# Adaptive Nonlinear $\mathcal{H}_\infty$ Control for Processes with Bounded Variations of Parameters — General Relative Degree Case —

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**Abstract**

A new class of adaptive nonlinear $\mathcal{H}_\infty$ control systems for processes with bounded variations of time-varying parameters, is proposed in this manuscript. Those control schemes are derived as solutions of particular nonlinear $\mathcal{H}_\infty$ control problems, where unknown system parameters are regarded as exogenous disturbances to the processes, and thus, in the resulting control systems, the $\mathcal{L}^2$ gains from system parameters to generalized outputs are made less than $\gamma$ ($> 0$). The control schemes are shown to be sub-optimal to some $\mathcal{H}_\infty$ cost functionals (or certain differential games), when the high-frequency gains are time-invariant.

## 1 Introduction

In the study of adaptive control, the main topics have been an asymptotic stability of adaptive control systems. So much attention has not been paid on the control performances such as transient performance and other performances [1], [2]. On the contrary, the recent researches on nonlinear $\mathcal{H}_\infty$ control and inverse optimality, could derive adaptive or nonlinear control systems which are optimal to certain meaningful cost functionals [4], [5].

Additionally, in the study of adaptive control, the stability analysis of adaptive systems have been focused on time-invariant processes mainly; no enough discussion for the case of time-varying systems has been made. Although several approaches have been examined for time-varying processes in the robust adaptive control schemes [2], those results could be applied to limited classes of time-varying systems, that is, only sufficiently small variations of time-varying parameters are accepted in those robust adaptive schemes.

The purpose of the present paper is to provide a new class of adaptive nonlinear $\mathcal{H}_\infty$ control systems for processes with bounded variations of parameters, where the control performances are discussed explicitly, and the stability analysis for time-varying systems are carried out successfully. Those control schemes are derived as solutions of particular nonlinear $\mathcal{H}_\infty$ control problems, where unknown system parameters are regarded as exogenous disturbances to the processes, and thus, in the resulting control systems, the $\mathcal{L}^2$ gains from system parameters to generalized outputs are made less than $\gamma$ ($> 0$) (the prescribed positive constant). The proposed control strategy can be applied to any time-varying (or time-invariant) systems, and the resulting control systems are bounded for arbitrarily large but bounded variations of time-varying parameters. Also, the control schemes are shown to be sub-optimal to some $\mathcal{H}_\infty$ cost functionals (or certain differential games), when the high-frequency gains are time-invariant. The former version of the present research [6] considered the case where relative degrees are 1 only. The present manuscript extends it to general relative degree cases by applying backstepping procedures [3].

## 2 Problem Statement

A single-input single-output nonlinear system is introduced

$$\frac{d}{dt}e(t) = \mathcal{L}(e(t)) + \mathcal{L}(f(e,t)) + b_0 u_{fn^*-1}(t) + \Phi^T\omega_1(t), \quad (1)$$

$$u_{fi}(t) = \frac{1}{(s+\lambda)^i}u(t), \quad (\lambda > 0), \quad (2)$$

where $e(t)$ is a control variable, $u(t)$ is a control input, and $f(e,t)$ is an unknown nonlinear term. $\omega_1(t)$ is a vector composed of measurable signals, $b_0$ and $\Phi$ are unknown system parameters which can be time-varying, and $\mathcal{L}(\cdot)$ is an unstructured element defined by

$$\mathcal{L}(v(t)) = G_0(s)v(t), \quad (G_0 \in \mathcal{RH}^\infty). \quad (3)$$

$\lambda$ ($> 0$) is a design parameter which is known, and $n^*$ is a relative degree of the controlled process. The following assumptions are introduced.

**Assumption 1** **1-1.** *The relative degree $n^*$ is known.* **1-2.** *Although $b_0$ can be time-varying, the sign of it remains unchanged ($b_0 > 0$ or $b_0 < 0$), and is known a priori. It is assumed that $b_0 > 0$ without loss of generality.* **1-3.** *The unknown nonlinear term $f(e,t)$ is evaluated by*

$$f(e,t)^2 \le f_0 \cdot \phi(e) \cdot e^2, \quad \phi(0) = 0, \quad (4)$$

*where $f_0$ is an unknown positive constant, and $\phi(e)(> 0)$ is a known function of $e$ which is $n^* - 1$ times differentiable with respect to $e$.* **1-4.** *The magnitude of $\omega_1(t)$ is evaluated as follows:*

$$\|\omega_1(t)\| \le M_1 \sup_{t \ge \tau} |e(\tau)| + M_2, \quad (M_1, M_2 > 0). \quad (5)$$

**1-5.** *The upper bound on the norm of parameter vector $\Phi_0$, which will be introduced later, is known. Also, the upper bound $\bar{b}_0$ and lower bound $\underline{b}_0$ on the high-frequency gain $b_0$ are known such that*

$$0 < \delta \le \underline{b}_0 \le b_0 \le \bar{b}_0 \le M < \infty, \quad (6)$$

*and the upper bound $\bar{p}$ and lower bound $\underline{p}$ on the parameter $p$ are known, too.*

$$p \equiv \frac{1}{b_0}, \quad \bar{p} \equiv \frac{1}{\underline{b}_0}, \quad \underline{p} \equiv \frac{1}{\bar{b}_0}. \tag{7}$$

The control problem is to determine a control input $u(t)$ adaptively such that the overall system is stabilized for arbitrary but bounded time-varying system parameters, and additionally, the control variable $e(t)$ converges to zero asymptotically in the ideal case (stability condition), while the resulting control system becomes optimal or sub-optimal to some meaningful cost functionals (optimality condition).

## 3 Adaptive Nonlinear $\mathcal{H}_\infty$ Control Scheme

The design of control systems are based on backstepping procedures [3] composed of step 1 $\sim$ step $n^*$, and in each steps, the control signals $v_i(t)$ are determined by applying nonlinear $\mathcal{H}_\infty$ control scheme. In the last step (step $n^*$), the actual control input $u(t)$ is obtained.

