# Any domain of attraction for a linear constrained system is a tracking domain of attraction

Franco Blanchini[†], Stefano Miani[‡],
[†]Dipartimento di Matematica ed Informatica
[‡]Dipartimento di Ingegneria Elettrica, Gestionale e Meccanica
Università degli Studi di Udine, Via delle Scienze 208, 33100 Udine, Italy
{blanchini,miani.stefano}@uniud.it

**Abstract**

We face the problem of determining a tracking domain of attraction, say the set of initial states starting from which it is possible to track reference signals in given class, for discrete-time systems with control and state constraints. We show that the tracking domain of attraction is exactly equal to the domain of attraction, say the set of states which can be brought to the origin by a proper feedback law. For constant reference signals we establish a connection between the convergence speed of the stabilization problem and tracking convergence which turns out to be independent of the reference signal. We also show that the tracking controller can be inferred from the stabilizing (possibly nonlinear) controller associated with the domain of attraction. We refer the reader to the full version [17], where the continuous-time case, proofs and extensions are presented.

## 1 Introduction

We consider linear systems with state and control constraints and we say that a certain convex and compact set including the origin in its interior is a domain of attraction to the origin if there exists a feedback control such that for any initial state in this set the state is driven asymptotically to the origin without constraints violation. Several previous references have dealt with the construction of such domains (see for instance [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]).

Here we consider the problem of tracking a reference signal including the zero one. We assume that the system is square (i.e. it has as many inputs as outputs) and that it is if free of invariant zeros at one.

We define a set of reference vectors which are constraints–compatible, in the sense that they are associated with state vectors which are in the interior of the domain of attraction to the origin and to feasible input vectors. The only condition required for the signal to be tracked is that it asymptotically converges to a value which is in this set. A signal satisfying the above requirement will be said admissible. We stress that the signal, during the transient, may assume values outside this set and it is a goal of the control to avoid constraints violations. For the zero reference signal, such a goal can be achieved only if the initial state is inside a domain of attraction to the origin. Any initial state from which we can track asymptotically an admissible reference signal is said to belong to a tracking domain of attraction. The question which hence arises is whether there are initial states inside the domain of attraction to the origin from which we cannot solve the problem of tracking an admissible reference signal. Put in other words this amounts to verify whether or not the largest tracking domain of attraction is a proper subset of the assigned domain of attraction. We show that the answer is negative.

The basic ideas we develop here are related to previously published results [12] [13], [14], [15], [16]. The main difference relies on the fact that in those references it is assumed that a stabilizing linear nominal compensator (often referred to as pre-compensator) is applied to the system. To this nominal compensator the so called reference governor is added which is a device which possibly attenuates the effect of the reference signal in order to avoid violations. This is basically equivalent to consider a stable system (i.e. a system to which a pre-compensator is applied) and to manage the reference for this system. Although this assumption is quite reasonable, it turns out that the resulting constructed invariant sets depend on the pre–compensator. Therefore, an unsuitable choice of the compensator can produce a very small domain of attraction.

Conversely, here *we do not assume the existence of any pre–compensator* and thus we may take as domain of attraction the largest one in absolute in the sense that, if the initial state is outside that region, constraints violation cannot be avoided, even for zero reference signal. We show that this is a tracking domain of attraction, in the sense that there is a (possibly non-linear) compensator, which is not a-priory fixed, that solves the tracking problem for any initial state in this region. Furthermore, we show that the control strategy can be inferred from the original stabilizing controller associated with the given domain of attraction. We use as a technical support a Lyapunov function suitably constructed by "reshaping" the one associated with such a domain.

Finally we show that, for constant reference signals and symmetric domains, the speed of convergence can be estimated a priori from the speed of convergence achieved by the stabilizing controller and it does not depend on the particular reference.

The results presented in this paper have been previously presented in the paper [17] to which we refer the reader for details and proofs.

