Generalized Design of Mixed H-infinity/Deadbeat Suboptimal Controllers for SISO Continuous-Time Servo Systems

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Abstract

This paper is concerned with a mixed $\mathcal{H}_\infty$/deadbeat suboptimal control problem for SISO continuous servo systems. By considering the deadbeat tracking, time domain performance may improve. Simultaneously, to improve frequency domain performance of the closed loop system, $\mathcal{H}_\infty$ norm constraint is introduced. This problem gives the deadbeat tracking control with $\mathcal{H}_\infty$ norm constraint and has been studied by Nobuyama *et al.* and Tsumura *et al.* individually. However, there is a structural difference between their designs. This paper proposes a new controller design of the mixed $\mathcal{H}_\infty$/deadbeat suboptimal control problem. The controller is more general since the structure of the controller covers those of the previous two designs.

1 Introduction

Recently, a deadbeat tracking problem for SISO continuous systems have been investigated by Nobuyama *et al.*[2],[3]. This control objective is to make the error signal deadbeatly zero within a given time and make it remain zero. Using this method with a short elimination time, we can expect the improvement of time domain performance. However, when designing the ordinary deadbeat controller, sometimes the closed loop system becomes unstable by noises or system errors. To improve robustness of the closed loop system, the design methods that guarantee frequency domain performance simultaneously have been studied by Nobuyama *et al.* and Tsumura *et al.* individually.
In the Nobuyama's design[5], an error signal of the servo system is restricted to

$$Q(s) = \frac{b_0(s) + b_1(s)e^{-sT}}{a(s)} \tag{1}$$

where $a(s)$, $b_0(s)$ and $b_1(s)$ are polynomial and $b_i(s)/a(s), (i = 1, 2)$ are proper. First, $a(s)$ with appropriate degree is fixed and $b_i(s), (i = 1, 2)$ with some free parameters are designed in order that the error signal may settle to zero and the controller may exist. And then, since the $\mathcal{H}_\infty$ optimal problem over those parameters is convex, this optimal solution can be given numerically. When the degree of $a(s)$ becomes larger, so does the number of free parameters. Therefore, if this solution does not achieve the desired norm, the degree of $a(s)$ is increased until its optimal solution achieves it.
In the Tsumura's design[6]-[8], at first, the $\mathcal{H}_\infty$ suboptimal control problem is solved. The free parameter of $\mathcal{H}_\infty$ controller is restricted to

$$Q(s) = f(\tilde{Q}(s)) \tag{2}$$

where f is a certain function and

$$\tilde{Q}(s) = q_0 + q_1 e^{-sT} + q_2 e^{-2sT} + \cdots + q_k e^{-ksT}. \tag{3}$$

They gave the sufficient norm condition of $\tilde{Q}(s)$ such as $\|Q(s)\|_\infty < 1$. Then, they designed $\tilde{Q}(s)$ to satisfy the conditions of the norm constraint and a finite time elimination by increasing time delay functions of (3). This method is little conservative because of the norm condition of $\tilde{Q}(s)$.
Comparing Nobuyama's design with Tsumura's, the structures of the free parameter are limited and different each other, although these two approaches are different. So, in this article, we propose the more generalized method of mixed $\mathcal{H}_\infty$/deadbeat suboptimal control by generalizing the structure of the controller to cover those of (1) and (3). Using this parameter, we can expect the controllers which guarantee better performance.
First of all, we use Youla parametrization and propose a structure of the free parameter in order to cover those of (1) and (3). We derive that the free parameter of its parametrization must be a deadbeat signal via appropriate design of Youla parametrization, and give the conditions such that the error signal settles to zero within finite time.
Secondly, it is shown that a set of the free parameter which satisfy the conditions for finite time elimination is affine with infinite dimension. Hence, the $\mathcal{H}_\infty$ optimization problem subject to those constraints is infinite dimensional convex problem. By employing a Ritz approximation, this problem on this approximation can be reduced to finite dimensional convex. Although this problem can be solved numerically, this

solution is not globally optimal. Hence, we adopt larger degree of the Ritz approximation if the solution over the certain Ritz approximation does not achieve the desired norm.
Finally we demonstrate that the optimal norm of a certain Ritz approximation is different from that of the another and the achieved norm of the closed loop system depends on the elimination time by numerical example.

