Effects of Radial shifts of Eigenvalues on Norms of Linear Systems

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Abstract

We show that for linear systems represented in the controllable canonical form there are simple and systematic changes in the input/output properties of the closed-loop system from the disturbance input to the system output when controls are applied to produce a radial displacement of all the closed-loop poles. In particular this is true for the transfer function and the impulse response of the systems. This in turn leads to a better understanding of how such changes in the control gains affect the induced L_2 norm, (the H_∞ norm) and the induced L_∞ norm (the L_1 norm) of the system.

1. Introduction

The role of the transfer function and of the impulse response in assessing how a linear system attenuates the energy, and the amplitude, of disturbance inputs, respectively, are well known (see, e.g., [1],[2]). Thus the prominent role of the "H_∞ problem"[3,10] and the "L_1 problem"[4,5] in modem control system design. Results are presented here that clarify the effect of radial shifts of all eigenvalues (via feedback control) on the input-output properties of the system from a disturbance input to the system output, providing new insight into the relationship between system eigenvalues and its input-output properties. At the core is the analytical relationship describing the effect of a radial shift of the spectrum on the transfer function for the class of so called *simple systems* . A single input single output (SISO) system, with a single disturbance input

$$\dot{x} = Ax + Bu + Gw, \quad y = Hx \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^1$, $w(t) \in R^1$, and $y(t) \in R^1$, is called a simple system if: (a) the pair $\{A_c, B_c\}$ is in controllable canonical form (and without loss of generality it is assumed that $A_c = A_0$ where A_0 has all zeros in the bottom row, and so (1) models a chain of integrators); and (b) the disturbance enters the system only through the j^{th} state equation, and the output of interest is the i^{th} state of the system. Therefore,

$$\begin{aligned} G_j &= [0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0]^T \\ H_i &= 0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0 \end{aligned} \tag{2}$$

where the one in G_j is in the j^{th} position and the one in H_i is in the i^{th} position. Such a system will be referred o a *simple system* of category {i,j}.

In many design methods the effectiveness of the selected control law is measured by how well it attenuates the energy of L_2 disturbances, based on the bound $\|y\|_2 \le \|G_{ij}(s)\|_\infty \|w\|_2$, and how well it attenuates the amplitude of L_∞ disturbances, based on the bound $\|y\|_\infty \le \|g_{ij}\|_1 \, w\|_\infty$. Here $\|G_{ij}(s)\|_\infty$ is the induced L_2 norm, i.e. the H_∞ norm of the transfer function $G_{ij}(s)$[3], and $\|g_{ij}\|_1$ is the induced L_∞ norm, i.e. the L_1 norm of the impulse response $g_{ij}(t)$ [4],[5].

Suppose a linear state feedback control u = - Kx is used resulting in a closed-loop system with the spectrum $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, with the transfer function $G_{ij}(s)$, and the impulse response $g_{ij}(t)$ from w to y. Suppose next, that the control is modified so that as a result all closed-loop eigenvalues are shifted radially along their nominal directions (connecting the eigenalues to the origin of the complex plane), resulting in the shifted spectrum $\overline{\Lambda} = \{\overline{\lambda}_1, \overline{\lambda}_2, \dots, \overline{\lambda}_n\} = \{r\lambda_1, r\lambda_2, \dots, r\lambda_n\}$. We will show that there is a simple relationship between the shift ratio r and the transfer function, and that this relationship depends only on r and the pair {i,j}, and is independent of the nominal locations of the eigenalues (i.e., independent of the nominal gain vector K).

This relationship allows the assessment of the effect of the radial shift on the H_∞ norm, and on the impulse response, and thus, on the induced L_1 norm. This clarifies how, in simple systems, the structural location of the disturbance input and of the system output, captured in the pair {i,j}, affect the ability of linear controls to reduce the relevant system norms.

2. Problem Formulation

Consider the system (1) and choose a stabilizing linear controller $u = -Kx = k_1x_1 - k_2x_2 - \dots - k_nx_n$ so that the resulting closed-loop system has the poles (eigenvalues) $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, and the transfer function

$$G(s) = H_i(sI - A_c)^{-1}G_j = \frac{n_{ij}(s)}{\phi(s)} \tag{3}$$

where $A_c = A - BK$, and $\phi(s)$ is the characteristic polynomial,

$$\phi(s) = s^n + k_ns^{n-1} + \dots + k_2s + k_1 = \prod_{i=1}^{n}(s - \lambda_i) \tag{4}$$

Let a radial shift be applied resulting in new, desired, closed-loop poles $\overline{\Lambda}_r = \{\overline{\lambda}_1, \overline{\lambda}_2, \dots \overline{\lambda}_n\}$ related to the initial poles by

$$\overline{\lambda}_q = r\lambda_q, \ q = 1,2,\dots,n \tag{5}$$

where r is called the *shift ratio.* The shift can be either to lower magnitudes of the poles, by using r < 1, or to larger magnitudes, by using r > 1. By pole-placement the controller gains, that will insure the closed-loop system has poles defined by (5), are

$$\overline{k}_i = r^{n+1-i}k_i, \ i = 1,2,\dots,n \tag{6}$$

The issue of interest is to describe the effects of such spectral shifts (and use of the appropriate control that achieves it) on the input/output characteristics of the system, for $0 < r < \infty$ and for all pairs {i,j}.