**Step 1)** Define $z_1(t)$, $z_2(t)$ by

$$z_1(t) \equiv e(t), \tag{8}$$

$$z_2(t) \equiv u_{fn^*-1}(t) - \alpha_1(t). \tag{9}$$

The virtual control $\alpha_1(t)$ is determined to stabilize $z_1(t)$.

$$\alpha_1(t) = -\hat{p}(t)\left[\{\hat{\theta}_1(t) + k_{11}\}z_1(t) + \hat{\theta}_2(t)\phi(z_1(t))z_1(t) + \hat{\Phi}(t)^T\omega_1(t) + v_1(t)\right]$$
$$\equiv -\hat{p}(t)[\hat{\Phi}_0(t)^T\omega_0(t) + k_{11}z_1(t) + v_1(t)] \equiv -\hat{p}(t)v_0(t), \tag{10}$$

$$\hat{\Phi}_0(t) = [\hat{\theta}_1(t),\, \hat{\theta}_2(t),\, \hat{\Phi}(t)^T]^T, \tag{11}$$

$$\omega_0(t) = [z_1(t),\, \phi(z_1(t))z_1(t),\, \omega_1(t)^T]^T, \tag{12}$$

$$v_0(t) = \hat{\Phi}_0(t)^T\omega_0(t) + k_{11}z_1(t) + v_1(t), \tag{13}$$

where $k_{11}$ is a positive constant, and $v_1(t)$ is to be determined later based on nonlinear $\mathcal{H}_\infty$ control strategy. In this manuscript, the projection type adaptive laws, where tuning parameters $\hat{\theta}$ are constrained to certain closed regions $\mathcal{S}$, are defined by

$$\dot{\hat{\theta}} = \Pr(\Gamma\phi\epsilon) \equiv \begin{cases} \Gamma\phi\epsilon & \text{Case I} \\ \Gamma\phi\epsilon - \Gamma\frac{\nabla g \nabla g^T}{\nabla g^T \Gamma \nabla g}\Gamma\phi\epsilon & \text{Case II} \end{cases}, \quad (\Gamma = \Gamma^T > 0), \tag{14}$$

where Case I : $\hat{\theta} \in \mathcal{S}^o$, or $\hat{\theta} \in \partial\mathcal{S}$ & $(\Gamma\phi\epsilon)^T\nabla g \le 0$, Case II : otherwise, $\mathcal{S} = \{\hat{\theta} : g(\hat{\theta}) \le 0\}$, $\mathcal{S}^o$ = interior of $\mathcal{S}$, $\partial\mathcal{S}$ = boundary of $\mathcal{S}$. By utilizing those descriptions, $\hat{\theta}_1(t)$, $\hat{\theta}_2(t)$, $\hat{p}(t)$ are tuned in the following ways:

$$\dot{\hat{\theta}}_1(t) = \Pr\{g_{11}z_1(t)^2\}, \quad \dot{\hat{\theta}}_2(t) = \Pr\{g_{12}\phi(z_1(t))z_1(t)^2\},$$
$$\dot{\hat{p}}(t) = \Pr\{g_{13}v_0(t)z_1(t)\}, \tag{15}$$

where $g_{11}$, $g_{12}$, $g_{13} > 0$, and each constraints are given by

$$g_{\theta_1}(\hat{\theta}_1) = \hat{\theta}_1^2 - M^2, \quad g_{\theta_2}(\hat{\theta}_2) = \hat{\theta}_2^2 - M^2,$$
$$g_p(\hat{p}) = \left(\hat{p} - \frac{\delta + M}{2}\right)^2 - \left(\frac{M-\delta}{2}\right)^2. \tag{16}$$

$M$ and $\delta$ are properly selected positive constants based on **Assumption 1-5**. Hereafter, we are to obtain the input signal $v_1(t)$ by applying nonlinear $\mathcal{H}_\infty$ control strategy. For this purpose, define $V_1(t)$ by

$$V_1(t) = \frac{1}{2}z_1(t)^2 + \frac{1}{2}\sum_{i=1}^{2}\{\hat{\theta}_i(t) - \theta_i^*\}^2/g_{1i} + \frac{\underline{b}_0}{2}\{\hat{p}(t) - \bar{p}\}^2/g_{13}$$
$$+\frac{1}{2}\{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\hat{\Phi}(t) - \Phi^*\}, \quad (G_{14} = G_{14}{}^T > 0), \tag{17}$$

where $\Phi^*$ is a nominal value (time-invariant) of the parameter $\Phi$ (time-varying), and $\theta_i^*$ are also nominal values (time-invariant) of the parameters $\theta_i$ (time-varying) determined later. We take the time derivative of it.

$$\dot{V}_1(t) = z_1(t)\{\mathcal{L}(z_1(t)) + \mathcal{L}(f(z_1,t)) + \Phi^T\omega_1(t)\}$$
$$-\Phi_0^{*T}\omega_0(t)z_1(t) - k_{11}z_1(t)^2$$
$$+(\underline{b}_0 - b_0)\hat{p}(t)\{\hat{\Phi}_0(t)^T\omega_0(t) + k_{11}z_1(t)\}$$
$$-\{1 + b_0\hat{p}(t) - \underline{b}_0\hat{p}(t)\}v_1(t)z_1(t) + b_0z_1(t)z_2(t)$$
$$+\{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(t) - \tau_{\phi 1}(t)\}, \tag{18}$$

$$\tau_{\phi 1}(t) = G_{14}\omega_1(t)z_1(t), \tag{19}$$

$$\Phi_0^* = [\theta_1^*,\, \theta_2^*,\, \Phi^{*T}]^T. \tag{20}$$

Since $\mathcal{L}(\cdot)$ is defined by (3), there exist bounded $\theta_{11}$, $\theta_{12}$ (positive) satisfying the next inequality.

$$\int_0^t z_1(\tau)\{\mathcal{L}(z_1(\tau)) + \mathcal{L}(f(z_1,\tau))\}d\tau$$
$$\le \int_0^t \{\theta_{11}(\tau)z_1(\tau)^2 + \theta_{12}(\tau)\phi(z_1(\tau))z_1(\tau)^2\}d\tau. \tag{21}$$

Then, the following relation is obtained by integrating $\dot{V}_1(t)$.

$$V_1(t) - V_1(0)$$
$$\le -k_{11}\int_0^t z_1(\tau)^2 d\tau + \int_0^t (\Phi_{01} - \Phi_0^*)^T\omega_0(\tau)z_1(\tau)d\tau$$
$$+\int_0^t (\underline{b}_0 - b_0)\hat{p}(\tau)\{(\hat{\theta}_1(\tau) + k_{11})z_1(\tau)$$
$$+\hat{\theta}_2(\tau)\phi(z_1(\tau))z_1(\tau) + \hat{\Phi}(\tau)^T\omega_1(\tau)\}z_1(\tau)d\tau$$
$$-\int_0^t \{1 + b_0\hat{p}(\tau) - \underline{b}_0\hat{p}(\tau)\}v_1(\tau)z_1(\tau)d\tau + \int_0^t b_0z_1(\tau)z_2(\tau)d\tau$$
$$+\int_0^t \{\hat{\Phi}(\tau) - \Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(\tau) - \tau_{\phi 1}(\tau)\}d\tau$$
$$\equiv -k_{11}\int_0^t z_1(\tau)^2 d\tau + \int_0^t \tilde{\Theta}_1^T\omega(\tau)d\tau$$
$$-\int_0^t \{1 + \tilde{b}_0(\tau)\}v_1(\tau)z_1(\tau)d\tau + \int_0^t b_0z_1(\tau)z_2(\tau)d\tau$$
$$+\int_0^t \{\hat{\Phi}(\tau) - \Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(\tau) - \tau_{\phi 1}(\tau)\}d\tau, \tag{22}$$

$$\Phi_{01} \equiv [\theta_{11},\, \theta_{12},\, \Phi^T]^T, \tag{23}$$

$$\tilde{\Theta}_1 \equiv [(\Phi_{01} - \Phi_0^*)^T,\, \underline{b}_0 - b_0]^T, \tag{24}$$

$$\tilde{b}_0 \equiv (b_0 - \underline{b}_0)\hat{p}(t)\ (\ge 0), \tag{25}$$

$$\omega(t) \equiv [\omega_0(t)^T,\, \hat{p}(t)\{\hat{\Phi}_0(t)^T\omega_0(t) + k_{11}z_1(t)\}]^T. \tag{26}$$