2 Preliminaries

2.1 Problem Statement

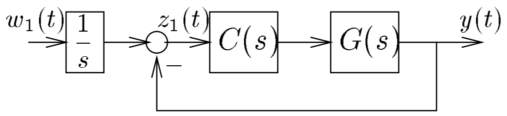

Figure 1: Servo System

Consider the SISO servo system realized as Figure 1. The state space realization of the plant $G(s)$ is represented as follows.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), x(0) = 0 \\ y(t) &= Cx(t) \end{aligned} \tag{4}$$

where $x(t) \in R^n$ is state vector, $w_1(t) \in R^1$ is scalar input and $y(t) \in R^1$ is scalar output. Suppose that A, B, C are real matrices of appropriate dimensions and (C, A, B) is minimal. One problem is to design deadbeat controller such that the error signal $z_1(t)$ settles zero within T_{settle}.
To improve frequency domain performance, let consider the generalized plant $P(s)$ illustrated in Figure 2 where $P(s)$ consists of a weighting functions and the plant $G(s)$ in the Figure 1.

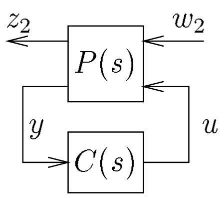

Figure 2: Generalized Plant

Second problem is to make infinity norm from w_2 to z_2, i.e. $\|F_{w_2 \to z_2}\|_\infty(s)$, less than γ. The purpose of this paper is to design the controllers which satisfy previous two problems simultaneously. It can be summarized as follows.

Problem 1 *Design the controller $C(s)$ which satisfies the following two conditions:*

- *The impulse response from w_1 to z_1 i.e. $F_{w_1 \to z_1}(t)$, settles to zero within $T_{\text{settle}} < \infty$.*
- *$\|F_{w_2 \to z_2}(s)\|_\infty < \gamma$ is satisfied.*

2.2 Condition of a Continuous Time Deadbeat Settling Signal

Continuous deadbeat control has been investigated by Nobuyama. A frequency-domain condition for the continuous time deadbeat settling signal is introduced in this section.

Lemma 1 [2]-[4] *Suppose the Laplace transform of a certain signal $\phi(t)$ is represented as*

$$\Phi(s) = \frac{b(s, e^{-sT_1}, \cdots, e^{-sT_q})}{a(s)}, \tag{5}$$

$$\begin{aligned} & b(s, e^{-sT_1}, \cdots, e^{-sT_q}) \\ & = b_0(s) + b_1(s)e^{-sT_1} + \cdots + b_q(s)e^{-sT_q}, \end{aligned} \tag{6}$$

where $T_i > 0 (i = 1, \cdots, q)$, $a(s)$,$b_i(s)$ are polynomials of s, and $b_i(s)/a(s), (i = 1, \cdots, q)$ is proper. And assume the next conditions

$$\left.\frac{\mathrm{d}^j}{\mathrm{d}s^j} b(s, e^{-sT_1}, \cdots, e^{-sT_q})\right|_{s=\alpha_i} = 0 \tag{7}$$

$$i = 1, \cdots, p, j = 0, \cdots, \nu_i - 1$$

where $\alpha_i (i = 1, \cdots, p)$ are zeros of $a(s)$ and $\nu_i (i = 1, \cdots, p)$ are their multiplicities, is satisfied. Then $\phi(t)$ settles deadbeatly to zero within $t = T_f$, where $T_f = \max(T_1, \cdots, T_q)$. That is

$$\phi(t) = \begin{cases} \phi_0(t), & 0 \le t < T_f \\ 0, & T_f \le t \end{cases}$$

□

Define $\theta_a(s, e^{-sT}), (a \in \boldsymbol{C}, T < \infty$ is a certain real number) as follows.

$$\theta_a(s, e^{-sT}) = \int_0^T e^{-at} e^{-st} dt \tag{8}$$

Then kth order differential of $\theta_a(s, e^{-sT})$ becomes

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k} \theta_a(s, e^{-sT}) = \int_0^T (-t)^k e^{-at} e^{-st} dt. \tag{9}$$

