On the Use of Switching Control for Systems with Bounded Disturbances

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Abstract

It has been known for some time that there are limits to the achievable performance in a linear time invariant feedback control system. Some of these limitations, however, may be ameliorated by the use of nonlinear and/or time varying feedback. Recently, there has been some interest in the use of switching control to improve the control performance for linear integrating plants with bounded input disturbances. In this paper we study asymptotic and BIBS stability properties of a class of switching feedback systems, and also examine one form of optimal control for such systems.

1 Introduction

It is well established that for a linear time invariant plant (LTI) with quadratic cost the optimal control law is linear [1]. It is also known that when linear control is used on linear plants there are fundamental limits on the achievable performance. These limits may be described in several ways, including frequency domain integral constraints (e.g. [2]), time domain integral constraints (e.g. [3]), singular quadratic optimal control (e.g. [4]), and limits on achievable H_∞ performance (e.g. [5]). For example, it follows from the time domain analysis that if a LTI plant has a LTI controller and each contains an integrator, then

$$\int_0^\infty e(t)dt = 0, \tag{1}$$

where $e(t)$ is the error in the closed loop step response. Constraint (1) implies, amongst other things, that there will always be overshoot in the step response of such a system.

In [6], it was noted that whilst some constraints are controller independent, others, such as the one above, are not. In the example above, the derivation of (1) uses the fact that there is a loop transfer function $L(s)$ which contains a double integrator. If, however, the controller is nonlinear or time varying, then this may no longer be the case, and it may be possible to circumvent the constraint (1). This leads naturally to the question of when this is possible, and what controllers achieve better performance.

This paper is concerned with a class of control problems for which switching control gives improved performance. We consider a LTI plant containing at least one integrator. The plant can be represented as a transfer function $G(s)$ and

$$\lim_{s\to 0} G(s) = \infty.$$

The input to the plant is disturbed by $d(t)$, where

$$\|d(.)\|_{L_\infty} \leq d_{max}.$$

Large changes are permitted in the reference signal $r(t)$ and the primary aim is to maintain good error performance with 'reasonable' control energy.

We consider a slight generalisation of the switched linear controller proposed in [6] where $K_1(s)$ is allowed to have dynamics. This is shown in Figure 1. The scheme is motivated by the observation that the controller integrator is not needed for setpoint tracking, but it is needed for disturbance rejection. Since the input disturbances are known to be small the controller integrator is switched out and reset for large error signals, whilst it is included when the error becomes small. $K_1(s)$ and K_2 are chosen so that the two LTI closed loop systems (which result when the switch is in a fixed position) are stable. ε is chosen so that it is larger than the steady state value of $|e(t)|$ with the integrator switched off. It should be noted the controller is heterogeneous as the switching operation changes the number of controller states.

In [6], the switched linear controller was used for a pure integrator. $K_1(s)$ was a constant, and hence the controller was switching between proportional and proportional plus integral control. The resulting closed loop was simulated. It was found that, for this problem, the switching control gives 'better' performance than any linear control.

In this paper the stability properties of the switched linear scheme are discussed in detail. The results are of a more general nature than those found in [7], where the related problem of the control of a second order plant with a first order reset element is studied. In Section 2, it is established that, generically, this particular type of switched controller is bounded input bounded state (BIBS) stable. However, perhaps not surprisingly, describing function analysis indicates that there may be cases of conditionally stable control loops where the scheme of [6] allows limit cycles in the response. An example for which this is indeed the case is presented. In Section 3, we then turn to the question of finding an 'optimal' controller for the class of plants considered. By posing an H_∞ optimisation including an L_∞ bound on the disturbance, we exhibit a piecewise linear optimal state feedback law for a first order plant. Section 4 concludes the paper with a discussion of directions for further work.

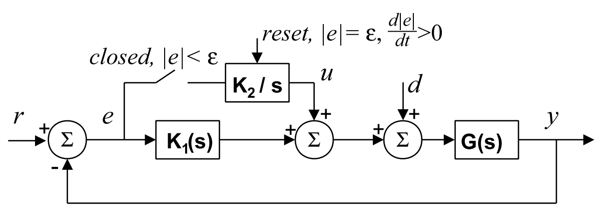

Figure 1: Control scheme of Goodwin, Feuer and Salgado.