## 2 Definitions and problem statement

In the sequel we denote by $int\mathcal{P}$ the interior of a set $\mathcal{P}$ and by $\partial\mathcal{P}$ its boundary. With the term C-set we denote a convex and compact set containing the origin as an interior point. It is known that any C-set $\mathcal{P}$ induces a positively homogeneous convex function which is known as Minkowski functional (see [18])

$$\psi_{\mathcal{P}}(x) = \inf\{\xi \geq 0 : \frac{x}{\xi} \in \mathcal{P}\} \tag{1}$$

The function $\psi_{\mathcal{P}}(x) \geq 0$ is such that $\psi_{\mathcal{P}}(x) = 0$ iff $x = 0$, and $\psi_{\mathcal{P}}(\lambda x) = \lambda\psi_{\mathcal{P}}(x)$ for any $\lambda \geq 0$. If $\mathcal{P}$ is 0-symmetric, (i.e. $x \in \mathcal{P}$ implies $-x \in \mathcal{P}$) then $\psi_{\mathcal{P}}$ is a norm. The index $\mathcal{P}$ will be omitted for brevity when the inducing set $\mathcal{P}$ of $\psi_{\mathcal{P}}$ is clear from the context.

We consider discrete-time square systems

$$\begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array} \tag{2}$$

The vector $y(t) \in \mathbb{R}^q$ is the system output, the vector $x(t) \in \mathbb{R}^n$ is the system state and $u(t) \in \mathbb{R}^q$ is the control input. We assume that $(A, B)$ is stabilizable and that $x$ and $u$ are subject to the constraints

$$x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U}$$

where $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^q$ are assigned C-sets.

Due to the presence of the constraints, the solution of the stabilization problem implies restrictions on the admissible initial conditions.

**Definition 2.1** *The C-set $\mathcal{S} \subset \mathcal{X}$ is a domain of attraction for the system (2) if there exists a continuous feedback control function $u(t) = \phi(x(t))$ such that any trajectory $x(t)$ with initial condition $x(0) \in \mathcal{S}$ is such that $x(t) \in \mathcal{S}$, $u(t) \in \mathcal{U}$ and*

$$\lim_{t\to\infty} x(t) = 0$$

It is well known that the above definition is fundamental for the stabilization problem under constraints. An initial condition can be driven to the origin without constraints violations if and only if it belongs to a domain of attraction. However, in practice, simple convergence is not enough but it is important to provide an index of the speed of convergence to the origin. Thus we introduce the following definition.

**Definition 2.2** *The C-set $\mathcal{S} \subset \mathcal{X}$ is a domain of attraction with speed of convergence $\lambda$ for the system (2) if there exists $0 \leq \lambda < 1$ and a continuous feedback control function $u(t) = \phi(x(t))$ such that any trajectory $x(t)$ with initial condition $x(0) \in \mathcal{S}$ is such that*

$$\psi_{\mathcal{S}}(x(t)) \leq \lambda^t \psi_{\mathcal{S}}(x(0))$$

*If we take $\lambda = 1$, the set $\mathcal{S}$ is simply said to be positively-invariant (often referred to as controlled–invariant). We say that $\mathcal{S}_\lambda \subset \mathcal{X}$ is the largest domain of attraction if for any domain of attraction $\mathcal{S}$ in $\mathcal{X}$ with speed of convergence $\lambda$ we have $\mathcal{S} \subset \mathcal{S}_\lambda$.*

A stabilizable system always admits a domain of attraction $\mathcal{P} \subset \mathcal{X}$. The knowledge of such a domain is fundamental in the stabilization problem under constraints, since if $x(0) \in \mathcal{P}$, then the conditions $x(t) \in \mathcal{P} \subset \mathcal{X}$, for $t \geq 0$, and $x(t) \to 0$, as $t \to \infty$ can be assured. Thus, once we have computed a domain of attraction $\mathcal{P}$ to solve the problem (possibly the largest one [2], [19], [20]), we can replace the constraint $x(t) \in \mathcal{X}$ by the new constraint

$$x(t) \in \mathcal{P}.$$

In this paper we deal with the problem of tracking a certain class of reference signals. To this aim we make the assumption that the system is free from zeros at one.