Using $\theta_a(s, e^{-sT})$, the signal $\Phi(s)$ in Lemma 1 can be written as follows.

$$\begin{aligned} \Phi(s) &= \sum_{i=1}^{N_1} \sum_{j=0}^{k_i} \sum_{k=0}^{k_{ij}} c_{ijk} e^{-sT_{ijk}} \frac{\mathrm{d}^j}{\mathrm{d}s^j} \theta_{\alpha_i}(s, e^{-sT_{1i}}) \\ &\quad + \sum_{i=0}^{N_2} c_i e^{-sT_{2i}} \end{aligned} \tag{10}$$

where α_i are zeros of $a(s)$ and k_i are their multiplicities. Furthermore, $c_{ijk}, c_i \in \boldsymbol{C}$, $0 \le T_{ijk}, T_{1i}, T_{2i} < \infty$ are real numbers and $N_1, N_2, k_{ij} < \infty$ are natural numbers.

3 Deadbeat Control for Continuous Time Systems

In this section we present the deadbeat control method for continuous systems which has been investigated by Nobuyama *et al.* [2]-[4] and improve the conditions of the deadbeat control by appropreate design of Youla parametrization. This method is effective to solve the first condition of Problem 1.

3.1 Class of Controllers

Let $f^T, h \in R^n$ be the matrixes such that $A_f = A + bf$ and $A_h = A + hc$ are Hurwitz respectively. Define

$$\begin{bmatrix} \tilde{X}(s) & -\tilde{Y}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} f \\ c \end{bmatrix} (sI - A_h)^{-1} \begin{bmatrix} -b & h \end{bmatrix} \quad (11)$$

$$\begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} f \\ c \end{bmatrix} (sI - A_f)^{-1} \begin{bmatrix} b & -h \end{bmatrix}. \quad (12)$$

Then these matrices provide a next double coprime factorization.

$$G(s) = N(s)M^{-1}(s) = \tilde{M}^{-1}(s)\tilde{N}(s), \quad (13)$$

$$\begin{bmatrix} \tilde{X}(s) & -\tilde{Y}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \begin{bmatrix} M(s) & Y(s) \\ N(s) & X(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (14)$$

All stabilizing controllers are given as follows.

$$\begin{aligned} K(s) &= (M(s)Q(s) - Y(s))(X(s) - N(s)Q(s))^{-1} \\ &= (\tilde{X}(s) - Q(s)\tilde{N}(s))^{-1}(Q(s)\tilde{M}(s) - \tilde{Y}(s)) \end{aligned} \quad (15)$$

where $Q(s)$ is a causal and stable transfer function.

3.2 Conditions for Deadbeat Controller

Using the controller (15), the error signal $F_{w_1 \to z_1}(s)$ becomes

$$\begin{aligned} F_{w_1 \to z_1}(s) &= \{X(s) - N(s)Q(s)\} \tilde{M}(s) \frac{1}{s} \\ &= \frac{X_0(s) - N_0(s)Q(s)}{\Delta(s)} g(s) \end{aligned} \quad (16)$$

where

$$\begin{aligned} X_0(s) &= \det(sI - A_f) - c \operatorname{adj}(sI - A_f)\, h \\ N_0(s) &= c \operatorname{adj}(sI - A_f)\, b \\ \Delta(s) &= \det(sI - A_f) \cdot \det(sI - A_h) \cdot s \\ g(s) &= \det(sI - A_h) + c \operatorname{adj}(sI - A_h)\, h \end{aligned}$$

If we design $Q(s)$ such that $F_{w_1 \to z_1}(s)$ satisfies the conditions of Lemma 1, $F_{w_1 \to z_1}(t)$ has the finite time settlement property. Until now, the structures of $Q(s)$ have been restricted to (1) or (3), so we propose the structures of $Q(s)$ as follows.

$$\begin{aligned} Q(s) &= \frac{b_0(s) + b_1(s)e^{-sT_1} + \cdots + b_m(s)e^{-sT_m}}{a(s)} \\ &\triangleq \frac{b(s)}{a(s)} \end{aligned} \quad (17)$$

where $b_i(s)$ and $a(s)$ are polynomials. Because $Q(s)$ is the causal and stable transfer function, $b_i(s)/a(s)$ must be proper and $a(s)$ must be Hurwitz polynomial. It is obvious that the structures of (17) cover these of (1) and (3).