2 Stability Properties of a Switched Linear Controller

In this section, the stability of the switched linear control scheme is studied. Recall that the scheme, which is shown in Figure 1, contains an integrator which is switched on when $e(t) < \varepsilon$ and is both switched off and reset at other times. This modified integrator will be referred to as a *resetting integrator*.

It is known that, in general, a switched linear system can be unstable even when the individual linear components are stable. For the particular system in this section, if the plant is a pure integrator and $K_1(s)$ is a constant gain, then the closed loop is exponentially stable. This holds because of the existence of a common Lyapunov function. Although the more general case is harder to analyse, BIBS stability holds as shown below.

Theorem 1 *Remove the resetting integrator from Figure 1 and let the state space description of the remaining system be*

$$\dot{x} = Ax + Bw, \quad y = Cx,$$

where x is the state and $w = [\,d\ r\,]^T$. Let

$$z = \begin{bmatrix} x \\ u \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & B\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ -K_2 C & 0 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} B \\ 0\ K_2 \end{bmatrix} \quad \text{and} \quad \bar{C} = [\,C\ 0\,].$$

If A and $\bar{A}$ are Hurwitz, then the switched linear system described by

$$\left.\begin{array}{ll} \dot{x} = Ax + Bw, & y = Cx \\ u = 0 & \end{array}\right\} \quad \text{when} \quad |e| \geq \varepsilon, \qquad (2)$$

$$\dot{z} = \bar{A}z + \bar{B}w, \quad y = \bar{C}z \qquad \text{when} \quad |e| < \varepsilon, \qquad (3)$$

is BIBS stable.

Proof See Appendix A.

Remark The resetting integrator scheme is described by (2) and (3). Since the component linear systems are stable, A and $\bar{A}$ are Hurwitz. Hence, by Theorem 1, the scheme of Figure 1 is BIBS stable.

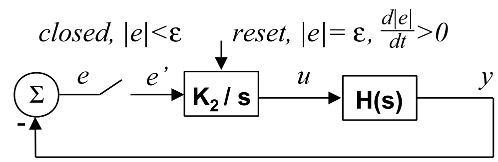

Figure 2: Simplified block diagram of Figure 1 with r and d set to zero.

It is of interest to know whether this BIBS stability result can be strengthened to asymptotic stability in general. In many cases, simulation of the closed loop suggests that this is true. However, analysis using describing functions suggests there may be examples of the scheme in Figure 1 which have a limit cycle.

Suppose the reference and disturbance are set to zero. Then Figure 1 is equivalent to Figure 2 with

$$H(s) = \frac{G(s)}{1 + K_1(s)G(s)}.$$

A limit cycle may exist in this system if the Nyquist plot of $H(s)/s$ crosses $\{b \in \mathbf{R} : b < -1/K_2\}$ more than once. This condition shall be called the *describing function condition*. It can be derived from the theory on limit cycles in [8] and [9]. Details are given in Appendix B. It may be deduced that if the Nyquist plot crosses the negative real axis twice, or if the phase of the Bode plot crosses $-180°$ twice, then K_2 can be chosen so that the describing function condition is satisfied. Unfortunately describing function analysis is approximate, and so, even if this condition is met, there may not be a limit cycle. This is illustrated in the following example.

Example 1

$$\frac{H(s)}{s} = \frac{(s^2 + 0.7s + 2)(s + 1)}{s(s^2 + 0.5s + 1)(s + \sqrt{2})^2}. \qquad (4)$$

Note that

$$G(s) = \frac{(s^2 + 0.7s + 2)(s + 1)}{s(s^3 + 2.33s^2 + 2.71 + 1.13)} \text{ and } K_1 = 1$$

will give (4). The Nyquist and Bode plots for $H(s)/s$ are shown in Figure 3. The plots cross the relevant boundaries twice, and describing function analysis suggests that there may be a limit cycle for $K_2 > 4$. However, extensive simulation studies and analysis using Lemma 1 below strongly suggest that, without exogenous disturbances, there is actually no limit cycle in this case. □