**Assumption 1** *The square matrix below is invertible.*

$$M_d = \begin{bmatrix} A - I & B \\ C & D \end{bmatrix}$$

Under Assumption 1 the system has the property that for any constant reference $r \in \mathbb{R}^q$ there is an unique state–input equilibrium pair $(\bar{x}, \bar{u})$ such that the corresponding equilibrium output is $r$. Such a pair is the unique solution of the equation

$$M_d \begin{bmatrix} \bar{x}(r) \\ \bar{u}(r) \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

Thus we can define the set of admissible constant reference vectors

$$\mathcal{R} = \{r \in \mathbb{R}^q : \ \bar{u}(r) \in \mathcal{U}, \ \bar{x}(r) \in \mathcal{P}\}.$$

Being $\mathcal{P}$ and $\mathcal{U}$ both bounded, $\mathcal{R}$ is also bounded. The set $\mathcal{R}$ is of fundamental importance. It is the set of all reference vectors for which the corresponding input and state equilibrium pairs do not violate the constraints $\bar{u} \in \mathcal{U}$ and $\bar{x} \in \mathcal{P}$. While the reason for imposing the former is obvious, the second deserves some explanations. Indeed we have imposed $x \in \mathcal{X}$. However, note that if $\bar{x}(r)$ is not included in a domain of attraction then the condition $x(t) \to \bar{x}(r)$ can cause a violation of the constraints if, for instance, after a sufficiently long period, the reference $r(t)$ switches to zero.

We are now going to introduce the set of all the admissible signals to be tracked, formed by the signals $r(t)$ having a finite limit $r_\infty$, with the condition that $r_\infty$ has some admissibility condition with respect to the constraints.

**Definition 2.3** *Assume that a small $0 < \epsilon < 1$ is given. A reference signal $r(t)$ is admissible if it is such that*

$$\lim_{t\to\infty} r(t) = r_\infty \in (1-\epsilon)\mathcal{R} \doteq \mathcal{R}_\epsilon.$$

The parameter $\epsilon$, as we will see later, is introduced to avoid singularities in the control. Such $\epsilon$ may be arbitrarily small and thus it does not practically affect the problem. We stress that an admissible reference signal *does not need to assume its values in $\mathcal{R}_\epsilon$*, but only its limit $r_\infty$ needs to do this. Now we can state the following basic definition.

**Definition 2.4** *The set $\mathcal{P} \subset \mathcal{X}$ is a tracking domain of attraction if there exists a (possibly nonlinear) feedback control*

$$u(t) = \Phi(x(t), r(t))$$

*such that for any $x(0) \in \mathcal{P}$ and for every admissible reference signal $r(t)$*

**i** *$x(t) \in \mathcal{P}$ and $u(t) \in \mathcal{U}$,*

**ii** *$y(t) \to r_\infty$ as $t \to \infty$.*

Since $r(t) = 0$ is an admissible reference signal, any tracking domain of attraction is a domain of attraction. The main result of this paper is that of showing that every domain of attraction $\mathcal{P}$ is also a tracking domain of attraction. The importance of this assertion lies in the fact that the tracking problem can be solved once one has found a domain of attraction. Since the latter operation is a well established topic this allows for the solution of a more general problem.

**Remark 2.1** *Note that under Assumption 1 the condition $y(t) \to r_\infty$ as $t \to \infty$ is equivalent to the two conditions $x(t) \to \bar{x}_\infty \doteq \bar{x}(r_\infty)$ and $u(t) \to \bar{u}_\infty \doteq \bar{u}(r_\infty)$.*

## 3 Main results

We will now introduce the functions which will be subsequently used for tracking every admissible reference signal. Suppose that a C-set $\mathcal{P}$, which is a domain of attraction, is given and for every $\bar{x} \in int\mathcal{P}$ and $x \in \mathcal{P}$ we introduce the following function:

$$\Psi_{\mathcal{P}}(x,\bar{x}) = \inf\{\alpha > 0 \ : \ \bar{x} + \frac{1}{\alpha}(x - \bar{x}) \in \mathcal{P}\} \qquad (3)$$

It is immediate that the just introduced function $\Psi$ recovers the values of $\psi$ when $\bar{x} = 0$, say $\Psi_{\mathcal{P}}(x, 0) = \psi_{\mathcal{P}}(x)$. For fixed $\bar{x}$, $\Psi_{\mathcal{P}}(x,\bar{x})$ is convex. Furthermore, the function $\Psi_{\mathcal{P}}(x,\bar{x})$ for $(x,\bar{x}) \in \mathcal{P} \times int\mathcal{P}$ is such that

$$\begin{aligned} \Psi_{\mathcal{P}}(x,\bar{x}) &= 0 \ \text{ iff } \ x = \bar{x} && (4)\\ \Psi_{\mathcal{P}}(x,\bar{x}) &< 1 \ \text{ iff } \ x \in \mathcal{P} && (5)\\ \Psi_{\mathcal{P}}(x,\bar{x}) &= 1 \ \text{ iff } \ x \in \partial\mathcal{P} && (6)\end{aligned}$$