Then, we derive the restrictions of $Q(s)$ in (17) such that $F_{w_1 \to z_1}(s)$ satisfies conditions of Lemma 1. Since The plant $G(s)$ is SISO system, the following property is satisfied.

$$\tilde{M}(s) = \frac{\det(sI - A)}{\det(sI - A_h)}, N(s) = \frac{c \operatorname{adj}(sI - A)\, b}{\det(sI - A_f)} \quad (18)$$

Let design f such that zeros of $\det(sI - A_f)$ cancel stable zeros of $c \operatorname{adj}(sI - A)\, b$ and design h such that zeros of $\det(sI - A_h)$ cancel stable zeros of $\det(sI - A)$. Then the pole-zero cancellation occurs inside $\tilde{M}(s)$ and $N(s)$. From the nature of double coprime factorization,

$$\begin{aligned} &\tilde{M}(s)X(s) - \tilde{N}(s)Y(s) \\ &= \frac{\det(sI - A)}{\det(sI - A_h)} \frac{\mathrm{Nu}(X(s))}{\det(sI - A_f)} \\ &\quad - \frac{c \operatorname{adj}(sI - A)\, b}{\det(sI - A_h)} \frac{\mathrm{Nu}(Y(s))}{\det(sI - A_f)} \\ &= 1 \end{aligned} \quad (19)$$

is satisfied. Multiply both side of (19) by $\det(sI - A_f)\det(sI - A_h)$ and substitute zeros of $c \operatorname{adj}(sI - A)\, b$ to its equation. (If $c \operatorname{adj}(sI - A)\, b$ has multiple zeros, partially differentiate the equation (19) with respect to s and substitute zeros.) Then it turns out that the denominator polynomial of $X(s)$ has the same stable zeros that $c \operatorname{adj}(sI - A)\, b$ has, and these zeros of denominator of $X(s)$ are canceled by its numerator. Substitute (17) to (16) and cancel zeros between denominator and numerator, (16) can be coordinated as

$$F_{w_1 \to z_1}(s) = \frac{X_0'(s)a(s) - N_0'(s)b(s)}{\Delta'(s)a(s)} g'(s) \quad (20)$$

where $N'(s), g'(s)$ do not have stable zeros. From the conditions of Lemma 1, the conditions for the finite time settlement of $F_{w_1 \to z_1}(t)$ become as follows.

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j} \{(X_0'(s)a(s) - N_0'(s)b(s))g'(s)\} \Big|_{s=\alpha_i} = 0 \quad (21)$$

$$i = 1, \cdots, p, j = 0, \cdots, \nu_i - 1$$

where $\alpha_i (i = 1, \cdots, p)$ are zeros of $\Delta'(s)a(s)$ and $\nu_i (i = 1, \cdots, p)$ are their multiplicities.
This constraint can be simplified.

Lemma 2 *The followings are equivalent:*

(a). $\displaystyle \frac{\mathrm{d}^j}{\mathrm{d}s^j} \{(X_0'(s)a(s) - N_0'(s)b(s))g'(s)\} \Big|_{s=\alpha} = 0$

(b). $\displaystyle \frac{\mathrm{d}^j}{\mathrm{d}s^j} \{X_0'(s)a(s) - N_0'(s)b(s)\} \Big|_{s=\alpha} = 0$

where $\mathrm{Re}(\alpha) < 0, j = 0, \cdots, p$ *and* $g'(s)$ *does not have stable zeros.*

(*Proof*)
(b)⇒(a) It is obvious.
(a)⇒(b) We employ the foundations of mathematics.