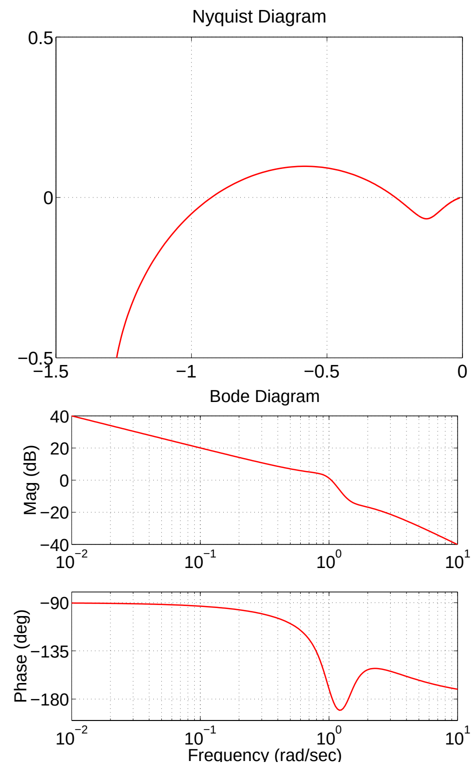

Figure 3: Nyquist and Bode Plots for Example 1

It will now be shown that for the system in Figure 2, there is a necessary and sufficient condition for the existence of a special type of limit cycle. Furthermore, if the condition is satisfied, then the method yields an initial condition which starts the oscillation.

Definition 1 A limit cycle of the system described in Theorem 1 shall be called *simple* if

(i) $z\left(t+\frac{T}{2}\right) = -z(t)$, where T is the period of oscillation and z is defined in Theorem 1,

(ii) the integrator is reset exactly twice each period. □

Note that (i) describes the symmetry of the oscillation. (ii) implies that y enters and leaves the interval $[-\varepsilon, \varepsilon]$ exactly twice per oscillation. An intuitive interpretation of this is that y crosses $[-\varepsilon, \varepsilon]$ twice. An example of a simple limit cycle is depicted in Figure 4.

Lemma 1 *The system shown in Figure 2 has a simple limit cycle iff* $\exists\, t_1, t_2, z_0$ *s.t.*

(i) z_0 *is an eigenvector of* $e^{A_2 t_2} R e^{A_1 t_1}$ *with a corresponding eigenvalue of -1,*

(ii) $-\varepsilon < \bar{C} e^{A_1 t} < \varepsilon$ *for* $0 < t < t_1$,

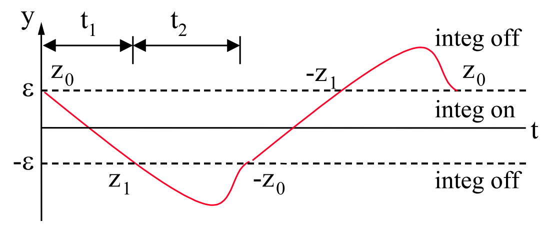

Figure 4: A simple limit cycle. States when $|y(t)| = \varepsilon$ are indicated.

(iii) $\bar{C} e^{A_2 t} R e^{A_1 t_1} < -\varepsilon$ *for* $0 < t < t_2$,

where

$$A_1 = \bar{A}, \quad A_2 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

and $dim(I) = dim(A)$*. If there is a limit cycle, then its period is* $2(t_1 + t_2)$ *and an initial condition of* z_0 *will start the oscillation.*

Proof When $|e(t)| < \varepsilon$, the integrator is active (the loop is closed) and

$$\dot{z} = A_1 z, \qquad y = \bar{C} z. \tag{5}$$

When $|e(t)| = \varepsilon$, the integrator is reset and $z(t^+) = Rz(t^-)$. When $|e(t)| > \varepsilon$, the integrator is inactive and

$$\dot{z} = A_2 z, \qquad y = \bar{C} z. \tag{6}$$

Now suppose the system has a simple limit cycle with period T. Let $|y(t)| = \varepsilon$ at $t = 0$, $t = t_1$ and $t = t_1 + t_2 = \frac{T}{2}$ as shown in Figure 4. Assume the integrator is on when $0 < t < t_1$. Combining the solutions of the state equations in (5) and (6) gives

$$z\left(\frac{T}{2}\right) = e^{A_2 t_2} R e^{A_1 t_1} z(0).$$

Since the limit cycle is simple, $z(\frac{T}{2}) = -z(0)$. Hence

$$-z(0) = e^{A_2 t_2} R e^{A_1 t_1} z(0),$$

from which it may be concluded that $z(0)$ is an eigenvector of $e^{A_2 t_2} R e^{A_1 t_1}$ with an eigenvalue of -1.

