Linear Systems with Prescribed Similarity Structural Invariants[1]

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Abstract

The problem of the existence of Linear Systems $\dot{x}(t) = Ax(t) + Bu(t)$ with prescribed structural invariants for system similarity is studied. Namely, we solve the problem of the existence of such a system with prescribed Controllability indices, Hermite indices and invariant factors when the invariant factors of A (which are also invariants under system similarity) are given.

1 Introduction, Notation and Preliminary Results

Let us assume that $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{n\times m}$, $\mathbb{F}$ being the field of real or complex numbers. In the study of the structure of linear control systems

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$

under systems similarity -(TAT^{-1}, TB), T invertible- several systems of invariants can be found. In [Popov (1978)] Popov gave a complete system of independent invariants for system similarity but this is not the only complete system of invariants that one can obtain (see for example [Kailath (1980), p. 494], [Chen (1984), p. 191] or [Zaballa (1997)]). An interesting feature of all these systems is that they are formed by two types of subsystems that, following [Mac Duffe (1934), p. 48] we can call *invariants of structure* and *numerical invariants*. The former ones are nonnegative integers and the later ones are real or complex numbers depending on the underlying field where the elements of A and B are.

The most mentioned structural invariants are the controllability and the Hermite indices ([Kailath (1980), ch. 6],[Chen (1984)]). Both come up when searching for a basis of the controllability subspace; i.e. the one generated by the columns of the controllability matrix: $\mathcal{C}(A,B) = [B \;\; AB \; \dots \; A^{n-1}B]$, in the following table:

$$\begin{array}{lllll} b_1 & b_2 & b_3 & \dots & b_m \\ Ab_1 & Ab_2 & Ab_3 & \dots & Ab_m \\ A^2b_1 & A_2b_2 & A^2b_3 & \dots & A^2b_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{n-1}b_1 & A^{n-1}b_2 & A^{n-1}b_3 & \dots & A^{n-1}b_m \end{array}$$

where $b_i \in \mathbb{F}^{n\times 1}$ is the i-th column of B. If $rank\mathcal{C}(A,B) = r$ and we select by columns (from left to right) the first r linearly independent columns of the table and we write them as:

$$b_1, \dots, A^{h_1-1}b_1, \dots, b_m, \dots, A^{h_m-1}$$

then $h_1, h_2, \dots h_m$ are the Hermite indices of the system. Actually, as pointed out in [Kailath (1980), p. 476], these indices are the degrees of the polynomials appearing in the diagonal of the Hermite normal form of the right denominator of the transfer $(sI_n - A)^{-1}B$. This is why they were called Hermite indices in [Zaballa (1997)] (see also [Siparis et*al.* (1991)]). If we proceed similarly but searching by rows from top to bottom and we rearrange the indices in nonincreasing order then we come up with the controllability indices [Brunovsky (1970)] or input structural indices as they are called in [Basile *et al.* (1992), p. 156].

It is also worth noting that system similarity implies similarity of the corresponding state matrices. It is well known (see for example [Gantmacher (1966)]) that two square matrices A_1 and A_2 are similar if and only if their corresponding characteristic matrices $sI - A_1$ and $sI - A_2$ are equivalent; i. e. they have the same invariant factors (the invariant factors of A being those of $sI - A$ as a polynomial matrix). Thus the invariant factors of the state matrix are invariant under system similarity. Furthermore, if we call invariant factors of (A, B), or of system (1), those of the polynomial matrix $[sI_n - A \quad B]$, then these polynomials are also invariants under system similarity. Notice that the invariant factors of system (1) are all equal to 1 if and only if the system is controllable, [Rosenbrock (1970)].

