

$$\rho(M + \delta M_0) < 1. \qquad (2.16)$$

With this δ we can choose T such that (2.14) holds and furthermore, we have the following relations

$$|\eta_i(t)| < \overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta, \qquad t \geq T, \quad i = 0, \ldots, m. \qquad (2.17)$$

Then (2.14) yields the following estimate for $t \geq T$

$$\tilde{f}(t) \leq \left(\sum_{i=0}^{m} (\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right) \max_{0\leq s\leq t} \tilde{x}(s). \qquad (2.18)$$

(ii) Next we prove that the solution of (2.1) is bounded for all initial function. Choose $T > 0$ such that (2.18) holds. For such T, formula (2.9) and standard estimates yield the inequality

$$\tilde{x}(t) \leq \tilde{y}(t) + \int_T^t \tilde{V}(t-s)\, \tilde{f}(s)\, ds, \qquad t \geq T. \qquad (2.19)$$

Combining (2.18) and (2.19), a change of variables, and the definition of M and M_0 yield

$$\tilde{x}(t) \leq \max_{0\leq u\leq t} \tilde{y}(u) + (M + \delta M_0) \max_{0\leq u\leq t} \tilde{x}(u). \qquad (2.20)$$

The right hand side of inequality (2.20) is monotone in t, therefore (2.20) yields that

$$\max_{0\leq u\leq t} \tilde{x}(u) \leq \max_{0\leq u\leq t} \tilde{y}(u) + (M + \delta M_0) \max_{0\leq u\leq t} \tilde{x}(u). \qquad (2.21)$$

Rearranging (2.21) and using that $y(t)$ is bounded by hypothesis (H4), we have that there exists a constant vector $z \geq 0$ such that

$$(I - (M + \delta M_0)) \max_{0\leq u\leq t} \tilde{x}(u) \leq \max_{0\leq u\leq t} \tilde{y}(u) \leq z, \qquad t \geq T. \qquad (2.22)$$

Inequality (2.16) and the fact that $M + \delta M_0$ has nonnegative components imply that $I - (M + \delta M_0)$ is a nonsingular M-matrix, therefore an application of Theorem 6.2.3 in [3] yields that $I - (M + \delta M_0)$ is a monotone matrix, hence

$$\max_{0\leq u\leq t} \tilde{x}(u) \leq (I - (M + \delta M_0))^{-1} z, \qquad t \geq T,$$

i.e., $x(t)$ is bounded for $t \geq 0$.

(iii) Next we show that $x(t)$ tends to 0 as $t \to \infty$, i.e., $\overline{\lim_{t\to\infty}} \tilde{x}(t) = 0$. Inequality (2.19) yields

$$\overline{\lim_{t\to\infty}} \tilde{x}(t) \leq \overline{\lim_{t\to\infty}} \tilde{y}(t) + \overline{\lim_{t\to\infty}} \int_T^t \tilde{V}(t-s)\, \tilde{f}(s)\, ds.$$

By (H4) we have $\overline{\lim_{t\to\infty}} \tilde{y}(t) = 0$, hence

$$\overline{\lim_{t\to\infty}} \tilde{x}(t) \leq \overline{\lim_{t\to\infty}} \int_T^t \tilde{V}(t-s)\, \tilde{f}(s)\, ds. \qquad (2.23)$$

For any $\delta > 0$ we can choose T such that (2.17) is satisfied and moreover, in (2.13) all arguments of $\tilde{x}(\cdot)$ in the integrals are large enough, i.e., we can estimate $\tilde{x}(\cdot)$ by $\overline{\lim_{t\to\infty}} \tilde{x}(t) + \delta \underline{1}$, where $\underline{1} = (1, 1, \ldots, 1)^T$, and consequently, for $t \geq T$, relation (2.13) implies the inequality
$\tilde{f}(t) \leq \left(\sum_{i=0}^{m} (\overline{\lim_{u\to\infty}} |\eta_i(u)| + \delta) \tilde{A}_i \right) \left(\sum_{i=0}^{m} \tilde{A}_i \right)$
$(\overline{\lim_{u\to\infty}} \tilde{x}(u) + \delta \underline{1})$.
Combining (2.23) and this last inequality we have

$$\overline{\lim_{t\to\infty}} \tilde{x}(t) \leq M \overline{\lim_{t\to\infty}} \tilde{x}(t). \qquad (2.24)$$