If the conditions of the lemma are met, then $z(t) = z_0$ implies $z(t + t_1 + t_2) = -z_0$. Thus, there is a simple limit cycle passing through z_0. □

The lemma can be used to show that many systems which satisfy the describing function condition do not have a simple oscillation. One such system was presented in Example 1. The system in Example 2 does contain a simple limit cycle.

Example 2

$$\frac{H(s)}{s} = \frac{s^2 + 1.5s + 9}{s(s^2 + 0.25s + 1)(s + 4)}. \tag{7}$$

Note that

$$G(s) = \frac{s^2 + 1.5s + 9}{s(s^2 + 3.81s + 1.33)} \quad \text{and} \quad K_1 = \frac{4}{9}$$

will give (7). The Bode diagram in Figure 5 shows that the phase of $H(s)/s$ crosses $-180°$ twice. The describing function condition is satisfied if $K_2 > 35$. When the lemma was applied to the system it was found that a simple limit cycle exists for $35 < K_2 < 70$.

□

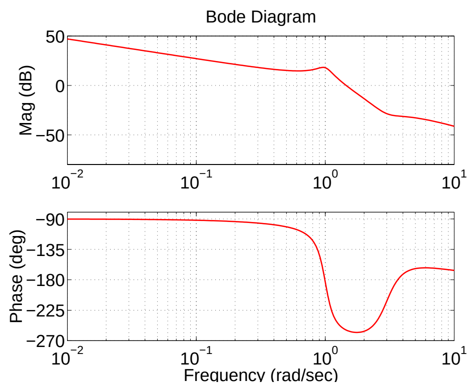

Figure 5: Bode Plot for Example 2.

Example 2 shows that the resetting integrator scheme is not always asymptotically stable. However, describing function analysis suggests problems only when the closed loop in Figure 2 is conditionally stable. Hence, it might be expected that, in practice, the scheme will often be asymptotically stable.

3 Optimal Control With Bounded L_2 Disturbance

The switched linear controller of Goodwin, Feuer and Salgado gives improved performance over linear control. A natural next question is what is the 'optimal' control for such systems? In this section a particular optimal control problem is solved for the following system:

$$\begin{aligned} \dot{x} &= u + d, \\ d &\in D = \{d \in L_2 : |d(t)| \leq 1\}. \end{aligned}$$

This is an integrator with a bounded L_2 input disturbance. The control law which minimises

$$J(u,d) = \left[\frac{1}{2}\int_0^\infty x^2(t) + \lambda^2 u^2(t) - \gamma^2 d^2(t) dt\right]$$

for the worst case disturbance in D will be found. This problem shall be referred to as the *constrained problem* because $|d(t)| \leq 1$. Removal of this condition yields the standard disturbance attenuation problem.

J is strictly convex in u and concave in d, and so, by Theorem 2.3 in [10], J has a unique saddle point. When $\gamma^2 > \lambda^2$ this saddle point occurs at

$$u^* = \begin{cases} -\frac{x}{\lambda^2\sqrt{\frac{1}{\lambda^2} - \frac{1}{\gamma^2}}}, & |x| \leq \alpha, \\ -\text{sgn}(x)\left(1 + \sqrt{1 + \frac{x^2 - \gamma^2}{\lambda^2}}\right), & |x| > \alpha, \end{cases} \tag{8}$$

$$\text{and} \quad d^* = \text{sat}\left(-\frac{\lambda^2}{\gamma^2}u^*\right), \tag{9}$$

where $\alpha = \gamma^2\sqrt{\frac{1}{\lambda^2} - \frac{1}{\gamma^2}}$ and

$$\text{sat}(x) = \begin{cases} x, & |x| \leq 1, \\ \text{sgn}(x), & |x| > 1. \end{cases}$$

u^* and d^* are the optimal control and the worst case disturbance, respectively. Details of the derivation of these are given in Appendix C.