The well known Rosenbrock's theorem [Rosenbrock (1970)] on eigenstructure assignment

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under state feedback can be seen as a result on the relationship between the invariant factors of the state matrix A, i.e., those of $sI_n - A$, and the controllability indices of a controllable system (A, B). Similarly the generalization of Rosenbrock's theorem to noncontrollable systems, [Zaballa (1988), Zaballa (1989)], provides a characterization of the possible controllability indices and invariant factors of system (A, B) for a given matrix A. Following with these ideas the study of the relationship between the controllability and the Hermite indices of a given pair as well as the relationship between the invariant factors of A and the Hermite indices and invariant factors of (A, B) for all possible choices of B was carried out in [Zaballa (1997)]. In this paper the four systems of invariants are considered together. Namely, we will deal with the following

Problem 1 *Let $A \in \mathbb{F}^{n\times n}$ and $\alpha_1 \mid \ldots \mid \alpha_n$ its invariant factors. Let $k_1 \geq \ldots \geq k_m > 0$ be positive integers, $h_1 \geq \ldots \geq h_m \geq 0$ nonnegative integers and $\gamma_1 \mid \ldots \mid \gamma_n$ monic polynomials. Under what conditions does there exist a matrix $B \in \mathbb{F}^{n\times m}$ such that (A, B) has $k_1, \ldots, k_m$ as controllability indices, $h_1, \ldots, h_m$ as Hermite indices and $\gamma_1, \ldots, \gamma_n$ as invariant factors?*

Our results will be of an algebraic nature and so we will not impose any restrictions on $\mathbb{F}$ that from now on will be considered arbitrary. We will use greek letters to denote polynomials, $\alpha \mid \beta$ will mean that α divides β and $d(\alpha)$ will be the degree of α.

As said before the controllability indices are invariant under system similarity but in the case when (A, B) is controllable, they form a complete system of invariants for the feedback equivalence. Following Brunovsky, [Brunovsky (1970)], two matrix pairs $(A, B), (\hat{A}, \hat{B}) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ are said to be feedback equivalent if there are nonsingular matrices $P \in \mathbb{F}^{n\times n}$ and $Q \in \mathbb{F}^{m\times m}$ and a matrix $R \in \mathbb{F}^{m\times n}$ such that $(\hat{A}, \hat{B}) = (PAP^{-1} + PBR, PBQ)$. If (A, B) is not completely controllable a complete system of invariants is given by the controllability indices and the invariant factors of (A, B).

The paper is organized as follows: in the next section we give a solution to the posed problem when the system is controllable, and in the following one when this condition is not satisfied.

If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are two partitions of nonnegative integers arranged in non increasing order, following ([Hardy *et al.* (1967)]) we say that a is majorized by b, and we write $a \prec b$ if

$$\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j, \; 1 \leq k < n$$

and

$$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j$$

From now on we will deal with Problem 1. A consequence of next Lemma is that we can substitute A by any matrix in its similarity class.

Lemma 1 .- *Let $A \in \mathbb{F}^{n\times n}$. Let $k_1 \geq \ldots k_m > 0$ and $h_1 \geq \ldots \geq h_m \geq 0$ be nonnegative integers and $\gamma_1, \ldots \gamma_n$ monic polynomials. Suppose that $A \overset{s}{\sim} \hat{A}$. Then there exists $B \in \mathbb{F}^{n\times m}$ such that (A, B) has $k_1, \ldots, k_m$ as controllability indices, $h_1, \ldots, h_m$ as Hermite indices and $\gamma_1, \ldots, \gamma_n$ as invariant factors if and only if there exists $\hat{B} \in \mathbb{F}^{n\times m}$ such that $(\hat{A}, \hat{B})$ has $k_1, \ldots, k_m$ as controllability indices, $h_1, \ldots, h_m$ as Hermite indices and $\gamma_1, \ldots, \gamma_n$ as invariant factors.*

Given a controllable matrix pair $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ we are going to give a canonical form for the similarity of matrix pairs associated to the Hermite indices, (see [Kailath (1980)]).

Lemma 2 .- *Let $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ a controllable pair and let $h_1 \geq \ldots \geq h_p > 0 = h_{p+1} = \ldots = h_m$ its Hermite indices. Then there exists a nonsingular matrix $P \in \mathbb{F}^{n\times n}$ such that*

$$(PAP^{-1}, PB) = (A_c, B_c)$$

where

$$A_c = (A_{ij}) \begin{array}{l} i = 1, \ldots, p \\ j = 1, \ldots, p \end{array} \qquad B_c = (B_{ij}) \begin{array}{l} i = 1, \ldots, p \\ j = 1, \ldots, m \end{array}$$

are the blocks

$$A_{ii} = \left(\begin{bmatrix} 0 & 0 & \ldots & 0 & x_{ii0} \\ 1 & 0 & \ldots & 0 & x_{ii1} \\ 0 & 1 & \ldots & 0 & x_{ii2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & x_{iih_i-1} \end{bmatrix} \right) \in \mathbb{F}^{h_i \times h_i},$$