3. Preliminaries

Recall the analytical expression for the resolvent of the closed-loop linear system when the feedback control is applied to (1). To that end introduce the notation

$$d^{n}(s) = \phi(s) \quad (7)$$

the characteristic polynomial of A, and

$$d^{n-j}(s) = s^{n-j} + k_n s^{n-j-1} + .. k^{j+2} s + k_{j+1}, \; j = 1,...,n \quad (8)$$

$$e^{j-1}(s) = k_j s^{j-1} + ... k_2 s + k_1, \; j = 1,...,n \quad (9)$$

$$f_j(s) = k_j s^{n-1} + k_{j-1} s^{n-2} + ... + k_1 s_{n-j}, \quad j = 1,2,...,n \quad (10)$$

Lemma 1. The resolvent $R(s) = [sI - A_c]^{-1} = \frac{1}{\phi(s)} W(s)$ where the elements of W(s) are given by

$$w_{ij}(s) = s^{i-1} d^{n-j}(s), \quad j \geq i, \; i, j = 1,..., n \quad (11)$$

$$w_{ij}(s) = -s^{i-j-1} e^{j-1}(s), \quad j < i, \quad i, j = 1,..., n \quad (12)$$

This determines the transfer function of a simple system for any control gain vector K and any pair {i,j}. From (12),

$$G_e^{ij}(s) = -\frac{s^{i-j-1} e^{j-1}(s)}{\phi(s)}, \; j<i \quad (13)$$

and, from (11),

$$G_e^{ij}(s) = \frac{s^{i-1} d^{n+1-j}(s)}{\phi(s)}, \quad j \geq i. \quad (14)$$

4. Effect of Spectral Shifts on the H_∞ norm

Let $G_{ij}(s)$ be the transfer function associated with the nominal eigenvalues, let $\overline{G}_{ij}(s)$ be the transfer function associated with the shifted eigenvalues (and so with the shifted controller gains $\{\overline{k}_0, \overline{k}_1,.., \overline{k}_n\}$) for a given pair {i,j}. Also, let $\overline{s} = rs$ define a *frequency shift* defined by the shift ratio r. The effect of the radial shift of all eigenvalues on the magnitude of the transfer function may then be summarized as follows:

Theorem 1. Suppose a radial shift of all closed-loop eigenvalues is achieved by the appropriate choice of new control gains. The transfer function after the shift is given by

$$G_{ij}(s) = r^{i-j-1} G_{ij}(\overline{s}) \quad (15)$$

Corollary 1. The H_p norms, $p = 2, \infty$, of the system after a frequency shift determined by r are given by

(i) $\| \overline{G}_{ij}(s) \|_2 = r^{i-j-1} \sqrt{r} \, \| G_{ij}(s) \|_2$ (16)

(ii) $\| \overline{G}_{ij}(s) \|_\infty = r^{i-j-1} \, \| G_{ij}(s) \|_\infty$ (17)

Theorem 2. The H_∞ norm of a *simple system* can be made arbitrarily small, by an appropriate radial shift for all pairs {i,j} except when the pair {i,j} satisfies the condition i-j-1 = 0. In this case the H_∞ norm is bounded by $\|G_e^{ij}(s)\|_\infty \geq 1$.

It is possible to also determine the effect of the radial shift on the transfer function from the disturbance to the control. Let $G_{uj}(s)$ be the transfer function from the disturbance input to the control output. The following results hold:

Theorem 3. (a) The transfer function $G_{uj}(s)$ of a simple system is given by

$$G_{uj}(s) = -\frac{f^{j}(s)}{\phi(s)} \quad (18)$$

(b) The transfer function $\overline{G}_{uj}(s)$ after a radial shift of all eigenvalues with a shift factor r is given by

$$\overline{G}_{uj}(s) = r^{n-j} G_{uj}(\overline{s}), \; j = 1,2,..., n \quad (19)$$

From Theorem 1 and Corollary 1 it follows that to reduce the H_∞ norm, a radial shift with $r > 1$ should be used whenever $i - j - 1 < 0$, and a radial shift with $r < 1$ should be used whenever $i - j - 1 > 0$. When $i - j - 1 = 0$ the H_∞ norm is invariant to radial shifts. It is of interest to note that when $i - j - 1 = 0$ the H_2 norm is increased for $r > 1$, and decreased for $r < 0$. Finally, it is of interest to note from Theorem 3 that the effect of radial shift on $G_{uj}(s)$ is independent of the system output (the index i). Furthermore, the application of a radial shift with $r < 1$, for $i - j - 1 > 0$, reduces the H_∞ norm of $G_{ij}(s)$ as well as the H_∞ norm of $G_{uj}(s)$. However, the application of a radial shift with $r > 1$, for $j \geq i-1$, reduces the H_∞ norm of $G_{ij}(s)$ but increases the H_∞ norm of $G_{uj}(s)$.