From that evaluation of $V_1(t)$, we introduce the following virtual process

$$\dot{z}_1 = -k_{11}z_1 + \omega^T\tilde{\Theta}_1 - (1 + \tilde{b}_0)v_1$$
$$\equiv f_1(z_1) + g_{11}d_1 + g_{12}v_1, \tag{27}$$

$$f_1(z_1) = -k_{11}z_1, \quad g_{11} = \omega^T, \quad d_1 = \tilde{\Theta}_1, \quad g_{12} = -(1 + \tilde{b}_0), \tag{28}$$

and stabilize this system via $v_1$ by applying nonlinear $\mathcal{H}_\infty$ control strategies, where the unknown system parameter $\tilde{\Theta}_1$ is regarded as an exogenous disturbance to the process. For this purpose, consider the Hamilton-Jacobi-Isaacs equation

$$\frac{\partial\tilde{V}_1}{\partial z_1}f_1 + \frac{1}{4}\left(\frac{\|g_{11}\|^2}{\gamma_1^{*2}} - \frac{g_{12}^2}{r_1}\right)\left(\frac{\partial\tilde{V}_1}{\partial z_1}\right)^2 + h_1z_1^2 \le 0, \tag{29}$$

where the solution $\tilde{V}_1$ is given by the next equation

$$\tilde{V}_1(t) = \frac{1}{2}z_1(t)^2. \tag{30}$$

$h_1$ and $r_1$ are positive functions to be determined from the inequality (29) based on inverse optimality for the given solution $\tilde{V}_1$ (30) and the positive constant $\gamma_1^*$. The substitution of (30) into (29) yields

$$-k_{11}z_1^2+\left\{\frac{\|\omega\|^2}{\gamma_1^{*2}}-\frac{(1+\tilde{b}_0)^2}{r_1}\right\}\frac{z_1^2}{4}+h_1z_1^2\le 0. \quad (31)$$

Since the unknown element $\tilde{b}_0(\ge 0)$ is included in the above inequality, we obtain $h_1$ and $r_1$ satisfying the next relation, which is a sufficient condition for (31).

$$-k_{11}z_1^2+\left(\frac{\|\omega\|^2}{\gamma_1^{*2}}-\frac{1}{r_1}\right)\frac{z_1^2}{4}+h_1z_1^2\le 0. \quad (32)$$

In order that the inequality holds, we choose $r_1$ such that

$$\frac{1}{r_1}=\frac{k_{12}+k_{13}\|\omega\|^2}{r_{01}} \Leftrightarrow r_1=\frac{r_{01}}{k_{12}+k_{13}\|\omega\|^2}, \quad (33)$$

where $k_{12}$, $k_{13}$, $r_{01}(>0)$ are design parameters. For that $r_1$, the corresponding $h_1$ is obtained as follows:

$$h_1z_1^2\le k_{11}z_1^2+\left\{\frac{k_{12}\gamma_1^{*2}+(k_{13}\gamma_1^{*2}-r_{01})\|\omega\|^2}{4r_{01}\gamma_1^{*2}}\right\}z_1^2,$$
$$(k_{13}\gamma_1^{*2}-r_{01}\ge 0). \quad (34)$$

From these $h_1$ and $r_1$, the control $v_1$ is derived such that

$$v_1^*=-\frac{1}{2r_1}g_{12}\frac{\partial\tilde{V}_1}{\partial z_1}=\frac{k_{12}+k_{13}\|\omega\|^2}{2r_{01}}(1+\tilde{b}_0)z_1. \quad (35)$$

Since the unknown element $\tilde{b}_0$ is included in the above equation, the actual input signal is replaced by

$$v_1^*=\frac{z_1}{2r_1}=\frac{k_{12}+k_{13}\|\omega(t)\|^2}{2r_{01}}z_1. \quad (36)$$

Then, we obtain the following relation for the original process (1), (2), $V_1(t)$ (17), and the input signal $v_1(t)$ which is not necessarily equal to $v_1^*(t)$ (36).

$$\begin{aligned}
V_1(t)-V_1(0)\le&\int_0^t[-k_{11}z_1(\tau)+\tilde{\Theta}_1^T\omega(\tau)\\
&-\{1+\tilde{b}_0(\tau)\}v_1(\tau)]z_1(\tau)d\tau+\int_0^t b_0z_1(\tau)z_2(\tau)d\tau\\
&+\int_0^t\{\hat{\Phi}(\tau)-\Phi^*\}^TG_{14}^{-1}\{\dot{\hat{\Phi}}(\tau)-\tau_{\phi 1}(\tau)\}d\tau\\
\le&\int_0^t\left\{-\frac{\|\omega(\tau)\|^2}{\gamma_1^{*2}}+\frac{1}{r_1}\right\}\frac{z_1(\tau)^2}{4}d\tau-\int_0^t h_1z_1(\tau)^2d\tau\\
&+\int_0^t z_1(\tau)\{\tilde{\Theta}_1^T\omega(\tau)-(1+\tilde{b}_0)v_1(\tau)\}d\tau\\
&+\int_0^t b_0z_1(\tau)z_2(\tau)d\tau\\
&+\int_0^t\{\hat{\Phi}(\tau)-\Phi^*\}^TG_{14}^{-1}\{\dot{\hat{\Phi}}(\tau)-\tau_{\phi 1}(\tau)\}d\tau\\
=&\int_0^t r_1\left\{-\frac{z_1(\tau)}{2r_1}+v_1(\tau)\right\}^2d\tau\\
&-\int_0^t\{h_1z_1(\tau)^2+r_1v_1(\tau)^2\}d\tau\\
&-\int_0^t\gamma_1^{*2}\left\|\tilde{\Theta}_1(\tau)-\frac{1}{2\gamma_1^{*2}}\omega(\tau)z_1(\tau)\right\|^2d\tau\\
&+\gamma_1^{*2}\int_0^t\|\tilde{\Theta}_1(\tau)\|^2d\tau\\
&-\int_0^t\tilde{b}_0(\tau)v_1(\tau)z_1(\tau)d\tau+\int_0^t b_0z_1(\tau)z_2(\tau)d\tau\\
&+\int_0^t\{\hat{\Phi}(\tau)-\Phi^*\}^TG_{14}^{-1}\{\dot{\hat{\Phi}}(\tau)-\tau_{\phi 1}(\tau)\}d\tau.
\end{aligned} \quad (37)$$