$1 \leq i \leq p$,

$$A_{ij} = \left(\begin{bmatrix} 0 & 0 & \ldots & 0 & x_{ji0} \\ 0 & 0 & \ldots & 0 & x_{ji1} \\ 0 & 0 & \ldots & 0 & x_{ji2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & x_{jih_i-1} \end{bmatrix} \right) \in \mathbb{F}^{h_i \times h_j},$$

$1 \leq i < j \leq p$,

$A_{ij} = 0 \in \mathbb{F}^{h_i \times h_j}$, $1 \leq j < i \leq p$.

$B_{ii} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T \in \mathbb{F}^{h_i \times 1},\ 1 \le i \le p$

$B_{ij} = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}^T \in \mathbb{F}^{h_i \times 1},\ \le i,j \le p,\ i \ne j,$

$B_{ij} = \begin{bmatrix} x_{ji0} & x_{ji1} & \dots & x_{jih_i-1} \end{bmatrix}^T \in \mathbb{F}^{h_i \times 1},$

$1 \le i \le p,\ p+1 \le j \le m.$

2 The controllable case

In this section we will deal with Problem 1 in the controllable case. The following two results provide necessary conditions for the Problem to have a solution; i.e. for the existence of a matrix $B \in \mathbb{F}^{n \times m}$ such that (A, B) has prescribed controllability and Hermite indices for a given $A \in \mathbb{F}^{n \times n}$.

Lemma 3 .- *[Zaballa (1997)] Let $A \in \mathbb{F}^{n\times n}$ and let $\alpha_1 \mid \dots \mid \alpha_n$ be its invariant factors. Let $h_1 \ge \dots \ge h_m \ge 0$ be nonnegative integers . Then there exists a matrix $B \in \mathbb{F}^{n\times m}$, $m \le n$, such that (A, B) is controllable and has $h_1, \dots, h_m$ as Hermite indices if and only if there are m monic polinomials $\beta_1, \dots, \beta_m$ such that $d(\beta i) = h_i,\ 1 \le i \le m$ and*

$$\alpha_i = 1, \quad 1 \le i \le n-m \tag{2}$$

$$\prod_{j=1}^{k} \alpha_{n-m+j} \mid g.c.d. \prod_{j=1}^{k} \beta_{i_j} : 1 \le i_1 < \dots < i_k \le m\},$$
$$1 \le k \le m-1 \tag{3}$$

$$\alpha_1 \dots \alpha_n = \beta_1 \dots \beta_m \tag{4}$$

Lemma 4 .- *[Zaballa (1997)] Let $(A, B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ be a controllable matrix pair. Let $k_1 \ge \dots \ge k_m \ge 0$ be its controllability indices and let $h_1, \dots, h_m$ be its Hermite indices. Then*

$$(k_1, \dots, k_m) \prec (h_1, \dots, h_m) \tag{5}$$

The rest of this section is dedicated to show that conditions (2)- (5) are also sufficient for a matrix $B \in \mathbb{F}^{n\times m}$ to exist so that for a given matrix $A \in \mathbb{F}^{n\times n}$ the pair (A, B) is controllable and has prescribed controllability and Hermite indices. This will be shown as a consequence of several lemmas.

The following lemma can be proved by induction on m with the help of the Theorem 3.2 of [Baragaña *et al.* (1997)] and the Lemma 2.4 of [Baragaña *et al.*(1999)].