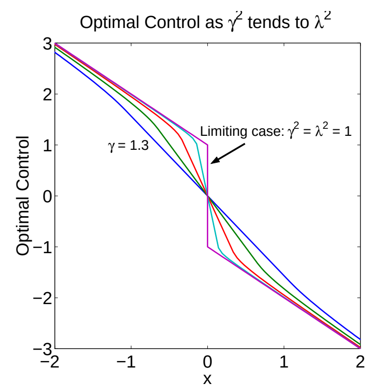

Figure 6: Optimal control for $\lambda = 1$, $\gamma = 1.3, 1.15, 1.05, 1.01, 1$

The optimal control for various values of γ is plotted in Figure 6. An interesting observation is that as $\gamma^2 \to \lambda^2$ (from above),

$$u^* \to -\frac{x}{\lambda} - \text{sgn}(x).$$

Hence, the limiting control law is piecewise affine. This is reminiscent of high gain control when x is small and low gain when it is large, and therefore matches qualitatively the control scheme of [6].

4 Conclusion and Future Research

In this paper the stability of the hybrid control scheme proposed by Goodwin, Feuer and Salgado was investigated. It was proved that the scheme is BIBS stable for plants containing an integrator. However, it is possible to construct examples in which the closed loop system has a limit cycle. Hence, the scheme is not, in general, asymptotically stable.

It was found that the nonlinear optimal control (with a particular H_∞ cost) for an integrator with bounded disturbance is piecewise affine with high gain for small errors and low gain for large errors.

Future research may involve extending the nonlinear optimal control problem to more general plants and also to include an observer. The inclusion of an observer allows more realistic modelling of a step disturbance, and will yield in general a nonlinear dynamic output feedback controller.

A Outline of Proof of Theorem 1†

In this proof the Lyapunov functions for the component linear systems are used to construct a positive definite function, U. It is then shown that this new function is bounded along the system trajectories. The properties of U imply that the state is bounded.

Note that $a_0, \dots, a_9$, σ_1, σ_2 are positive constants.

Define $\tau_1, \tau_2, \dots$ and $t_1, t_2, \dots$ as sequences of times which satisfy $\tau_k < t_k < \tau_{k+1}$ and $\tau_1 > 0$. Let the integrator be off when $t \in [\tau_k, t_k]$, and on when $t \in (t_k, \tau_{k+1})$.

When the integrator is off, $|e(t)| \geq \varepsilon$, the dynamics are described by (2), and we can find a matrix $P = P^T > 0$ such that a Lyapunov function for (2) is $V(x) = x^T P x$.

When the integrator is on, the dynamics are described by (3), and we can find $\bar{P} = \bar{P}^T > 0$, a_1, a_2 such that a Lyapunov function for (3) is $\bar{V}(x,u) = z^T \bar{P} \mathrm{z}$, and $\bar{V}(x,u) \leq a_1 V(x) + a_2 u^2$.

Let $q_k(t) = e^{-\delta(t-t_k)}$, with δ a small positive constant, and define

$$U(x,u,t) = \begin{cases} V(x), & t \in [\tau_k, t_k], \\ q_k(t)V(x) + \frac{1-q_k(t)}{a_1}\bar{V}(x,u), & t \in (t_k, \tau_{k+1}), \end{cases} \quad (10)$$

It can be seen that U is a uniformly positive definite function of x. It should also be noted that $U(x(t), u(t), t)$ is continuous for $t \in (\tau_k, \tau_{k+1})$, but it may be discontinuous at $t = \tau_{k+1}$. It is necessary to analyse the behaviour of U on segments of a trajectory. There are three cases to consider: when the integrator is on, when it is off, and when it is reset.

Case 1: Integrator off, $t \in [\tau_k, t_k]$.

When $U = V$ is evaluated along the trajectories of (2), we have

$$\dot{U} = \quad \leq \quad -\sigma_1 U + a_3\sqrt{U}, \quad (11)$$

Case 2: Integrator on, $t \in (t_k, \tau_{k+1})$.