**Step i)** $(2\le i\le n^*)$ Take the time derivative of $z_i(t)$.

$$z_i(t)\equiv u_{fn^*-i+1}(t)-\alpha_{i-1}(t), \quad (38)$$

$$\begin{aligned}
\dot{z}_i(t)=&-\lambda u_{fn^*-i+1}(t)+u_{fn^*-i}(t)-\beta_{i-1}(t)\\
&-\gamma_{i-1}(t)\{\mathcal{L}(z_1(t))+\mathcal{L}(f(z_1,t))+b_0u_{fn^*-1}(t)+\Phi^T\omega(t)\}\\
&-\gamma_{Ki-1}(t)\Pr\{G_1\tilde{v}_1(t)z_1(t)\}-\gamma_{\phi i-1}(t)\dot{\hat{\Phi}}(t)-\gamma_{bi-1}(t)\dot{\hat{b}}_0(t),
\end{aligned} \quad (39)$$

$$\beta_{i-1}(t)=\frac{\partial\alpha_{i-1}}{\partial\omega_{i-1}}\dot{\omega}_{i-1}(t)+\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\{-\lambda u_{fn^*-j+1}(t)$$
$$+u_{fn^*-j}(t)-\beta_{j-1}(t)\},\ (i\ge 3), \quad (40)$$

$$\gamma_{i-1}(t)=\frac{\partial\alpha_{i-1}}{\partial z_1}-\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\gamma_{j-1}(t),\ (i\ge 3), \quad (41)$$

$$\gamma_{Ki-1}(t)=\frac{\partial\alpha_{i-1}}{\partial\hat{K}_1}-\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\gamma_{Kj-1}(t),\ (i\ge 3), \quad (42)$$

$$\gamma_{\phi i-1}(t)=\frac{\partial\alpha_{i-1}}{\partial\hat{\Phi}}-\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\gamma_{\phi j-1}(t),\ (i\ge 3), \quad (43)$$

$$\gamma_{bi-1}(t)=\frac{\partial\alpha_{i-1}}{\partial\hat{b}_0}-\sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial z_j}\gamma_{bj-1}(t),\ (i\ge 3), \quad (44)$$

$$\left(\beta_1(t)=\frac{\partial\alpha_1}{\partial\omega_1}\dot{\omega}_1(t),\ \gamma_1(t)=\frac{\partial\alpha_1}{\partial z_1},\ \gamma_{K1}(t)=\frac{\partial\alpha_1}{\partial\hat{K}_1},\right.$$
$$\left.\gamma_{\phi 1}(t)=\frac{\partial\alpha_1}{\partial\hat{\Phi}},\ \gamma_{b1}(t)=0\right), \quad (45)$$

$$\hat{K}_1(t)=[\hat{\theta}_1(t),\hat{\theta}_2(t),\hat{p}(t)]^T, \quad (46)$$

$$\tilde{v}_1(t)=[z_1(t),\phi(z_1(t))z_1(t),v_0(t)]^T, \quad (47)$$

$$G_1=\mathrm{diag}(g_{11},g_{12},g_{13}), \quad (48)$$

$\omega_{i-1}(t)\equiv$[vector signals composed of the elements $\{u_{fn^*-1}(t)\sim u_{fn^*-i+2},\omega_{i-2}(t),\dot{\omega}_{i-2}(t)\}$], $(i\ge 3)$. (49)

For $z_i(t)$, we introduce $z_{i+1}(t)$ and determine the virtual control $\alpha_i(t)$ so as to stabilize $z_i(t)$.

$$z_{i+1}(t)\equiv u_{fn^*-i}(t)-\alpha_i(t),\quad (3\le i\le n^*-1), \quad (50)$$

$$\begin{aligned}
\alpha_i(t)=&\lambda u_{fn^*-i+1}(t)+\beta_{i-1}(t)-c_{i1}(t)z_{i-1}(t)\\
&+\gamma_{i-1}(t)\hat{\Phi}(t)^T\omega_1(t)+\hat{b}_0(t)\gamma_{i-1}(t)u_{fn^*-1}(t)\\
&-k_{i1}z_i(t)-k_{i2}\gamma_{i-1}(t)^2z_i(t)-k_{i3}\gamma_{Ki-1}(t)\|\tilde{v}_1(t)\|^2z_i(t)\\
&+v_i(t)+\tilde{\alpha}_i(t),\quad (k_{i1},k_{i2},k_{i3}>0),
\end{aligned} \quad (51)$$

$$c_{i1}(t)=\hat{b}_0(t)\ (i=2),\quad 1\ (i\ge 3), \quad (52)$$

where $\tilde{\alpha}_i(t)$ is an auxiliary signal to be determined later, and the input signal $v_i(t)$ is to be obtained by applying nonlinear $\mathcal{H}_\infty$ control strategy in **Step 1**. Define $V_i(t)$ by

$$V_2(t)=\frac{1}{2}z_2(t)^2+\frac{1}{2}\{\hat{b}_0(t)-b_0^*\}^2/g_2, \quad (53)$$

$$V_i(t)=\frac{1}{2}z_i(t)^2,\ (i\ge 3), \quad (54)$$

where $b_0^*$ (time-invariant) is a nominal value of $b_0$ (time-varying), and take the time derivative of it. Then, we get the following inequality.

$$\begin{aligned}
&\int_0^t c_{i2}z_{i-1}(\tau)z_i(\tau)d\tau\\
&+\int_0^t\{\hat{\Phi}(\tau)-\Phi^*\}^TG_{14}^{-1}\{\dot{\hat{\Phi}}(\tau)-\tau_{\phi i-1}(\tau)\}d\tau\\
&+\int_0^t\{\hat{b}_0(\tau)-b_0^*\}\{\dot{\hat{b}}_0(\tau)-\tau_{bi-1}(\tau)\}d\tau+\int_0^t\dot{V}_i(\tau)d\tau\\
&\le\int_0^t z_i(\tau)z_{i+1}(\tau)d\tau
\end{aligned}$$