A sketch of the level surfaces of the function $\Psi_{\mathcal{P}}(x,\bar{x})$ for fixed $\bar{x}$ is in Fig. 1. One further relevant property of this function is that $\Psi_{\mathcal{P}}$ is Lipschitz in $x$ and positively homogeneous of order one with respect to the variable $z = x - \bar{x} \in \mathbb{R}^n$, $\bar{x} \in int\mathcal{P}$ i.e

$$\Psi_{\mathcal{P}}(\xi z + \bar{x}, \bar{x}) = \xi \Psi_{\mathcal{P}}(z + \bar{x}, \bar{x}) \qquad (7)$$

In view of property (4) we have that the function is a suitable Lyapunov candidate for tracking, and from (5-6) we have that this function is suitable to prevent constraints violations, as we will show later.

Let us now consider the function $\Psi_{\mathcal{P}}(x,\bar{x})$ and for every $x \in \mathcal{P}$ and $\bar{x} \in int\mathcal{P}$, with $x \neq \bar{x}$, set

$$\tilde{x} \doteq \bar{x} + (x - \bar{x})\frac{1}{\Psi_{\mathcal{P}}(x,\bar{x})} \in \partial\mathcal{P}.$$

The vector $\tilde{x}$ is the intersection of $\partial\mathcal{P}$ with the half line starting from $\bar{x}$ and passing through $x$ (see Fig. 1). Assume that $\mathcal{P}$ is a given domain of attraction and

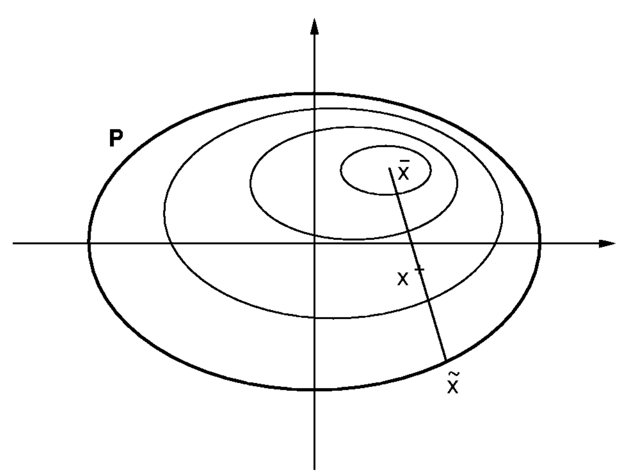


**Figure 1:** The function $\Psi_{\mathcal{P}}(x,\bar{x})$ for fixed $\bar{x}$

that $\phi(x)$ is the control law associated with this set. As shown in [19], [20] there exists always a positively homogeneous control of order one which can be associated with such a domain. Therefore, without restriction, we introduce the following assumption.

**Assumption 2** *The stabilizing control law $\phi(x)$ associated with the domain of attraction $\mathcal{P}$ is Lipschitz and positively homogeneous of order 1, i.e. $\phi(\alpha x) = \alpha\phi(x)$ for $\alpha \geq 0$.*

The next step in the derivation of a feedback control law $\Phi(x, r)$ is the definition of a saturation map $\Gamma : \mathbb{R}^q \to \mathcal{R}_\epsilon$ as follows:

$$\Gamma(r) = \begin{cases} r\psi_{\mathcal{R}_\epsilon}(r)^{-1} & \text{when} \quad \psi_{\mathcal{R}_\epsilon}(r) > 1 \\ r & \text{otherwise} \end{cases}$$

say

$$\Gamma(r) = r * sat\left(\frac{1}{\psi_{\mathcal{R}_\epsilon}(r)}\right)$$

where $sat(\cdot)$ is the saturation function. Note that $\Gamma(r)$ is the identity if we restrict $r \in \mathcal{R}_\epsilon$. Conversely, for $r \notin \mathcal{R}_\epsilon$, $\Gamma(r)$ is the intersection of $\partial\mathcal{R}_\epsilon$ and the segment having extrema 0 and $r$.