Lemma 5 .- *Let $h_1 \ge \dots \ge h_p > 0$ be positive integers and let $(A_{ii}, B_{ii}) \in \mathbb{F}^{h_i\times hi} \times \mathbb{F}^{h_i\times 1}$, $i = 1, \dots, p$ where*

$$A_{ii} = \begin{bmatrix} 0 & 0 & \dots & 0 & x_{ii0} \\ 1 & 0 & \dots & 0 & x_{ii1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & x_{iih_i-1} \end{bmatrix},\ B_{ii} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let $k_1 \ge \dots \ge k_m > 0$ be positive integers. If condition (5) is satisfied then there exist matrices $X_{ij} \in \mathbb{F}^{h_i\times h_j}$, $Y_{ij} \in \mathbb{F}^{h_i\times 1}$, $1 \le i \le p-1$, $i+1 \le j \le p$ and $Y_{ip+1} \in \mathbb{F}^{h_i\times(m-p)}$, $1 \le i \le p$ such that $(\tilde{A}, \tilde{B})$ has $k_1, k_2, \dots, k_m$ as controllability indices, where

$$\tilde{A} = \begin{bmatrix} A_{11} & X_{12} & \dots & X_{1p} \\ 0 & A_{22} & \dots & X_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{pp} \end{bmatrix}$$

and

$$\tilde{B} = \begin{bmatrix} B_{11} & Y_{12} & \dots & Y_{1p} & Y_{1p+1} \\ 0 & B_{22} & \dots & Y_{2p} & Y_{2p+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_{pp} & Y_{pp+1} \end{bmatrix}.$$

Lemma 6 .- *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ 0 & A_{22} & \dots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{pp} \end{bmatrix}$$

and

$$B = \begin{bmatrix} B_{11} & 0 & \dots & 0 \\ 0 & B_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{pp} \end{bmatrix}$$

with $(A_{ii}, B_{ii}) \in \mathbb{F}^{n_i\times n_i} \times \mathbb{F}^{n_i\times m_i}$ controllable, $1 \le i \le p-1$ and $(A_{pp}, B_{pp}) \in \mathbb{F}^{n_p\times n_p} \times \mathbb{F}^{n_p\times m_p}$. Let $m \ge \sum_{i=1}^{p} m_i$ and $k_1 \ge \dots \ge k_m \ge 0$ be nonnegative integers. Then there exist matrices $Z_{ij} \in \mathbb{F}^{n_i\times m_j}$, $1 \le i \le p-1$, $i+1 \le j \le p$ and $Z_{ip+1} \in \mathbb{F}^{n_i\times(m-\sum_{i=1}^{p} m_i)}$, $1 \le i \le p$ such that if

$$\overline{B} = \begin{bmatrix} B_{11} & Z_{12} & \dots & Z_{1p} & Z_{1p+1} \\ 0 & B_{22} & \dots & Z_{2p} & Z_{2p+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_{pp} & Z_{pp+1} \end{bmatrix}$$

then $(A, \overline{B})$ has $k_1, \dots, k_m$ as controllability indices if and only if there exist matrices $X_{ij} \in \mathbb{F}^{n_i\times n_j}, Y_{ij} \in \mathbb{F}^{n_i\times m_j}$, $1 \le i \le p-1$, $i+1 \le j \le p$ and $Y_{ip+1} \in \mathbb{F}^{n_i\times(m-\sum_{i=1}^{p} m_i)}$, $1 \le i \le p$ such that $(\tilde{A}, \tilde{B})$ has $k_1, \dots, k_m$ as controllability indices.

Finally, the following lemma, whose proof is straightforward, allow us to perform some transformations on matrix B without altering in the Hermite indices of pair (A, B).

Lemma 7 .- *Let $A \in \mathbb{F}^{n\times n}$ and $b_i \in \mathbb{F}^{n\times 1}, 1 \leq i \leq m$. Let $h_1, \dots, h_m$ be the Hermite indices of the pair (A, B) where $B = [b_1, \dots, b_m]$. Let t be a nonnegative integer and $l \in \{2, \dots, m\}$. Put $b_l' = b_l + \sum_{j=0}^{t} \sum_{i=1}^{l-1} c_{ij} A^j b_i$ where $c_{ij} \in \mathbb{F}$ are arbitrary.. Then, $h_1, \dots, h_m$ are the Hermite indices of the matrix pair (A, B') where $B' = [b_1, \dots, b_l', \dots, b_m]$.*

Now we can prove our main result

Theorem 1 .- *Let $A \in \mathbb{F}^{n\times n}$ and let $\alpha_1 \mid \dots \mid \alpha_n$ be its invariant factors. Let $k_1 \geq \dots \geq k_m > 0$ and $h_1 \geq \dots \geq h_m \geq 0$ be nonnegative integers. Then there exists a matrix $B \in \mathbb{F}^{n\times m}$ such that (A, B) is controllable and has $k_1, \dots, k_m$ as controllability indices and $h_1, \dots, h_m$ as Hermite indices if and only if there are m monic polynomials $\beta_1, \dots, \beta_m$ such that $d(\beta_i) = h_i,\ 1 \leq i \leq m$ and conditions (2)-(5) are satisfied.*

Proof.- The necessity is a direct consequence of Lemmas 3 and 4.