When $j=0$, (b) is equivalent to (a) because $g'(\alpha) \neq 0$.
Suppose (b) is equivalent to (a) from $j=0$ to $j=k-1$. Then

$$\begin{aligned} &\frac{\mathrm{d}^k}{\mathrm{d}s^k}\left\{(X_0'(s)a(s)-N_0'(s)b(s))g'(s)\right\}\Big|_{s=\alpha} \\ &= \sum_{i=0}^{k} {}_kC_i \frac{\mathrm{d}^{k-i}}{\mathrm{d}s^{k-i}}\left(X_0'(s)a(s)-N_0'(s)b(s)\right)\frac{\mathrm{d}^i}{\mathrm{d}s^i}g'(s)\Big|_{s=\alpha} \\ &= \frac{\mathrm{d}^k}{\mathrm{d}s^k}\left\{X_0'(s)a(s)-N_0'(s)b(s)\right\}\Big|_{s=\alpha} g'(\alpha)=0 \end{aligned}$$

(b) is also equivalent to (a) when $j=k$. Therefore (b) is equivalent to (a). □
Via Lemma 2, the conditions (21) are simplified as

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(X_0'(s)a(s)-N_0'(s)b(s)\right)\Big|_{s=\alpha_i}=0 \qquad (22)$$

$$i=1,\cdots,p, j=0,\cdots,\nu_i-1$$

where $\alpha_i (i=1,\cdots,p)$ are zeros of $\Delta'(s)a(s)$ and $\nu_i (i=1,\cdots,p)$ are their multiplicities.
This conditions require that the free parameter $Q(s)$ should satisfy the conditions of Lemma 1.

Theorem 1 *Let the double coprime factorization be given by (11),(12),(13) and (14), where zeros of $\det(sI-A_f)$ and $\det(sI-A_h)$ cancel by stable zeros of $c\ \mathrm{adj}(sI-A)\ b$ and $\det(sI-A)$ respectively. Then, the sufficient condition for the conditions (22) is that the free parameter $Q(s)$ in (17) satisfies conditions of Lemma 1.*

(*Proof*)
For simplicity, assume that $a(s)$ doesn't have multiple root. From the conditions (22), the following conditions are necessary.

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(X_0'(s)a(s)-N_0'(s)b(s)\right)\Big|_{s=\alpha_i}=0 \qquad (23)$$

where $\mathrm{Re}(\alpha_i)<0 (i=1,\cdots,k)$ are all zeros of $a(s)$. Since $N_0'(s)$ is designed such as unstable polynomial, $N_0'(\alpha_i)\neq 0$. Clearly $a(\alpha_i)=0$ is satisfied. Thus

$$b(\alpha_i)=0 \qquad (24)$$

where α_i is zeros of $a(s)$, is satisfied. This implies that $Q(s)$ in (17) satisfies the conditions of Lemma 1. That is, $Q(t)$ settles to zero within $\max\{T_1,\cdots,T_m\}$. □
The previous theory shows that to make $F_{w_1\to z_1}(t)$ zero within finite time, it is necessary that $Q(s)$ is the deadbeat signal of Lemma 1. By proof of Theorem 1, if $Q(s)$ is designed such as the signal of Lemma1, the remained conditions of (22) are these for zeros of $\Delta(s)$. Hence, the conditions of (22) with settlement time T_{settle} becomes as followings.

Theorem 2 *$F_{w_1\to z_1}(t)$ settles to zero within finite time T_{settle} if the free parameter $Q(s)$ in (15) satisfies followings.*

- *$Q(s)$ is the stable deadbeat signal of Lemma 1. Furthermore $Q(t)$ settles to zero within T_{settle}.*
- $\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left\{X_0'(s)a(s)-N_0'(s)b(s)\right\}\Big|_{s=\alpha_i}=0$
  $i=1,\cdots,k; j=\nu_{ai},\cdots,\nu_i+\nu_{ai}-1$
  where α_i are zeros of $\Delta(s)$, ν_i are their multiplicities and ν_{ai} are the multiplicities of $a(s)$.

4 Design of $\mathcal{H}_\infty$/Deadbeat Suboptimal Controllers

The previous section presented the restrictions of $Q(s)$ for the deadbeat control. This section presents the design procedure of the controllers which satisfy $\mathcal{H}_\infty$ norm constraint subject to those restrictions. $\mathcal{H}_\infty$ norm of a certain transfer matrix is known as convex[1]. Applying the controller (15) to the system, $F_{w_2\to z_2}(s)$ can be written as

$$F_{w_2\to z_2}(s)=F_{21}(s)+F_{22}(s)Q(s)F_{23}(s). \qquad (25)$$

Therefore, if a set of $Q(s)$ which satisfies those restrictions is convex or affine, the $\mathcal{H}_\infty$ optimization problem subject to those restrictions becomes convex on this set. So, we show that this set is affine at first.