This case is more complicated to analyse due to the structure of U during this period. However, after some lengthy analysis we obtain

$$\dot{U} \leq -\sigma_2 U + a_4\sqrt{U} + a_5\sqrt{U}\sqrt{q_k(t)}|u| + a_6\delta q_k(t)u^2, \quad (12)$$

†See [11] for further details

Over the interval (t_k, τ_{k+1}) the error satisfies $|e(t)| < \varepsilon$. Thus,

$$|u(t)| \leq \int_{t_k}^{t} K_2 \varepsilon d\tau = (t - t_k)a_7,$$

and therefore

$$\sqrt{q_k(t)}|u| \leq \frac{2a_7}{\delta}e^{-1}.$$

Substituting these inequalities into the right hand side of Inequality (12) yields

$$\dot{U} \leq -\sigma U + a_8\sqrt{U} + a_9. \quad (13)$$

Case 3: Resetting of the integrator at τ_{k+1}.

Let $U_{k+1}^- = U(x(\tau_{k+1}^-), u(\tau_{k+1}^-), \tau_{k+1}^-)$ and let U_{k+1}^+ be defined in a similar way. After some analysis we find that $\exists\ 0 < \beta \leq 1$ such that

$$U_{k+1}^+ \leq \frac{U_{k+1}^-}{\beta} \quad (14)$$

By taking δ sufficiently small, further arguments based on this analysis yield the desired BIBS result.

B Proof of Describing Function Condition

Let $$f(x) = \begin{cases} x, & |x| < \varepsilon; \\ 0, & \text{elsewhere.} \end{cases}$$

Injecting $A\sin\omega t$ into a resetting integrator is the same as injecting $f(A\sin\omega t)$. Furthermore, in steady state, the output of the resetting integrator is

$$u(t) = \int_0^t K_2 f(A\sin\omega\tau)d\tau$$

because $u(t) = 0$ when $|A\sin\omega t| = \varepsilon$. Hence, resetting the integrator has no effect.

The describing function of f is

$$N_f(A) = \begin{cases} 1, & A < \varepsilon; \\ \frac{2}{\pi}\sin^{-1}\left(\frac{1}{A}\right) - \frac{1}{A}\sqrt{1 - \frac{1}{A^2}}, & A \geq \varepsilon. \end{cases}$$

The linearity of integration implies that the fundamental component of $u(t)$ is the integral of the fundamental component of $K_2 f(A\sin\omega t)$. It follows that the describing function $N(A,\omega)$ of the resetting integrator is $\frac{K_2 N_f(A)}{j\omega}$. A limit cycle may exist if $N(A,\omega)H(j\omega) = -1$ for some A and ω [9],[8]. This is equivalent to

$$\frac{-1}{K_2 N_f(A)} = \frac{H(jw)}{j\omega}.$$

So a limit cycle may exist if the Nyquist plot of $\frac{H(s)}{s}$ intersects $\{b \in \mathbf{R} : b \leq -1/K_2\}$. The phase of $\frac{H(jw)}{j\omega}$

tends to $-90°$ as ω tends to 0. Suppose $\frac{H(jw)}{j\omega}$ crosses the negative real axis once. Then $\frac{-1}{K_2}$ must be left of the crossing to ensure stability of the PI controller (Nyquist stability theorem). This implies that $\frac{H(jw)}{j\omega}$ cannot intersect $\frac{-1}{K_2 N_f(A)}$. It follows that $\frac{H(jw)}{j\omega}$ must cross the negative real axis at least twice.

Remark. Suppose $\frac{H(jw)}{j\omega}$ crosses the negative real axis exactly twice, at a and b, $a < b < 0$. Then $\frac{-1}{K_2} < a$ or $b < \frac{-1}{K_2} < 0$ for stability. If $b < \frac{-1}{K_2} < 0$ describing function analysis predicts two limit cycles. An unstable one at b and a stable one at a [9].