$$+\frac{1}{k_{i2}}\int_0^t \{\theta_{21}(\tau)z_1(\tau)^2+\theta_{22}(\tau)\phi(z_1(\tau))z_1(\tau)^2\}d\tau$$
$$+\frac{1}{4k_{i3}}\|G_1\|^2\int_0^t z_1(\tau)^2 d\tau$$
$$-k_{i1}\int_0^t z_i(\tau)^2 d\tau+\int_0^t \tilde{\Theta}_2(\tau)\tilde{\omega}_i(\tau)d\tau$$
$$+\int_0^t \{\hat{\Phi}(\tau)-\Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(\tau)-\tau_{\phi i}(\tau)\}d\tau$$
$$+\int_0^t \{\hat{b}_0(\tau)-b_0^*\}\{\dot{\hat{b}}_0(\tau)-\tau_{bi}(\tau)\}d\tau$$
$$-\int_0^t \gamma_{\phi i-1}(\tau)\dot{\hat{\Phi}}(\tau)z_i(\tau)d\tau-\int_0^t \gamma_{bi-1}(\tau)\dot{\hat{b}}_0(\tau)z_i(\tau)d\tau$$
$$+\int_0^t z_i(\tau)\{v_i(\tau)+\tilde{\alpha}_i(\tau)\}d\tau, \tag{55}$$
$$\tilde{\omega}_2(t)=[z_1(t)-\gamma_1(t)u_{fn^*-1}(t),\ -\gamma_1(t)\omega_1(t)^T]^T, \tag{56}$$
$$\tilde{\omega}_i(t)=[-\gamma_{i-1}(t)u_{fn^*-1}(t),\ -\gamma_{i-1}(t)\omega_1(t)^T]^T,\quad (i\geq 3), \tag{57}$$
$$\tau_{\phi i}(t)=\tau_{\phi i-1}(t)-G_{14}\gamma_{i-1}(t)\omega_1(t)z_i(t), \tag{58}$$
$$\tau_{b2}(t)=g_2\{z_1(t)-\gamma_1(t)u_{fn^*-1}(t)\}, \tag{59}$$
$$\tau_{bi}(t)=\tau_{bi-1}(t)-g_2\gamma_{i-1}(t)u_{fn^*-1}(t)z_i(t),\quad (i\geq 3), \tag{60}$$
$$c_{i2}=b_0\ (i=2),\quad 1\ (i\geq 3), \tag{61}$$

where $\theta_{21}$, $\theta_{22}$, are positive functions satisfying the following inequalities, and $k_{i2}$, $k_{i3}$ are arbitrary positive constants.

$$-\int_0^t \gamma_{i-1}(\tau)\{\mathcal{L}(z_1(\tau)+\mathcal{L}(f(z_1,\tau))\}z_i(\tau)d\tau$$
$$\leq k_{i2}\int_0^t \gamma_{i-1}(\tau)^2 z_i(\tau)^2 d\tau$$
$$+\frac{1}{k_{i2}}\int_0^t \{\theta_{21}(\tau)z_1(\tau)^2+\theta_{22}(\tau)^2\phi(z_1(\tau))z_1(\tau)^2\}d\tau, \tag{62}$$
$$-\int_0^t \gamma_{Ki-1}(\tau)\Pr\{G_1\tilde{v}_1(\tau)z_1(\tau)\}z_i(\tau)d\tau$$
$$\leq \int_0^t \|\gamma_{Ki-1}(\tau)\|\|G_1\|\|\tilde{v}_1(\tau)\||z_1(\tau)||z_i(\tau)|d\tau$$
$$\leq k_{i3}\int_0^t \|\gamma_{Ki-1}(\tau)\|^2\|\tilde{v}_1(\tau)\|^2 z_i(\tau)^2 d\tau$$
$$+\frac{1}{4k_{i3}}\|G_1\|^2\int_0^t z_1(\tau)^2 d\tau. \tag{63}$$

From that relation (55), we introduce the virtual process

$$\dot{z}_i=-k_{i1}z_i+\tilde{\omega}_i^T\tilde{\Theta}_2+v_i\equiv f_i(z_i)+g_{i1}d_i+g_{i2}v_i, \tag{64}$$
$$f_i(z_i)=-k_{i1}z_i,\quad g_{i1}=\tilde{\omega}_i^T,\quad d_i=\tilde{\Theta}_2,\quad g_{i2}=1, \tag{65}$$

and stabilize this system via $v_i$ by applying nonlinear $\mathcal{H}_\infty$ control strategies, where the unknown parameter $\tilde{\Theta}_2$ is regarded as an exogenous disturbance to the process. We set the corresponding Hamilton-Jacobi-Isaacs equation

$$\frac{\partial\tilde{V}_i}{\partial z_i}f_i+\frac{1}{4}\left(\frac{\|g_{i1}\|^2}{\gamma_i^{*2}}-\frac{g_{i2}^2}{r_i}\right)\left(\frac{\partial\tilde{V}_i}{\partial z_i}\right)^2+h_i z_i^2\leq 0, \tag{66}$$

where the solution $\tilde{V}_i$ is given by the next equation

$$\tilde{V}_i(t)=\frac{1}{2}z_i(t)^2. \tag{67}$$

Positive functions $h_i$ and $r_i$ are determined from the inequality (66) based on inverse optimality for the given solution $\tilde{V}_i$ (67) and the positive constant $\gamma_i^*$. The substitution of (67) into (66) yields

$$-k_{i1}z_i^2+\left\{\frac{\|\tilde{\omega}_i\|^2}{\gamma_i^{*2}}-\frac{1}{r_i}\right\}\frac{z_i^2}{4}+h_i z_i^2\leq 0. \tag{68}$$

In order that the above inequality holds for any $z_i$ and $\tilde{\omega}_i$, $r_i$ should be

$$\frac{1}{r_i}=\frac{k_{i4}+k_{i5}\|\tilde{\omega}_i\|^2}{r_{0i}}\ \Leftrightarrow\ r_i=\frac{r_{0i}}{k_{i4}+k_{i5}\|\tilde{\omega}_i\|^2}, \tag{69}$$

where $k_{i4}, k_{i5}, r_{0i}(>0)$ are design parameters. For that $r_i$, the corresponding $h_i$ and the control input $v_i(t)$ are obtained as follows:

$$h_i z_i^2\leq k_{i1}z_i^2+\left\{\frac{k_{i4}\gamma_i^{*2}+(k_{i5}\gamma_i^{*2}-r_{02})\|\tilde{\omega}_i\|^2}{4r_{0i}\gamma_i^{*2}}\right\}z_i^2,$$
$$(k_{i5}\gamma_i^{*2}-r_{0i}\geq 0), \tag{70}$$
$$v_i^*=-\frac{1}{2r_i}g_{i2}\frac{\partial\tilde{V}_i}{\partial z_i}=-\frac{z_i}{2r_i}=-\frac{k_{i4}+k_{i5}\|\tilde{\omega}_i\|^2}{2r_{0i}}z_i. \tag{71}$$

Then, we derive the following relation for the original process (1), (2), $V_1(t)\sim V_i(t)$ ((17), (53), (54)) and the input signals $v_1(t)\sim v_i(t)$ which are not necessarily equal to $v_1^*(t)\sim v_i^*(t)$ ((36), (71)).