The control we propose has the following form

$$\Phi(x, r) = \phi(\tilde{x})\Psi_{\mathcal{P}}(x, \bar{x}) + (1 - \Psi_{\mathcal{P}}(x, \bar{x}))\bar{u} \qquad (8)$$

where

$$\begin{bmatrix} \bar{x}(\bar{r}) \\ \bar{u}(\bar{r}) \end{bmatrix} \doteq M_d^{-1} \begin{bmatrix} 0 \\ \bar{r} \end{bmatrix} \qquad (9)$$

and

$$\bar{r} = \Gamma(r). \qquad (10)$$

Note that, for $r \in \mathcal{R}_\epsilon$, (10) does not play any role. Note also that, since $\bar{r} = \Gamma(r) \in \mathcal{R}_\epsilon$, then $\bar{x} \in int\mathcal{P}$, thus the term $\Psi_{\mathcal{P}}(x, \bar{x})$ in (8) is defined. However, the expression (8) is not defined for $x = \bar{x}$ because of the critical term $\phi(\tilde{x})\Psi_{\mathcal{P}}(x, \bar{x})$. Nevertheless, in view of the homogeneity of $\phi$ and from the expression of $\tilde{x}$, we have that

$$\begin{aligned} \phi(\tilde{x})\Psi_{\mathcal{P}}(x, \bar{x}) &= \phi(\Psi_{\mathcal{P}}(x, \bar{x})\tilde{x}) = \\ &= \phi\left(x + (\Psi_{\mathcal{P}}(x, \bar{x}) - 1)\bar{x}\right) \end{aligned} \qquad (11)$$

$\phi(\tilde{x})\Psi_{\mathcal{P}}(x, \bar{x}) \to 0$ as $x \to \bar{x}$ so that we can extend the function by continuity by assuming $\phi(\tilde{x})\Psi_{\mathcal{P}}(\bar{x}, \bar{x}) = 0$.

The introduced control law inherits most of the properties from $\phi(x)$ according to the next proposition which assures existence and uniqueness of the solution of (2) when the control $\Phi(x, r)$ is used, provided that the admissible reference signal $r(t)$ is measurable.

**Proposition 1** *Suppose $\phi(x)$ is Lipschitz and homogeneous of order 1. Then $\Phi(x, r) : \mathcal{P} \times \mathbb{R}^q \to \mathcal{U}$ defined as in (8)-(10) is continuous and it is Lipschitz w.r.t. $x$.*

To have an idea on how this control works, note that as long as the condition

$$\Psi_{\mathcal{P}}(x(t), \bar{x}(t)) \leq 1 \qquad (12)$$

is satisfied, we have that $x(t) \in \mathcal{P}$. Moreover, for $\bar{x} \in int\mathcal{P}$, the control is just a *convex combination* of the control $\bar{u}(\bar{r})$ and $\phi(\tilde{x})$. By construction, $\bar{u}(\bar{r}) \in \mathcal{U}$ and $\phi(\tilde{x}) \in \mathcal{U}$ thus $\Phi(x, r) \in \mathcal{U}$.

Our effort will be devoted in proving that the proposed control law guarantees (12) as well as the limit condition

$$\Psi_{\mathcal{P}}(x(t), \bar{x}(r_\infty)) \to 0, \qquad (13)$$

where $\bar{x}(r_\infty)$ is the steady state associated with $r_\infty \in \mathcal{R}_\epsilon$ (note that $\Gamma(r_\infty) = r_\infty$). Indeed such a limit condition implies $x(t) \to \bar{x}(r_\infty)$ and, from (8) and (11), $\Phi(x(t), r(t)) \to \Phi(\bar{x}(r_\infty), r_\infty) = \bar{u}(r_\infty)$. Therefore if (13) holds, we have that $y(t) \to r_\infty$.