Hence

$$(I - M) \overline{\lim_{t\to\infty}} \tilde{x}(t) \leq 0. \qquad (2.25)$$

By assumption $\rho(M) < 1$, M has nonnegative components, and therefore $I - M$ is a nonsingular M-matrix, therefore by Theorem 6.2.3 in [3] $I - M$ is monotone, hence (2.25) yields that $\overline{\lim_{t\to\infty}} \tilde{x}(t) \leq 0$. On the other hand $\overline{\lim_{t\to\infty}} \tilde{x}(t) \geq 0$, therefore $\lim_{t\to\infty} \tilde{x}(t) = 0$.
The proof of the theorem is complete.

The following corollary is an easy consequence of the theorem.

Corollary 2.4 *Let M_0 defined by* (2.15). *If*

$$\overline{\lim_{t\to\infty}} |\eta_i(t)| < \frac{1}{\rho(M_0)}, \qquad i = 0, \ldots, m,$$

then the trivial solution of (2.1) *is asymptotically stable.*

If the fundamental solution $V(t)$ of (2.4) is nonnegative, (i.e., each component $v_{ij}(t)$ of $V(t)$ is nonnegative and therefore $V(t) = \tilde{V}(t)$), then it is easy to compute the integral in (2.15). In particular, we have the following result.

Proposition 2.5 *If the trivial solution of* (2.4) *is asymptotically stable, then the fundamental solution of* (2.4) *satisfies*

$$\left(\sum_{i=0}^{m} A_i \right) \int_0^\infty V(s)\, ds = -I,$$

where I is the identity matrix.

Proof: Let $V(t)$ be the fundamental solution of (2.4) corresponding to $T = 0$. By integrating (2.5) from 0 to $t > 0$ we get

$$V(t) - V(0) = \sum_{i=0}^{m} A_i \int_0^t V(s - r_i)\, ds.$$

A change of variables in the integrals and the assumed initial condition $V(t) = 0$ for $t < 0$ yield

$$\begin{aligned} V(t) - V(0) &= \sum_{i=0}^{m} A_i \int_{-r_i}^{t-r_i} V(s)\, ds \\ &= \sum_{i=0}^{m} A_i \int_{0}^{t-r_i} V(s)\, ds. \end{aligned}$$

Using $V(0) = I$ and the fact $V(t) \to 0$ as $t \to \infty$ we obtain the equality

$$-I = \left(\sum_{i=0}^{m} A_i\right) \int_0^\infty V(s)\,ds,$$

which proves the proposition.

Remark 2.6 *In the case when $V(t)$ is nonnegative, and $\sum_{i=0}^m A_i$ is nonsingular, Proposition 2.5 implies that*

$$M_0 = -\left(\sum_{i=0}^{m} A_i\right)^{-1} \left(\sum_{i=0}^{m} \tilde{A}_i\right)^2, \qquad (2.26)$$

therefore our stability condition in Corollary 2.4 is given in terms of the coefficient matrices.