$$\sum_{j=1}^{i}\{V_j(t)-V_j(0)\}\leq\int_0^t r_1\left\{-\frac{z_1(\tau)}{2r_1}+v_1(\tau)\right\}^2 d\tau$$
$$-\int_0^t \{h_1 z_1(\tau)^2+r_1 v_1(\tau)^2\}d\tau$$
$$-\int_0^t \gamma_1^{*2}\left\|\tilde{\Theta}_{1i}(\tau)-\frac{1}{2\gamma_1^{*2}}\omega(\tau)z_1(\tau)\right\|^2 d\tau$$
$$+\gamma_1^{*2}\int_0^t \|\tilde{\Theta}_{1i}(\tau)\|^2 d\tau-\int_0^t \tilde{b}_0(\tau)v_1(\tau)z_1(\tau)d\tau$$
$$+\sum_{j=2}^{i}\left[\int_0^t r_j\left\{-\frac{z_j(\tau)}{2r_j}+v_j(\tau)\right\}^2 d\tau\right.$$
$$-\int_0^t \{h_j z_j(\tau)^2+r_j v_j(\tau)^2\}d\tau$$
$$-\int_0^t \gamma_j^{*2}\left\|\tilde{\Theta}_2(\tau)-\frac{1}{2\gamma_j^{*2}}\tilde{\omega}_j(\tau)z_j(\tau)\right\|^2 d\tau$$
$$\left.+\gamma_j^{*2}\int_0^t \|\tilde{\Theta}_2(\tau)\|^2 d\tau\right]$$
$$+\int_0^t z_i(\tau)z_{i+1}(\tau)d\tau$$
$$+\int_0^t \{\hat{\Phi}(\tau)-\Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(\tau)-\tau_{\phi i}(\tau)\}d\tau$$
$$+\int_0^t \{\hat{b}_0(\tau)-b_0^*\}\{\dot{\hat{b}}_0(\tau)-\tau_{bi}(\tau)\}/g_2 d\tau$$
$$-\sum_{j=2}^{i}\int_0^t \gamma_{\phi j-1}(\tau)\dot{\hat{\Phi}}(\tau)z_j(\tau)d\tau$$
$$-\sum_{j=2}^{i}\int_0^t \gamma_{bj-1}(\tau)\dot{\hat{b}}_0(\tau)z_j(\tau)d\tau+\sum_{j=2}^{i}\int_0^t \tilde{\alpha}_j(\tau)z_j(\tau)d\tau, \tag{72}$$
$$\tilde{\Theta}_{1i}=[(\Phi_{0i}-\Phi_0^*)^T,\ \underline{b}_0-b_0]^T, \tag{73}$$
$$\Phi_{0i}=\left[\theta_{11}+\sum_{j=2}^{i}\left(\frac{\theta_{21}}{k_{j2}}+\frac{\|G_1\|^2}{4k_{j3}}\right),\ \theta_{12}+\sum_{j=2}^{i}\frac{\theta_{22}}{k_{j2}},\ \Phi^T\right]^T. \tag{74}$$

In **Step** $n^*$), the actual control input is obtained as

$$u(t)=\alpha_{n^*}(t), \tag{75}$$

and $v_{n^*}$ is derived in the same way as (71). From (74), $\theta_1$, $\theta_2$ are defined by

$$\theta_1 = \theta_{11} + \sum_{j=2}^{n^*}\left(\frac{\theta_{21}}{k_{j2}} + \frac{\|G_1\|^2}{4k_{j3}}\right), \quad \theta_2 = \theta_{12} + \sum_{j=2}^{n^*}\frac{\theta_{22}}{k_{j2}}. \quad (76)$$

$\theta_1^*$, $\theta_2^*$ in (17), are nominal values (time-invariant) of $\theta_1$, $\theta_2$ in (76).

**Step $n^*+1$)** The tuning laws of $\hat{\Phi}(t)$, $\hat{b}_0(t)$ and the auxiliary signals $\tilde{\alpha}_i(t)$ are determined such that

$$\dot{\hat{\Phi}}(t) = \Pr\{\tau_{\phi n^*}(t)\}, \quad \dot{\hat{b}}_0(t) = \Pr\{\tau_{bn^*}(t)\}, \quad (77)$$

$$\begin{aligned}\tilde{\alpha}_i(t) = &-k_{i6}\|\gamma_{\phi i-1}(t)\|^2\|\omega_1(t)\|^2 z_i(t)\\ &-(n^*-1)k_{i7}\|\gamma_{\phi i-1}(t)\|^2\|\omega_1(t)\|^2 z_i(t)\\ &-\sum_{j=2}^{n^*}\frac{\|G_{14}\|^2}{4k_{j7}}\gamma_{i-1}(t)^2 z_i(t)\\ &-k_{i8}\gamma_{bi-1}(t)^2 z_i(t) - (n^*-1)k_{i9}\gamma_{bi-1}(t)^2 u_{fn^*-1}(t)^2 z_i(t)\\ &-\sum_{j=3}^{n^*}\frac{g_2^2}{4k_{j9}}\gamma_{j-1}(t)^2 z_i(t), \quad (2 \le i \le r+1),\end{aligned} \quad (78)$$

where the constraints are defined by

$$g_\phi(\hat{\Phi}) = \|\hat{\Phi}\|^2 - M^2, \quad g_b(\hat{b}_0) = \|\hat{b}_0\|^2 - M^2. \quad (79)$$

$M(>0)$ is determined similarly to (16). Then, the following inequality is derived by utilizing the property of projection type adaptive laws.

$$\begin{aligned}&\{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\dot{\hat{\Phi}}(t) - \tau_{\phi n^*}(t)\}\\ &+\{\hat{b}_0(t) - b_0^*\}\{\dot{\hat{b}}_0(t) - \tau_{bn^*}(t)\}/g_2 d\tau\\ &= \{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\Pr(\tau_{\phi n^*}(t)) - \tau_{\phi n^*}(t)\}\\ &+\{\hat{b}_0(t) - b_0^*\}\{\Pr(\tau_{bn^*}(t)) - \tau_{bn^*}(t)\}/g_2 d\tau \le 0. \quad (80)\end{aligned}$$

Also, from the evaluation of $\Pr(\cdot)$, it follows that

$$\begin{aligned}&\|\Pr(\tau_{\phi n^*}(t))\|\\ &= \|\Pr\{G_{14}\omega_1(t)z_1(t) - \sum_{j=2}^{n^*} G_{14}\gamma_{j-1}(t)\omega_1(t)z_j(t)\}\|\\ &\le \|G_{14}\|\|\omega_1(t)\||z_1(t)| + \sum_{j=2}^{n^*}\|G_{14}\||\gamma_{j-1}(t)|\|\omega_1(t)\||z_j(t)|, \quad (81)\end{aligned}$$

$$\begin{aligned}&\|\Pr(\tau_{bn^*}(t))\| = \|\Pr\{g_2 z_1(t) - \sum_{j=2}^{n^*} g_2\gamma_{j-1}(t)u_{fn^*-1}(t)z_j(t)\}\|\\ &\le g_2|z_1(t)| + \sum_{j=2}^{n^*} g_2|\gamma_{j-1}(t)||u_{fn^*-1}(t)||z_j(t)|, \quad (82)\end{aligned}$$

and the next relation is obtained.

$$\begin{aligned}&-\sum_{j=2}^{n^*}\gamma_{\phi j-1}(t)\dot{\hat{\Phi}}(t)z_j(t) - \sum_{j=3}^{n^*}\gamma_{bj-1}(t)\dot{\hat{b}}_0(t)z_j(t) + \sum_{j=2}^{n^*}\tilde{\alpha}_j(t)z_j(t)\\ &\le \sum_{j=2}^{n^*}\frac{\|G_{14}\|^2}{4k_{j6}}z_1(t)^2 + \sum_{j=3}^{n^*}\frac{g_2^2}{4k_{i8}}z_1(t)^2. \quad (83)\end{aligned}$$