For evident practical reasons, we introduce an extended concept of speed of convergence, appropriate for the condition (13). In the discrete-time case, given a *fixed* $\bar{x} \in int\mathcal{P}$, we say that the speed of convergence is $\lambda < 1$ if

$$\Psi_{\mathcal{P}}(Ax + Bu, \bar{x}) \leq \lambda\Psi_{\mathcal{P}}(x, \bar{x}).$$

We start by considering a special case, namely the one in which the reference signal is constant. In this case we can show that, if we have a domain of attraction to the origin with a certain speed of convergence, we can achieve the tracking goal without constraints violation for all the initial states in such domain. Furthermore, for symmetric domains, we can guarantee a speed of convergence which is *independent of the reference input.*

**Theorem 3.1** *Let $\mathcal{P}$ be a domain of attraction with speed of convergence $\lambda$ for system (2) associated with the control $\phi(x)$ satisfying Assumption 2. Then, for every admissible constant reference signal $r(t) = \bar{r}$, the control law (8)–(9), is such that for every initial condition $x(0) \in \mathcal{P}$ we have that $x(t) \in \mathcal{P}$ and $u(t) \in \mathcal{U}$, for every $t \geq 0$ and $lim_{t\to\infty}y(t) = \bar{r}$. Moreover, if $\mathcal{P}$ is 0-symmetric, the speed of convergence $\lambda_{TR} = \frac{\lambda+1}{2}$ is guaranteed.*

The proposed control law can be successfully used even when the reference $r(t)$ is allowed to vary provided that it is admissible according to Definition 2.3.

**Theorem 3.2** *Let $r(t)$ be admissible as in Definition 2.3. Any domain of attraction $\mathcal{P}$, with speed of convergence $0 \leq \lambda < 1$, for system (2) is a tracking domain of attraction. Moreover, the control law in (8)–(10) assures the conditions (i) and (ii) of Definition 2.4.*

**Remark 3.1** *If we consider a constant reference $r \in \mathcal{R}_\epsilon$ and the corresponding steady state vectors derived* *from $\bar{x}$ and $\bar{u}$ by means of (9), then we can apply a state and control translation by considering the system $\delta\hat{x} = A\hat{x}+B\hat{u}$ with the new constraints $\hat{u} = u-\bar{u} \in \mathcal{U}-\bar{u} = \hat{\mathcal{U}}$ and $\hat{x} = x - \bar{x} \in \mathcal{X} - \bar{x} = \hat{\mathcal{X}}$. From this algebraic point of view, our result amounts to prove that the largest domain of attraction of the translated problem is just achieved by translating the original largest domain of attraction as $\hat{\mathcal{P}} = \mathcal{P} - \bar{x}$.*

## 4 Example

Consider the discrete-time system

$$x(k+1) = \begin{bmatrix} 1 & .3 \\ -1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} .5 \\ 1 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} -.1 & .3 \end{bmatrix} x(k)$$

with state and control constraint given by $\mathcal{X} = \{x : \|x\|_\infty \le 1\}$, $\mathcal{U} = \{u : |u| \le 1\}$ For this system we computed a symmetric polyhedral domain of attraction $\mathcal{P} = \{x : \|Fx\|_\infty \le 1\}$ with speed of convergence $\lambda = .9$ (the boundary of $\partial\mathcal{P}$ is depicted in Figure 2),

$$F = \begin{bmatrix} 0 & 1 \\ 1.8160 & -0.2421 \\ 1.3140 & 0.1932 \end{bmatrix}$$

In this case the constraints on the reference value derive from the constraint that $\bar{x} \in \mathcal{P}$ and translate in $|\bar{r}| \le .27$. The linear variable structure controller associated with $\mathcal{P}$ is given by $u(x) = k_i x$ with $i = \arg\max_j |f_j x|$, where $f_i$ and $k_i$ are the $i$-th rows of $F$ and $K$ respectively, where

$$K = \begin{bmatrix} -0.4690 & -0.7112 \\ -0.6355 & -0.7806 \\ -1.3140 & -0.1931 \end{bmatrix}.$$

The function $\Psi(x, \bar{x})$, is given by:

$$\Psi(x, \bar{x}) \doteq \max_i \frac{F_i(x-\bar{x})}{1 - F_i\bar{x}}, \quad x \in (1-\epsilon)\mathcal{P}$$

where $\epsilon > 0$ is a small number. Note that $1 - F_i\bar{x} \ge \epsilon$, so singularities are avoided. We applied the control law with $\epsilon = .01$, starting from zero initial state, for the reference signal

$$r(t) = 0.2 + 0.4 * sin(0.01 * t) * e^{-0.005*t}.$$

Figure 3 depicts the time evolution of the output and the reference value and Figure 2 shows the corresponding state-space trajectory.