To conclude the section in the next Proposition we give a sufficient condition for positivity of the fundamental solution of (2.4). We shall need the following notations. Let $A_i = \left[a_{jk}^{(i)}\right]$, $V(t) = \left[v_{jk}(t)\right]$, $\alpha_{jj}^{(i)} \equiv \max\left\{-a_{jj}^{(i)}, 0\right\}$, $\beta_{jj}^{(i)} \equiv \max\left\{a_{jj}^{(i)}, 0\right\}$. Then we can rewrite initial value problem (2.5)-(2.6) in terms of the components for $t \geq 0$
$\dot{v}_{jk}(t) = -\sum_{i=0}^m \alpha_{jj}^{(i)} v_{jk}(t-r_i)$
$+\sum_{i=1}^m \sum_{\substack{l=1\\ l\neq j}}^n a_{jl}^{(i)} v_{lk}(t-r_i) + \sum_{i=1}^m \beta_{jj}^{(i)} v_{jk}(t-r_i)$,
and
$v_{jk}(t) = \begin{cases} \delta_{jk}, & t = 0,\\ 0, & t < 0,\end{cases}$
(where δ_{jk} is the Kronecker-delta), $j,k = 1,2,\ldots,n$. Consider the following two initial value problems associated to the negative parts of the components in the main diagonals of A_i:

$$\begin{aligned} \dot{w}_{jk}(t) &= -\sum_{i=0}^{m} \alpha_{jj}^{(i)} w_{jk}(t-r_i), t \geq 0, && (2.27)\\ w_{jk}(t) &= \begin{cases} \delta_{jk}, & t = 0,\\ 0, & t < 0,\end{cases} && (2.28)\end{aligned}$$

and

$$\begin{aligned} \dot{u}_{j}(t) &= -\sum_{i=0}^{m} \alpha_{jj}^{(i)} u_{j}(t-r_i), \qquad t \geq 0, && (2.29)\\ u_{j}(t) &= \begin{cases} 1, & t = 0,\\ 0, & t < 0,\end{cases} && (2.30)\end{aligned}$$

$j,k = 1,2,\ldots,n$. Clearly, we have that for all $t \geq 0$

$$w_{jk}(t) = \begin{cases} 0, & j \neq k,\\ u_j(t), & j = k.\end{cases} \qquad (2.31)$$

Proposition 2.7 *Assume that*
(i) $a_{jk}^{(i)} \geq 0$ for all $j,k = 1,2,\ldots,n$, $j \neq k$.
(ii) $\sum_{i=0}^m \alpha_{jj}^{(i)} r_i \leq \frac{1}{e}$ for all $j = 1,2,\ldots,n$.
Then $v_{jk}(t) \geq 0$ for all $t \geq 0$ and $j,k = 1,2,\ldots,n$.

Proof: Let $w_{jk}(t)$ and $u_j(t)$ be the solutions of initial value problems (2.27)-(2.28) and (2.29)-(2.30), respectively, $(j,k = 1,2,\ldots,n)$. By Theorem 3.31 in [9] it follows that $u_j(t) \geq 0$ for all $j = 1,2,\ldots,n$. (The above theorem applies for solutions corresponding to continuous initial functions. To use that result for IVP (2.29)-(2.30) we approximate the initial function in (2.30) by appropriate continuous initial functions, $u_j^l(t)$, $t \leq 0$, $l = 1,2,\ldots$, and by arguing that the corresponding solutions $u_j^l(t)$, $t \geq 0$ approximate $u_j(t)$ uniformly on compact time intervals we get that the limit $u_j(t) = \lim_{l\to\infty} u_j^l(t)$, is also nonnegative.) Nonnegativeness of $u_j(t)$ and relation (2.31) yield that $w_{jk}(t) \geq 0$ for $t \geq 0$, $j,k = 1,2,\ldots,n$ as well. The variation-of-constant formula implies the relation

$$v_{jk}(t) = w_{jk}(t) + \sum_{i=1}^{m} \sum_{\substack{l=1\\ l\neq j}}^{n} a_{jl}^{(i)} \int_0^t u_j(t-s) v_{lk}(s-r_i)\,ds$$
$$+\sum_{i=0}^{m} \beta_{jj}^{(i)} \int_0^t u_j(t-s) v_{jk}(s-r_i)\,ds.$$

Using the nonnegativeness of $w_{jk}(t)$, $u_j(t)$, $\beta_{jj}^{(i)}$, $a_{jl}^{(i)}$ $(l \neq j)$, and the initial condition on $v_{jk}(t)$ it is easy to see the nonnegativeness of $v_{jk}(t)$.