Finally, we derive the following evaluation of $V_i(t)$ ((17), (53), (54)) by utilizing (80), (81), (82), (83), where $v_i(t)$ are not necessarily equal to $v_i^*(t)$((36), (71)).

$$\begin{aligned}\sum_{i=1}^{n^*}\{V_i(t) - V_i(0)\} \le &\int_0^t r_1\left\{-\frac{z_1(\tau)}{2r_1} + v_1(\tau)\right\}^2 d\tau\\ &-\int_0^t\{h_1 z_1(\tau)^2 + r_1 v_1(\tau)^2\}d\tau\\ &-\int_0^t \gamma_1^{*2}\left\|\tilde{\Theta}_{1n^*}(\tau) - \frac{1}{2\gamma_1^{*2}}\omega(\tau)z_1(\tau)\right\|^2 d\tau\\ &+\gamma_1^{*2}\int_0^t\|\tilde{\Theta}_{1n^*}(\tau)\|^2 d\tau - \int_0^t \tilde{b}_0(\tau)v_1(\tau)z_1(\tau)d\tau\\ &+\sum_{i=2}^{n^*}\left[\int_0^t r_i\left\{-\frac{z_i(\tau)}{2r_i} + v_i(\tau)\right\}^2 d\tau\right.\\ &-\int_0^t\{h_i z_i(\tau)^2 + r_i v_i(\tau)^2\}d\tau\\ &-\int_0^t\gamma_i^{*2}\left\|\tilde{\Theta}_2(\tau) - \frac{1}{2\gamma_i^{*2}}\tilde{\omega}_i(\tau)z_i(\tau)\right\|^2 d\tau\\ &\left.+\gamma_i^{*2}\int_0^t\|\tilde{\Theta}_2(\tau)\|^2 d\tau\right]\end{aligned} \quad (84)$$

Then, we have the following main theorems.

**Theorem 1** *The adaptive control system described above (where $v_1^*(t) \sim v_{n^*}^*(t)$ ((36), (71)) are included) is uniformly bounded for arbitrary bounded variation of system parameters $b_0$, $p$, $\Phi_0$. Additionally, the control errors $z_1(t) \sim z_{n^*}(t)$ can be made arbitrarily small by proper design parameters $k_{12}$, $k_{13}$, $k_{i4}$, $k_{i5}$, $r_{0j}$ $(2 \le i \le n^*, 1 \le j \le n^*)$.*

**Proof:** By introducing state variables $v(t)$ of the stable systems (state-space representation $(F, G)$), the unstructured $\mathcal{L}(z_1(t))$ and $\mathcal{L}(f(z_1,t))$ are written in the following:

$$\dot{v}(t) = Fv(t) + G\begin{bmatrix} z_1(t)\\ f(z_1,t)\end{bmatrix}, \quad (85)$$

$$\begin{aligned}&\|\mathcal{L}(z_1(t))\|^2 + \|\mathcal{L}(f(z_1,t))\|^2\\ &\quad\le M_1\|v(t)\|^2 + M_2 z_1(t)^2 + M_3\phi(z_1(t))z_1(t)^2, \quad (86)\end{aligned}$$

$$PF + F^TP = -I \quad (P = P^T > 0). \quad (87)$$

Adding $v(t)$, $\bar{V}(t)$ is defined by

$$\bar{V}(t) = \frac{1}{2}\sum_{i=1}^{n^*} z_i(t)^2 + v(t)^T P v(t). \quad (88)$$

Then, we have

$$\begin{aligned}\dot{\bar{V}}(t) \le &M\|\omega(t)\|\cdot|z_1(t)| + \sum_{i=2}^{n^*} M\|\tilde{\omega}_i(t)\|\cdot|z_i(t)|\\ &-\underline{b}_0\delta\frac{k_{12} + k_{13}\|\omega(t)\|^2}{2r_{01}}z_1(t)^2 - \sum_{i=2}^{n^*}\frac{k_{i4} + k_{i5}\|\tilde{\omega}_i(t)\|^2}{2r_{0i}}z_i(t)^2\\ &-\delta\|v(t)\|^2\\ \le &-\frac{\underline{b}_0\delta k_{12}}{2r_{01}}z_1(t)^2 - \sum_{i=2}^{n^*}\frac{k_{i4}}{2r_{0i}}z_i(t)^2 - \delta\|v(t)\|^2\\ &+\frac{r_{01}}{2\underline{b}_0\delta k_{13}}M^2 + \sum_{i=2}^{n^*}\frac{r_{0i}}{2k_{i5}}M^2\\ \le &-\delta_0\bar{V}(t) + \frac{r_{01}}{2\underline{b}_0\delta k_{13}}M^2 + \sum_{i=2}^{n^*}\frac{r_{0i}}{2k_{i5}}M^2, \quad (89)\end{aligned}$$

where $0 < \delta$, $\delta_i$, $M < \infty$. Hence, we show that the adaptive system remains bounded for any bounded variations of system parameters. Additionally, we can make $\bar{V}(t)$ and then

$z_1(t) \sim z_{n^*}(t)$ arbitrarily small by choosing sufficiently large $k_{12}$, $k_{13}$, $k_{i4}$, $k_{i5}$ and sufficiently small $r_{0i}$. ∎

**Theorem 2** *For that adaptive control systems (including $v_1 \sim v_{n^*}$), we assume that $b_0$ is time-invariant. Then, $v_1^* \sim v_{n^*}^*$ ((36), (71)) are sub-optimal control inputs which minimize the upper bound on the following cost functional.*

$$J(t) \equiv \sup_{\tilde{\Theta}_{1n^*}, \tilde{\Theta}_2 \in \mathcal{L}^2} \Bigg[ \sum_{i=1}^{n^*} \int_0^t \{h_i z_i(\tau)^2 + r_i v_i(\tau)^2\} d\tau + \frac{1}{2}\sum_{i=1}^{n^*} z_i(t)^2 + \frac{1}{2}\sum_{i=1}^{2}\{\hat{\theta}_i(t) - \theta_i^*\}^2/g_{1i} + \frac{b_0}{2}\{\hat{p}(t) - \bar{p}\}^2/g_2 + \frac{1}{2}\{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\hat{\Phi}(t) - \Phi^*\} + \frac{1}{2}\{\hat{b}_0(t) - b_0^*\}^2/g_2 - \gamma_1^{*2}\int_0^t \|\tilde{\Theta}_{1n^*}(\tau)\|^2 d\tau - \sum_{i=2}^{n^*}\gamma_i^{*2}\int_0^t \|\tilde{\Theta}_2(\tau)\|^2 d\tau \Bigg]. \quad (90)$$