## 5 Conclusions

In this paper it was shown that every set of initial states which can brought to the origin while assuring that no control and state constraints are violated is indeed a tracking domain. This amounts to say that it is possible to track a given reference value only if its corresponding state and input equilibrium pair belongs to a domain of attraction, whereas if this condition is not met the state is not "trackable". This result was provided without assuming the existence of a stabilizing compensator, by furnishing a Lyapunov function which is derived by the domain of attraction and which allow to derive a tracking strategy which avoids state and input constraints yet guaranteeing an a-priori speed of convergence. Future directions in this area concern the possibility of deriving stabilizing tracking strategies capable of maintaining the tracking error within pre-specified bounds and of improving the speed of convergence.

The presented approach can be applied to systems with control constraints only by computing the domain of attraction obtained by considering a fictitious (sufficiently large) state constraints region and by applying the tech-

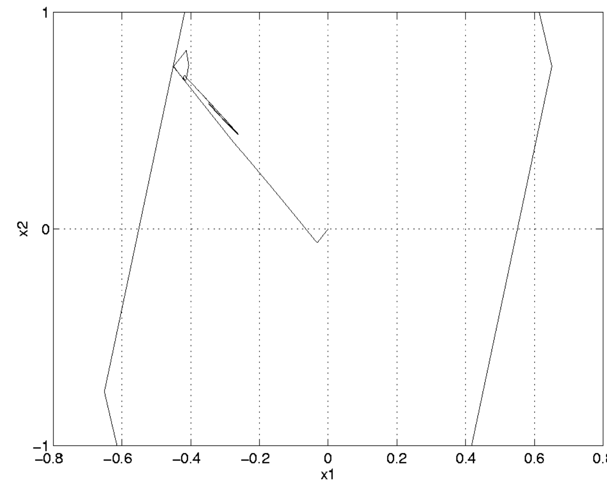


**Figure 2:** Domain of attraction and state-space evolution

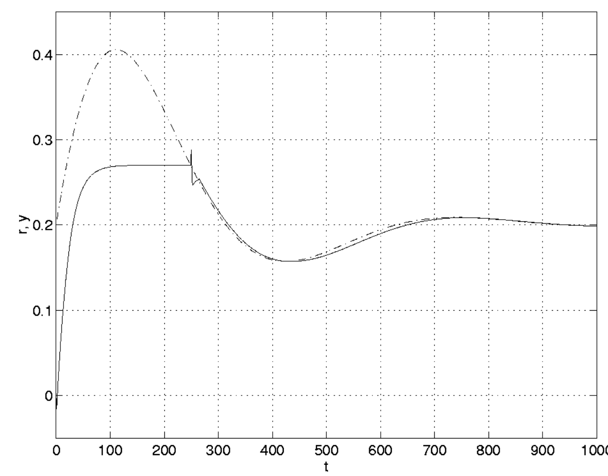


**Figure 3:** Output and reference time-evolution

niques suggested in [5], [8], [22]. We finally remind that for systems with no state constraints and with no eigenvalues outside the closed unit disk a completely different approach can be used for global stabilization [25] or semi–global stabilization [26], [27]. In this case those approaches can be successfully applied to the solution of the tracking problem with an arbitrarily large initial condition set.

## References


[1] Benzaouia A. and Burgat C. *The regulator problem for a class of linear systems with constrained control.* Syst. & Contr. Letters, Vol.10, pp.357–363, 1988.

[2] Blanchini F. *Set invariance in control – a survey.* Automatica, Vol35, No.11, pp.1747–1768, 1999.

[3] Burgat C. and Tarbouriech S. *Positively invariant sets for constrained continuous-time systems with cone properties.* IEEE Trans. Automat. Contr., Vol.39, No.2, pp.401–405, 1994.

[4] Castelan E.B. and Hennet J.C. *Eigenstructure assignment for state constrained linear continuous time systems.* Automatica, Vol.28, No.3, pp. 605–611, 1992.

[5] Gutman P.O. and Cwikel M. *Admissible sets and feedback control for discrete-time linear systems with bounded control and states.* IEEE Trans. Automat. Contr., Vol.31, No.4, pp.373–376, 1986.