4.1 Affine Property of $Q(s)$

It is shown that the set $Q(s)$ which satisfy the conditions of Theorem 2 is affine. To begin with, a definition of affine is introduced.

Definition 1 [1] *$\mathcal{H}_1\subseteq\mathcal{H}$ is affine if for any H, $\tilde{H}\in\mathcal{H}_1$, and any $\lambda\in\boldsymbol{R}$, $\lambda H+(1-\lambda)H\in\mathcal{H}_1$.*

Theorem 3 *The set of the free parameters $Q(s)$ which satisfies Theorem 2 is affine.*

(*Proof*)
The error signal $F_{w_1\to z_1}(s)$ can be written as

$$F_{w_1\to z_1}(s)=F_{11}(s)+F_{12}(s)Q(s)F_{13}(s). \qquad (26)$$

where $Q(s)$ is designed such that $F_{w_1\to z_1}(t)$ settles zero within finite time. Suppose that $Q_1(s)$ and $Q_2(s)$ are different free parameter which satisfy Theorem 2, and $\lambda Q_1(s)+(1-\lambda)Q_2(s)$ is applied to the controller (15). Then the error signal becomes

$$\begin{aligned} F_{w_1\to z_1}(s) &= F_{11}+F_{12}(\lambda Q_1(s)+(1-\lambda)Q_2(s))F_{13} \\ &= \lambda(F_{11}+F_{12}Q_1(s)F_{13}) \\ &\quad +(1-\lambda)(F_{11}+F_{12}Q_2(s)F_{13}) \end{aligned} \qquad (27)$$

Since the signals $F_{11}+F_{12}Q_1(s)F_{13}$ and $F_{11}+F_{12}Q_2(s)F_{13}$ settle to zero within finite time, the signal (27) also settles to zero within finite time. It is obvious that $\lambda Q_1(s)+(1-\lambda)Q_2(s)$ is the stable deadbeat signal of Lemma 1. Hence $\lambda Q_1(s)+(1-\lambda)Q_2(s)$ satisfies the conditions of Theorem 2. □
Since this affine set is infinite dimension, it is not easy to obtain the optimal solution. So, we introduce the Ritz approximations.

4.2 Ritz Approximations [1]

It is not easy to obtain the optimal solution of the infinite dimensional convex problems. By employing the finite dimension subset of the infinite dimensional set, that infinite problems are reduced to finite. Let $\mathcal{H}$ be the infinite dimensional vector space. Then the Nth Ritz approximation is given as follows.

$$\mathcal{H}_N \triangleq \left\{ R_0 + \sum_{1\le i\le N} x_i R_i \middle| x_i \in \boldsymbol{R}, 1 \le i \le N \right\} \tag{28}$$

where $R_0, R_1, R_2, \cdots \in \mathcal{H}$.
This set is the finite dimensional affine subset of $\mathcal{H}$. If the original problem is convex, the Ritz approximation yields convex problem, since it is the original problem with the affine specification adjoined.

4.3 Formulation of The Problems and Design Procedures

To begin with, the elements of the Ritz approximation are derived. Then the optimization problems on the Ritz approximation are formulated. Finally we give the algorithms of the controller design.
From the first condition of Theorem 2, it is necessary that $Q(s)$ satisfies the conditions of Lemma 1. So $Q(s)$ is composed of the next signals. (See 2.2)

$$\begin{array}{l} e^{-sT_{1i}} \dfrac{\mathrm{d}^k}{\mathrm{d}s^k}\theta_a(s, e^{-sT_{2i}}), (\mathrm{Re}(a) < 0) \\ e^{-sT_{3i}} \end{array} \tag{29}$$

where $\theta_a(s, -sT_{2i})$ is defined in (8). The convex optimization problem with the Ritz approximation becomes

$$\begin{aligned} &\min \|F_{w_2\to z_2}(s)\|_\infty \\ &= \min_{x_1,\cdots,x_N} \|F_{21}(s) + F_{22}(s)Q_x(s)F_{23}(s)\|_\infty \end{aligned} \tag{30}$$