*Also we have the next inequality for those $v_1^* \sim v_{n^*}^*$.*

$$\sum_{i=1}^{n^*} \int_0^t \{h_i z_i(\tau)^2 + r_i v_i^*(\tau)^2\} d\tau + \frac{1}{2}\sum_{i=1}^{n^*} z_i(t)^2 + \frac{1}{2}\sum_{i=1}^{2}\{\hat{\theta}_i(t) - \theta_i^*\}^2/g_{1i} + \frac{b_0}{2}\{\hat{p}(t) - \bar{p}\}^2/g_2 + \frac{1}{2}\{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\hat{\Phi}(t) - \Phi^*\} + \frac{1}{2}\{\hat{b}_0(t) - b_0^*\}^2/g_2$$
$$\leq \frac{1}{2}\sum_{i=1}^{n^*} z_i(0)^2 + \frac{1}{2}\sum_{i=1}^{2}\{\hat{\theta}_i(0) - \theta_i^*\}^2/g_{1i} + \frac{b_0}{2}\{\hat{p}(0) - \bar{p}\}^2/g_2 + \frac{1}{2}\{\hat{\Phi}(0) - \Phi^*\}^T G_{14}^{-1}\{\hat{\Phi}(t) - \Phi^*\} + \frac{1}{2}\{\hat{b}_0(0) - b_0^*\}^2/g_2 + \gamma_1^{*2}\int_0^t \|\tilde{\Theta}_{1n^*}(\tau)\|^2 d\tau + \sum_{i=2}^{n^*}\gamma_i^{*2}\int_0^t \|\tilde{\Theta}_2(\tau)\|^2 d\tau. \quad (91)$$

*Especially, if $\tilde{\Theta}_{1n^*}$, $\tilde{\Theta}_2 \in \mathcal{L}^2$, then $z_1(t) \sim z_{n^*}(t) \to 0$.*

**Theorem 3** *For that adaptive system (including $v_1^* \sim v_{n^*}^*$ ((36), (71))), we assume that $b_0$ is not time-invariant. Then, the following inequality is derived.*

$$\sum_{i=1}^{n^*} \int_0^t \{h_i z_i(\tau)^2 + r_i v_i^*(\tau)^2\} d\tau + \frac{1}{2}\sum_{i=1}^{n^*} z_i(t)^2 + \frac{1}{2}\sum_{i=1}^{2}\{\hat{\theta}_i(t) - \theta_i^*\}^2/g_{1i} + \frac{b_0}{2}\{\hat{p}(t) - \bar{p}\}^2/g_2 + \frac{1}{2}\{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\hat{\Phi}(t) - \Phi^*\} + \frac{1}{2}\{\hat{b}_0(t) - b_0^*\}^2/g_2 + \int_0^t \frac{\tilde{b}_0(\tau)}{2r_1} z_1(\tau)^2 d\tau$$
$$\leq \frac{1}{2}\sum_{i=1}^{n^*} z_i(0)^2 + \frac{1}{2}\sum_{i=1}^{2}\{\hat{\theta}_i(0) - \theta_i^*\}^2/g_{1i} + \frac{b_0}{2}\{\hat{p}(0) - \bar{p}\}^2/g_2 + \frac{1}{2}\{\hat{\Phi}(0) - \Phi^*\}^T G_{14}^{-1}\{\hat{\Phi}(t) - \Phi^*\} + \frac{1}{2}\{\hat{b}_0(0) - b_0^*\}^2/g_2 + \gamma_1^{*2}\int_0^t \|\tilde{\Theta}_{1n^*}(\tau)\|^2 d\tau + \sum_{i=2}^{n^*}\gamma_i^{*2}\int_0^t \|\tilde{\Theta}_2(\tau)\|^2 d\tau, \quad (92)$$

*where $\frac{\tilde{b}_0(t)}{2r_1} z_1(t)^2 \geq 0$. Especially, if $\tilde{\Theta}_{1n^*}$, $\tilde{\Theta}_2 \in \mathcal{L}^2$, then, it holds that $z_1(t) \sim z_{n^*}(t) \to 0$, and the adaptive system converges to the sub-optimal control system in Theorem 2.*

**Proof:** From the evaluation of $\sum_{i=1}^{n^*} V_i(t)$ (84), Theorem 2 and 3 are easily derived. ∎

**Remark :** The sub-optimality in Theorem 2 comes from inequalities (21), (22), (32), (34), (37), (55), (62), (63), (68), (70), (72), (80), (81), (82), (83), (84).

**Theorem 4** *Assume that $(\Phi_0 - \Phi_0^*)$, $(b_0 - b_0^*)$, $(p - p^*) \in \mathcal{L}^2$ and $b_0 \to b_0^*$, $p \to p^*$ for certain constant $\Phi_0^*$, $b_0^*$, $p^*$ ($b_0^* p^* = 1$). Then, we have $z_1(t) \sim z_{n^*}(t) \to 0$ (as $t \to \infty$).*

**Proof:** The positive function $V_1(t)$ is newly re-defined by

$$V_1(t) = \frac{1}{2}z_1(t)^2 + \frac{1}{2}\sum_{i=1}^{2}\{\hat{\theta}_i(t) - \theta_i^*\}^2/g_{1i} + \frac{b_0^*}{2}\{\hat{p}(t) - p^*\}^2/g_{13} + \frac{1}{2}\{\hat{\Phi}(t) - \Phi^*\}^T G_{14}^{-1}\{\hat{\Phi}(t) - \Phi^*\}. \quad (93)$$

Then, we have the similar evaluation of $\sum_{i=1}^{n^*} V_i(t)$, where $\tilde{\Theta}_{1n^*}(t)$ and $\tilde{b}_0(t)$ are differently defined by

$$\tilde{\Theta}_{1n^*}(t) \equiv [(\Phi_{0n^*} - \Phi_0^*)^T, b_0^* - b_0]^T,$$
$$\tilde{b}_0(t) \equiv (b_0 - b_0^*)\hat{p}(t). \quad (94)$$

Contrary to the previous $\tilde{b}_0(t)$ (25), for this new $\tilde{b}_0(t)$, it does not hold that $\tilde{b}_0(t) \geq 0$. However, since $\tilde{b}_0(t)$ converges to zero asymptotically, it holds that

$$h_1 z_1(t)^2 + \frac{\tilde{b}_0(t)}{2r_1} z_1(t)^2 \geq \delta z_1(t)^2, \quad (\forall t \geq T), \quad (95)$$

for sufficiently large $T > 0$. Then, rewriting (92) with the initial time $T$, we show that $z_1(t) \sim z_{n^*}(t) \to 0$, where the boundedness of the adaptive system is also considered. ∎

## 4 Concluding Remarks

The proposed nonlinear control schemes can be seen as the specified form of $\kappa$-term compensation in integrator backstepping procedures [3], but the difference is that those are derived as solutions of particular $\mathcal{H}_\infty$ control problems, and that the $\mathcal{L}^2$ gains from unknown system parameters to the generalized outputs are prescribed explicitly.

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