Note that in the ODE case, i.e., when $m = 0$, $r_0 = 0$, condition (ii) of the previous proposition is satisfied automatically, and then condition (i) is also necessary for the positivity of $v_{jk}(t)$. (See Theorem 3 in Chapter 10 of [2].)

3. Examples

Consider the scalar version of (2.1).

$$\dot{x}(t) = \sum_{i=0}^{m} a_i x(t - r_i - \eta_i(t)), \qquad t \geq 0 \qquad (3.1)$$

with initial condition

$$x(t) = \varphi(t), \qquad -r \leq t \leq 0, \qquad (3.2)$$

where $\varphi : [-r, 0] \to \mathbb{R}$ is a continuous function. The corresponding equation with unperturbed delays is

$$\dot{y}(t) = \sum_{i=0}^{m} a_i y(t - r_i), \qquad t \geq 0. \qquad (3.3)$$

Let $v(t)$ be the fundamental solution of (3.3), i.e.

$$\begin{aligned} \dot{v}(t) &= \sum_{i=0}^{m} a_i v(t-r_i), \qquad t \geq 0 && (3.4)\\ v(t) &= \begin{cases} 1, & t = 0,\\ 0, & t < 0.\end{cases} && (3.5)\end{aligned}$$

The scalar version of Theorem 2.3 is the following.

Theorem 3.1 *Assume that the trivial solution of* (3.3) *is asymptotically stable and the functions* $\eta_i(\cdot)$ *(*$i = 0, \dots, m$*) satisfy*

$$\sum_{i=0}^{m} |a_i| \overline{\lim_{t\to\infty}} |\eta_i(t)| < \frac{1}{\left(\sum_{i=0}^m |a_i|\right) \int_0^\infty |v(t)|\, dt}. \qquad (3.6)$$

Then the trivial solution of (3.1) *is asymptotically stable.*

Note that if the fundamental solution is nonnegative, then Remark 2.6 yields that condition (3.6) is equivalent to

$$\sum_{i=0}^{m} |a_i| \overline{\lim_{t\to\infty}} |\eta_i(t)| < \frac{-\sum_{i=0}^m a_i}{\sum_{i=0}^m |a_i|}. \qquad (3.7)$$

In the general case we would need an upper estimate of $\int_0^\infty |v(t)|\, dt$ to get an easily verifyable condition on the allowable perturbation. Such an estimate at this time is known (see [8]) only for the single-delay equation of the form

$$\dot{x}(t) = -bx(t - \tau), \qquad (3.8)$$

where $b > 0$ and $b\tau < \pi/2$ (hence the trivial solution is asymptotically stable). For this equation it can be shown (see [8]) that there exists a unique characteristic root $\lambda_0 = \alpha_0 + \beta_0 i$ of equation (3.8) , i.e., a solution of $\lambda = -be^{-\lambda\tau}$, satisfying $\beta_0 \in \left[0, \frac{\pi}{2\tau}\right)$. Then the fundamental solution of (3.8) satisfies

$$\int_0^\infty |v(t)|\, dt \le \frac{1}{b} \frac{\alpha_0^2 + \beta_0^2}{\alpha_0^2}. \qquad (3.9)$$

In the general case, the practical importance of our result can be argued as follows:
i) it is easy to obtain numerical approximation of the fundamental solution, ii) using the fact that the fundamental solution exponentially converges to 0 if the trivial solution is asymptotically stable, it is easy to obtain good numerical approximation of the integral $\int_0^\infty |v(t)|\, dt$, and iii) using the numerical value of the integral and condition (3.6) get approximate bounds for the allowable perturbations.

The following examples show applications of this method.