where $Q_x(s) \in \mathcal{H}_N$,

$$\mathcal{H}_N \triangleq \left\{ Q_0(s) + \sum_{1\le i\le N} x_i Q_i(s) \middle| x_i \in \boldsymbol{R}, 1 \le i \le N \right\}, \tag{31}$$

$$\begin{aligned} &Q_0(s), \cdots, Q_N(s) \\ &\in \mathrm{span}\{e^{-sT_{1i}} \frac{\mathrm{d}^k}{\mathrm{d}s^k}\theta_a(s, e^{-sT_{2i}}), e^{-sT_{3i}}\}, (\mathrm{Re}(a) < 0) \end{aligned}$$

and $Q_x(s)$ satisfies the second conditions of Theorem 2. Since the second conditions of Theorem 2 are equations, we can eliminate the parameter x_i in (28) by number of restrictions. For example, assume that jth Ritz approximation's elements are chosen as $0, Q_0(s), \cdots, Q_j(s)$ and zeros of $\Delta(s)$ are $\alpha_i, \cdots, \alpha_k$ without multiple zeros. Thus the second constrains of Theorem 2 becomes

$$\begin{bmatrix} Q_0(\alpha) & \cdots & Q_j(s) \\ \vdots & \vdots & \vdots \\ Q_0(s) & \cdots & Q_j(s) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_j \end{bmatrix} = \begin{bmatrix} \dfrac{X_0'(\alpha_1)}{N_0'(\alpha_1)} \\ \vdots \\ \dfrac{X_0'(\alpha_k)}{N_0'(\alpha_k)} \end{bmatrix} \tag{32}$$

(32) has solution, if the matrix of left side of above equation is $j \ge k$ and full row rank. If the matrix does not have full row rank, we can make it full row rank by modifying the time delay or the settlement time of the $Q_i(s)$. Using $(j-k)$th free parameters, the solution can be written as

$$\begin{bmatrix} x_1 \\ \vdots \\ x_j \end{bmatrix} = K_1 + K_2 \begin{bmatrix} x_1' \\ \vdots \\ x_{j-k}' \end{bmatrix}. \tag{33}$$

Thus jth Ritz approximation can be written as follows.

$$\begin{aligned} &\begin{bmatrix} Q_0(s) & \cdots & Q_j(s) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_j \end{bmatrix} \\ &= \begin{bmatrix} Q_1'(s) & \cdots & Q_{j-k}'(s) \end{bmatrix} \begin{bmatrix} x_1' \\ \vdots \\ x_{j-k}' \end{bmatrix} \triangleq \boldsymbol{Q}'(s)\boldsymbol{x}'. \end{aligned} \tag{34}$$

Therefore the optimization problem with jth Ritz approximation becomes

$$\min_{\boldsymbol{x}'} \|F'(\boldsymbol{x}')_{w_2\to z_2}(s)\|_\infty \tag{35}$$

where

$$F'(\boldsymbol{x}')_{w_2\to z_2}(s) = F_{21}(s) + F_{22}(s)\boldsymbol{Q}'(s)\boldsymbol{x}'F_{23}(s). \tag{36}$$

If this optimal solution satisfies desire norm, we accept its free parameter. On the other hand, if this optimal solution does not satisfy the norm condition, we use the larger degree of the Ritz approximation and repeat the same procedure. Finally, we summarize the above arguments as following design procedures.

(i). Design the controller (15) and cancel stable zeros and poles between the plant and the controller.

(ii). Solve the conditions such that $F_{w_1\to z_1}(t)$ settles to zero within finite time T_{settle}.

(iii). Let k be the number of the restriction in (ii). Design the kth Ritz approximation of $Q(s)$ which elements settle to zero within time T_{settle} and decide the parameters such that $Q(s)$ satisfies the restrictions of (ii). If $\|F_{w_2\to z_2}(s)\|_\infty < \gamma$, go to (vi) else go to (iv).

(iv). Increase the degree of Ritz approximation by one, solve the constraints in (ii) and obtain the convex optimal problem (35).

(v). Solve the convex optimal problem. If its solution is less than γ, go to (vi) else return to (iv).