Assume that a matrix $A \in \mathbb{F}^{n\times n}$ is given with $\alpha_1 \mid \dots \mid \alpha_n$ as invariant factors. By Lemma 3 conditions (2)-(4) are sufficient for the existence of a matrix $B \in \mathbb{F}^{n\times m}$ such that (A, B) has $h_1, \dots, h_m$ as Hermite indices. By Lemma 2 (A, B) is similar to (A_c, B_c) where this pair has the form exhibited in that lemma. From Lemma 5 we have that if condition (5) is fulfilled then there are $X_{ij} \in \mathbb{F}^{h_i\times h_j}, Y_{ij} \in \mathbb{F}^{h_i\times 1},\ 1 \leq i \leq p-1,\ i+1 \leq j \leq p$ and $Y_{ip+1} \in \mathbb{F}^{h_i\times(m-p)}$ such that $(\tilde{A}, \tilde{B})$ has $k_1, \dots, k_m$ as controllability indices.

By Lemma 6 there exist$Z_{ij} \in F^{h_i\times 1},\ 1 \leq i \leq p-1,\ i+1 \leq j \leq p$ and $Z_{ip+1} \in F^{h_i\times(m-p)},\ 1 \leq i \leq p$ such that if

$$\overline{B}_c = \begin{bmatrix} B_{11} & Z_{12} & \dots & Z_{1p} & Z_{1p+1} \\ 0 & B_{22} & \dots & Z_{2p} & Z_{2p+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & B_{pp} & Z_{pp+1} \end{bmatrix}$$

then $(A_c, \overline{B}_c)$ has $k_1, \dots, k_m$ as controllability indices. Finaly, from Lemma 7 we have that the Hermite indices of $(A_c, \overline{B}_c)$ are $h_1, \dots, h_m$.

As A and A_c are similar matrices, there is a nonsingular matrix $P \in \mathbb{F}^{n\times n}$ such that $A = PA_cP^{-1}$. Put $B = PB_c'$. Then (A, B) has $k_1, \dots, k_m$ and $h_1, \dots, h_m$ as controllability and Hermite indices respectively, and the theorem follows. □

It is worthnoting that if $\mathbb{F}$ is algebraically closed in [Zaballa (1997)] it has been shown that conditions (2), (3) and (4) are equivalent to (2) and

$$(h_1, \dots, h_m) \prec (d(\alpha_n), \dots, d(\alpha_{n-m+1})) \qquad (6)$$

Thus in this case we have the following consequence of Theorem 1:

Corollary 1 .- *Let $\mathbb{F}$ an algebraically closed field. Let $A \in \mathbb{F}^{n\times n}$ and let $\alpha_1 \mid \dots \mid \alpha_n$ be its invariant factors. Let $k_1 \geq \dots \geq k_m > 0$ and $h_1 \geq \dots \geq h_m \geq 0$ be nonnegative integers. Then there exists a matrix $B \in \mathbb{F}^{n\times m}$ such that (A, B) is controllable and has $k_1, \dots, k_m$ as controllability indices and $h_1, \dots, h_m$ as Hermite indices if and only if conditions (2), (6) and (5) are satisfied.*

3 The noncontrollable case

Now we will generalize Theorem 1 to the noncontrollable case. It is well-known (see for example [Kailath (1980), p. 361]) that if (A, B) is not completely controllable then there is a nonsingular matrix P such that

$$PAP^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \quad PB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad (7)$$

where $(A_1, B_1) \in \mathbb{F}^{r\times r}$ is controllable and $r = rank\ C(A, B)$. Furthermore, the invariant factors of (A, B) different from one and those of A_3 coincide ([Zaballa (1988)]).