Example 3.2 Consider the scalar equation

$$\dot{x}(t) = -0.1x(t) + 2x(t-1) - 2x\left(\left[\frac{t-1.3}{h}\right] h\right), \qquad (3.10)$$

where $[\cdot]$ denotes the greatest integer function and $h > 0$ is the sampling period. The piecewise constant delay in the last term can be considered as a perturbation of $t - 1.3$ with

$$\eta(t) = t - 1.3 - \left[\frac{t-1.3}{h}\right] h.$$

Then we have that $|\eta(t)| \le h$ for all $t \ge 0$. The corresponding unperturbed delay equation is

$$\dot{x}(t) = -0.1x(t) + 2x(t-1) - 2x(t-1.3). \qquad (3.11)$$

We show a numerical approximation of the fundamental solution of Equation (3.11) on Figure 1. The picture indicates that the fundamental solution exponentially tends to zero, i.e., the trivial solution of (3.11) is asymptotically stable. Numerical approximation gives that $\int_0^\infty |v(t)|\, ds = 10.5914$. Therefore using Theorem 3.1 if $h < \frac{1}{10.5914 \cdot 8.2} = 0.0115$, then the trivial solution of (3.10) is asymptotically stable.

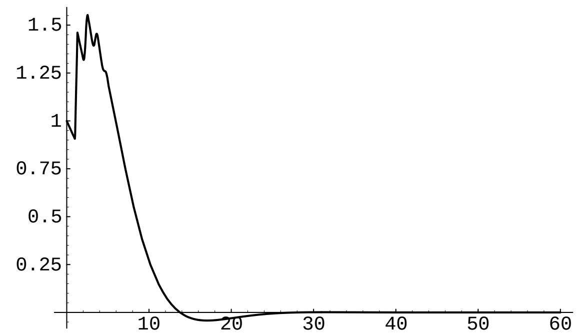

Figure 1

Example 3.3 Consider the following system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - 1 - \eta_1(t)) + A_2 x(t - 1.5 - \eta_2(t)), \qquad (3.12)$$

where $x(t) \in \mathbb{R}^2$,

$$A_0 = \begin{pmatrix} -0.1 & 0.3 \\ -0.5 & 0.0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.7 & -0.4 \\ 0.5 & -0.8 \end{pmatrix} \quad \text{and}$$

$$A_2 = \begin{pmatrix} -1.0 & 0.1 \\ 0.1 & 0.4 \end{pmatrix}.$$

The corresponding unperturbed equation is

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_2 x(t-1.5). \qquad (3.13)$$

On Figure 2 we display the components of the numerical solutions of the fundamental matrix solution. This picture indicates that every component function tends to zero exponentially as $t \to \infty$, therefore the trivial solution of (3.13) is asymptotically stable. Numerical approximation of the components of $\int_0^\infty \tilde{V}(t)\, dt$ gives the following numerical values for the matrix M_0

$$M_0 = \begin{pmatrix} 18.699 & 10.800 \\ 16.441 & 10.641 \end{pmatrix},$$

therefore $\rho(M_0) = 28.591$. Corollary 2.4 implies that if the perturbations of the delays satisfy $\overline{\lim_{t\to\infty}} |\eta_i(t)| < 0.035$ $(i = 1, 2)$, then the trivial solution of (3.12) is asymptotically stable.

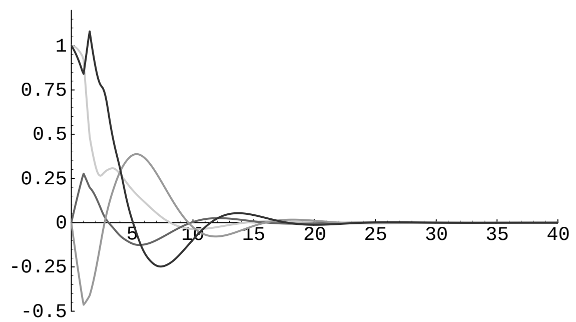

Figure 2

4. References

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