[6] Gutman P.O. and Hagander P. *A new design of constrained controllers for linear systems.* IEEE Trans. Automat. Contr., Vol.30, pp.22–33, 1985.

[7] Keerthy S.S. and Gilbert E.G. *Computation of minimum-time feedback control laws for discrete-time systems with state and control constraints.* IEEE Trans. Automat. Contr., Vol.32, No.5, pp.432–435, 1987.

[8] Lasserre J. B. *Reachable, controllable sets and stabilizing control of constrained systems.* Automatica, Vol.29, No.2, pp.531–536, 1993.

[9] Lin Z., Stoorvogel A.A., and Saberi A. *Output regulation for linear systems subject to input saturation.* Automatica, Vol.32, No.1, pp.29–47, 1996.

[10] Sznaier M. and Damborg M.J. *Heuristically enhanced feedback control of constrained discrete-time systems.* Automatica, Vol.26, No.3, pp.521–532, 1990.

[11] Vassilaki M. and Bitsoris G. *Constrained regulation of linear continuous-time dynamical systems.* Syst. & Contr. Letters, Vol.47, pp.247-252, 1989.

[12] Angeli D. *Control strategy for constrained systems.* Doctor of Philosophy Dissertation, Dipartimento di Sistemi e Informatica, Università di Firenze, Italy, Dec. 1999.

[13] Bemporad A., Casavola A., and Mosca E. *Nonlinear control of constrained linear systems with predictive reference management.* IEEE Trans. Automat. Contr., Vol.42, No.3, pp.340–349, 1997.

[14] Gilbert E.G. and Tan K.T. *Linear systems with state and control constraints: the theory and the application s of the maximal output admissible sets.* IEEE Trans. Automat. Contr., Vol.36, No.9, pp. 1008-1020, 1991.

[15] Gilbert E.G., Kolmanowsky I., and Tan K. T. *Discrete–time reference governors and the nonlinear control of systems with state and control constraints.* Int. J. Rob. Nonlin. Contr., pp. 487-504, 1995.

[16] Graettinger T.J. and Krogh B. H. *On the computation of reference signal constraints for guaranteed tracking performance.* Automatica, Vol.28, pp. 1125–1141, 1992.

[17] Blanchini F. and Miani S. *Any Domain of Attraction for a Linear Constrained System is a Tracking Domain of Attraction.* SIAM J. Contr. Optim, Vol.38, No.3, pp. 971–994, 2000.

[18] Luenberger D. G. *Optimization by vector space methods.* John Wiley & Sons, New York, 1969.

[19] Blanchini F. and Miani S. *Constrained stabilization for continuous-time systems.* Syst. & Contr. Letters, Vol.28, No.2, pp.95–102, 1996.

[20] Blanchini F. *Ultimate boundedness control for discrete-time uncertain system via set-induced Lyapunov functions.* IEEE Trans. Automat. Contr., Vol.39, No.2, pp.428–433, 1994.

[21] Boyd S., El Ghaoui L., Feron E., and Balakrishnan V. *Linear Matrix Inequalities in System and Control Theory.* SIAM Studies in Applied Mathematics, Philadelphia, 1994.

[22] Blanchini F. and Miani S. *Constrained stabilization via smooth Lyapunov functions.* Syst. & Contr. Letters, Vol.35, No.3, pp.155–163, 1998.

[23] Clarke F. *Optimization and Nonsmooth Analysis.* John Wiley & Sons, New York, 1983.

[24] Rockafellar R. T. *Convex Analysis.* Princeton University Press, Princeton, New Jersey, 1972.

[25] Sussmann H.J., Sontag E.D., and Yang Y.D. *A general result on the stabilization of linear systems using bounded controls.* IEEE Trans. Automat. Contr., Vol.39, No.12, pp.2411–2425, 1994.

[26] Saberi A., Lin Z., and Teel A.R. *Control of linear systems with saturating actuators.* IEEE Trans. Automat. Contr., Vol.41, No.3, pp.368–378, 1996.

[27] Teel A.R. *Global stabilization and restricted tracking for multiple integrators with bounded controls.* Syst. & Contr. Letters, Vol.18, No.3, pp.165–171, 1992.