C Derivation of Optimal Control and Worst Case Disturbance

By Theorem 2.6 in [10] $u^* = \tilde{u}$ and $d^* = \tilde{d}$ if there exists a C^1 solution to

$$
\begin{aligned}
0 &= \min_{u \in L_2} \max_{d \in D} \left[\frac{x^2(t) + \lambda^2 u^2(t) - \gamma^2 d^2(t)}{2} \right. \\
&\qquad \left. + \frac{dV^*}{dx}(u(t) + d(t)) \right] \\
&= \frac{x^2 + \lambda^2 \tilde{u}^2 - \gamma^2 \tilde{d}^2}{2} + \frac{dV^*}{dx}(\tilde{u} + \tilde{d}) \qquad (15)
\end{aligned}
$$

This is the Hamilton-Jacobi-Isaacs equation (HJIE) for this problem. The HJIE is, in general, a PDE but in this case it simplifies to an ODE because x is a scalar and the problem has an infinite horizon.
$\tilde{u}$ and $\tilde{d}$ are given by

$$
\begin{aligned}
\tilde{u} &= -\frac{1}{\lambda^2} \frac{dV^*}{dx} \qquad (16) \\
\tilde{d} &= \begin{cases} \frac{1}{\gamma^2} \frac{dV^*}{dx} & \left| \frac{1}{\gamma^2} \frac{dV^*}{dx} \right| \leq 1 \\ \operatorname{sgn}\left(\frac{dV^*}{dx} \right) & \text{otherwise.} \end{cases} \\
&= \operatorname{sat}\left(\frac{1}{\gamma^2} \frac{dV^*}{dx} \right). \qquad (17)
\end{aligned}
$$

Substituting $\tilde{u}$ and $\tilde{d}$ into (15) yields a first order, non-linear, ODE for $V^*(x)$. Since the problem is symmetric about $x = 0$ it is sufficient to solve the ODE for $x > 0$. The solution exists for $\gamma > \lambda > 0$ and is given by

$$
V^*(x) = \begin{cases} V_1^*(x), & |x| \leq \alpha, \\ V_2^*(x), & x > \alpha, \end{cases}
$$

where

$$
\begin{aligned}
\alpha &= \frac{\gamma}{\lambda}\beta, \qquad \beta = \sqrt{\gamma^2 - \lambda^2}, \\
0 &= x^2 - \frac{\beta}{\gamma^2 \lambda^2} \left(\frac{dV_1^*}{dx} \right)^2, \qquad V_1(0) = 0, \qquad (18) \\
0 &= \frac{x^2}{2} - \frac{1}{2\lambda^2} \left(\frac{dV_2^*}{dx} \right)^2 + \frac{dV_2^*}{dx} - \frac{\gamma^2}{2}, \\
&\qquad V_2(\alpha) = V_1(\alpha). \qquad (19)
\end{aligned}
$$

Note that Equation (18) is the ODE associated with the unconstrained problem. It holds when $\tilde{d} = \frac{1}{\gamma^2} \frac{dV^*}{dx}$. Equation (19) comes from letting $\tilde{d} = 1$ in Equation (15). Solving the ODE in this 'piecewise' manner is valid because $\frac{dV_1^*}{dx} \leq \gamma^2$ only when $x \in [-\alpha, \alpha]$, and $\frac{dV_2^*}{dx} > \gamma^2$ only when $x > \alpha$. It can also be seen that $\frac{dV_1^*}{dx}$ is continuous and hence, $u^*(x) = \tilde{u}(x)$ and $d^*(x) = \tilde{d}(x)$. Solving for V_1^* and V_2^* yields

$$
\begin{aligned}
\frac{dV_1^*}{dx} &= \frac{\gamma\lambda}{\beta} x \\
V_1^*(x) &= \frac{\gamma\lambda}{2\beta} x^2 \\
\frac{dV_2^*}{dx} &= \lambda^2 \left[1 + \frac{1}{\lambda} \sqrt{x^2 - \beta^2} \right] \\
V_2^*(x) &= \lambda^2 x + \frac{\lambda}{2} \left[x\sqrt{x^2 - \beta^2} - \beta^2 \cosh^{-1} \frac{x}{\beta} \right] + c,
\end{aligned}
$$

where the constant c ensures V is continuous. The positive solutions for $\frac{dV_1^*}{dx}$ and $\frac{dV_2^*}{dx}$ are taken so that u^* is stabilising.

Substitution of the solutions for $\frac{dV_1^*}{dx}$ and $\frac{dV_2^*}{dx}$ into Equations (16) and (17) gives the optimal control and worst case disturbance in (8) and (9).

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