(vi). Obtain the free parameter $Q(s)$ in (15).

We now show that the structure (1) or (3) can be represented as a special case of the Ritz approximation.

Corollary 1 *The structure (1) consists of linear combination of* $1, e^{-sT}$ *and signals* (9) *with the settlement time* T*, since (1) contains only one time delay. That is, the sequence of the Ritz approximation are restricted to*

$$0, 1, e^{-sT}, \frac{1 - e^{\alpha_1 T}e^{-sT}}{s - \alpha_1}, \frac{1 - e^{\alpha_2 T}e^{-sT}}{s - \alpha_2}, \cdots \tag{37}$$

Corollary 2 *The structure (3) consists of linear combination of* $1, e^{-sT}, e^{-2sT}, \cdots$. *Therefore, the sequence of the Ritz approximation are restricted to*

$$0, 1, e^{-sT}, e^{-2sT}, \cdots, e^{-qsT} \tag{38}$$

5 Numerical Example

Let consider the following plant.

$$G(s) = \frac{1}{(s-1)(s+2)} \tag{39}$$

First of all, we design the controller such that the error signal $F_{w_1 \to z_1}(t)$ settles to zero within time 2. Then, we observe the variation of robustness against additional error of the plant by change of the degree of the Ritz approximation. Let consider next generalized plant and observe the variation of optimal norm of $\|F_{w_2 \to z_2}(s)\|_\infty$ by the degree of the Ritz approximation.

$$\begin{bmatrix} z_2 \\ y \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} w_2 \\ u \end{bmatrix} \tag{40}$$

The solution of $\mathcal{H}_\infty$ optimal problem is $\|F_{w_2 \to z_2}(s)\|_\infty = 6.02$.
Let design the controller (15) to cancel stable pole -2 of the plant. This time, we use the sequence of Ritz approximations as (37) or (38). Since there are four constraints for the deadbeat control, the degree of the Ritz approximation begins with four.

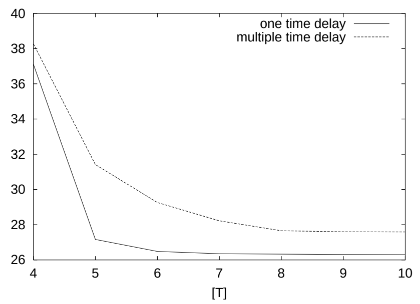

Figure 3: The degree of Ritz approximation versus optimal solution

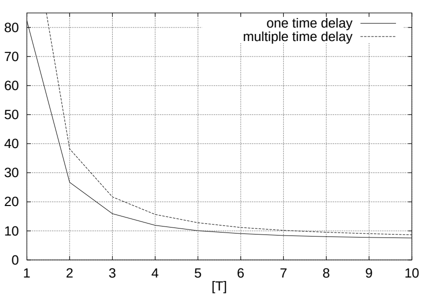

Figure 4: The elimination time versus achieved norm

Figure 3 presents the variation of the optimal solutions by change of the degree of approximations. This result appears that to make $\|F_{w_2 \to z_2}(s)\|_\infty$ less than 26 is difficult. For this reason, we can considere that the deadbeat elimination time is restricted two. So we demonstrate relationship between the elimination time and achieved norm of the closed loop system in Figure 4. It seems that the achieved norm of the closed loop system is inversely as the elimination time and approaches to $\mathcal{H}_\infty$ optimal norm when the elimination time becomes larger. On the other hand, Figure 3 also show that the achieved norm is affected by the selection of the the sequence of the Ritz approximation.

6 Conclusion

In this paper, we suggest the design method of mixed $\mathcal{H}_\infty$/deadbeat control. This controller guarantees not only the time domain performance but also the frequency performance. The free parameter of Youla parameterization has more general structure than these of the other studies. So we can expect the controllers which guarantee better performance.
However, there are some problems. One problem is the choice of the sequence of the Ritz approximation. The optimal norm of the certain approximation is different from that of the another. Second problem is that the controller may not exist. In the example, it turns out that the achieved norm of the closed loop system depend on settlement time.
As the future work, to solve above problems and the deadbeat control problems with other norm constraints are remained.

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