As the Hermite indices and the controllability indices are invariant under similarity tranformations, it follows that the Hermite indices and the controllability indices of (A, B) are those of the matrix pair (A_1, B_1) in the decomposition of (A, B) as in (7).

The generalization of Lemma 3 to the noncontrollable case is given by

Lemma 8 .- *([Zaballa (1997)]) Let $A \in \mathbb{F}^{n\times n}$ and let $\alpha_1 \mid \dots \mid \alpha_n$ be its invariant factors. Let $h_1, \dots, h_m$ and $\gamma_1, \dots, \gamma_n$ be nonnegative integers and monic polynomials, respectively, such that $d(\gamma_1) + \dots + d(\gamma_n) = q$. Let $r = n - q$. Then there exists a matrix $B \in \mathbb{F}^{n\times m},\ \ m \leq r$, such that (A, B) has $h_1, \dots, h_m$ as Hermite indices and $\gamma_1, \dots, \gamma_n$ as invariant factors if and only if there are m monic polinomials $\tau_1, \dots, \tau_m$ such that $d(\tau_i) = h_i,\ 1 \leq i \leq m$ and*

$$\alpha_{i-m} \mid \gamma_i \mid \alpha_i,\ 1 \le i \le n \tag{8}$$

$$\prod_{k=1}^{r-m+j} \sigma_k \mid g.c.d.\{\prod_{k=1}^{j} \tau_{i_k} : 1 \le i_1 < \ldots < i_j \le m\}, \quad 1 \le j \le m \tag{9}$$

$$\sigma_1 \ldots \sigma_r = \tau_1 \ldots \tau_m \tag{10}$$

where we agree that $\alpha_i := 1$ *for* $i < 1$,

$\sigma_j = \frac{\beta^j}{\beta^{j-1}}$, $1 \le j \le r$ *and*

$$\beta^j = \prod_{i=1}^{n+j} l.c.m.(\gamma_{i-j}, \alpha_{i-r}) \ ,\ 0 \le j \le r$$

With the help of this result we can generalize the Theorem 1 to give a complete solution to the Problem 1.

Theorem 2 .- *Let* $A \in \mathbb{F}^{n\times n}$ *and let* $\alpha_1 \mid \ldots \mid \alpha_n$ *be its invariant factors. Let* $k_1 \ge \ldots \ge k_m > 0$ *and* $h_1 \ge \ldots \ge h_m \ge 0$ *be nonnegative integers and* $\gamma_1, \ldots, \gamma_n$ *monic polynomials, such that* $d(\gamma_1)+\ldots+d(\gamma_n) = q$. *Let* $r = n - q$. *Then there exists a matrix* $B \in \mathbb{F}^{n\times m}$, $m \le r$, *such that* (A, B) *has* $h_1, \ldots, h_m$ *as Hermite indices,* $k_1, \ldots, k_m$ *as controllability indices and* $\gamma_1, \ldots, \gamma_n$ *as invariant factors if and only if there are m monic polinomials* $\tau_1, \ldots, \tau_m$ *such that* $d(\tau_i) = h_i$, $1 \le i \le m$, *and the conditions (8)-(10) and (5) are satisfied.*

Corollary 2 .- *Let* $A \in \mathbb{F}^{n\times n}$ *and let* $\alpha_1 \mid \ldots \mid \alpha_n$ *be its invariant factors,* $\mathbb{F}$ *algebraically closed. Let* $k_1 \ge \ldots \ge k_m > 0$, $h_1 \ge \ldots \ge h_m \ge 0$ *be nonnegative integers and* $\gamma_1, \ldots, \gamma_n$ *monic polynomials such that* $d(\gamma_1) + \ldots + d(\gamma_n) = q$. *Let* $r = n - q$. *Then there exists a matrix* $B \in \mathbb{F}^{n\times m}$, $m \le r$, *such that* (A, B) *has* $k_1, \ldots, k_m$ *as controllability indices,* $h_1, \ldots, h_m$ *as Hermite indices and* $\gamma_1, \ldots, \gamma_n$ *as invariant factors if and only if the conditions (8), (5) and*

$$(h_1, \ldots, h_m) \prec (d(\sigma_r), \ldots, d(\sigma_1)) \tag{11}$$

are satisfied.

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