5. Effect of Spectral Shifts on the Impulse Response and the induced L_1 norm

Consider now the effect of the radial shift on the induced L_∞ norm. Recall that the induced L_∞ norm of the linear system, denoted by L_g, with impulse response g(t) is

$$\| L_g \|_\infty = \sup_w \frac{\|y\|_\infty}{\|w\|_\infty} = \int_0^\infty | g(t) | dt = \| g(t) \|_1 \quad (20)$$

where $|g(t)\|_1$ denotes the L_1 norm of g(t).

For completeness we also state the effect on the norms $\|g(t)\|_2$ and $\|g(t)\|_\infty$. The first will not have a direct application here, but the second plays a role in the bound

$$\| y \|_\infty \leq \| g \|_\infty \| w \|_1 \quad (21)$$

and has a role in considering the bound on $\|y\|_\infty$ for linear systems with nonzero initial conditions.

To determine how a radial shift affects the L_1, L_2 and L_∞ norm of g(t) recall that $\overline{G}_{ij}(s) = r^{i-j-1} G(\overline{s})$ for simple systems, and define $\overline{g}_{ij}(t) = \mathscr{L}^{-1}\{\overline{G}_{ij}(s)\}$ to be the impulse response of the simple system after the radial shift. Finally, define a *time scaling* by $\overline{t} = rt$.

Theorem 4. The impulse response of a simple system after a radial shift of all eigenvalues by a shift ratio r is given by

$$\overline{g}_{ij}(t) = r^{i-j} g_{ij}(rt) \quad (22)$$

where $g_{ij}(t)$ is the impulse response of the system before the shift.

Corollary 2. The L_p, p = 1, 2, ∞, norms of the impulse response after a radial shift with shift ratio r are:

(i) $\| \bar{g}_{ij} \|_1 = r^{i-j-1} \| g_{ij} \|_1$ (23)

(ii) $\| \bar{g}_{ij} \|_2 = r^{i-j-1} \sqrt{r} \| g_{ij} \|_2$ (24)

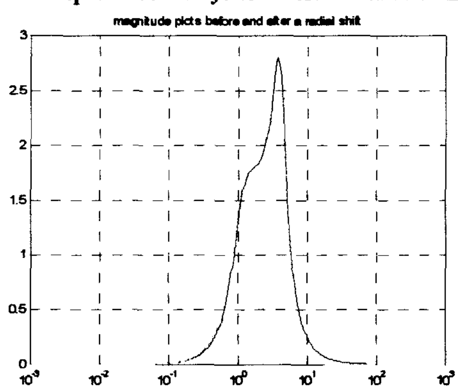
(iii) $\| \bar{g}_{ij} \|_\infty = r^{i-j} \| g_{ij} \|_\infty$ (25)

Example 2. Consider the linear system

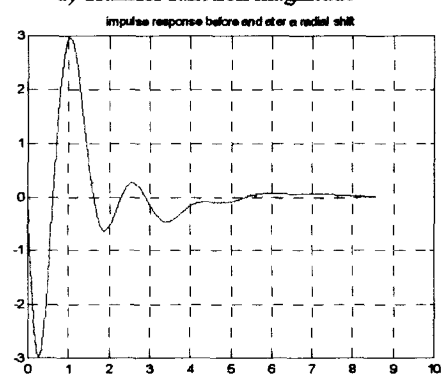
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} w$$

$$y = [0 \quad 0 \quad 0 \quad 1] x$$

Therefore, i = 4, j = 1, and place the poles at $\{-0.5 \pm j, -1 \pm j4\}$ by choosing the linear control $u = -21.25x_1 - 19.5x_2 - 20.25x_3 - 3x_4$. Suppose now a radial shift is applied and the control is modified to be $\bar{u} = -1.3281x_1 - 2.4375x_2 - 5.0625x_3 - 1.5x_4$ which insures that the new closed-loop system has eigenvalues $\{-0.25 \pm j0.5, -0.5 \pm j2$ implying $r = 0.5$. Magnitude of the transfer function of the system before and after the shift are shown in Figure 1.a. , and show a four-fold reduction of the peak magnitude (which also shifts to a lower frequency). The impulse response of the system before and after the radial shift



a) Transfer function magnitude



b) Inpulse response

Figure 1. Effect of shift on the transfer function and the impulse response

are shown in Figure 1.b. The impulse response before the radial shift has a peak value $\|g_{41}(t)\|_\infty=2.9795$, while the impulse response after the radial shift has a peak value of 0.3724, the ratio of the two peaks being $0.3724/2.9795 = 1/8=(0.5)^3 = r^{i-j}$, as predicted. Observe also that the peak for the original design occurs at t = 1.005[sec], while the peak for the modified design occurs at t = 2.01 [sec], again as predicted.

6. Application to general SISO systems - Bounds on the H_∞ norm

Consider now the general case of a controllable SISO system with a single disturbance input. Let the original system model be of the form

$$\dot{z} = \underline{A}z + \underline{B}u + \underline{G}w \qquad y = \underline{H}z \qquad (26)$$

where $\underline{G}$, $\underline{H}$ may have the form as G_j, H_i in (2). This system may be reduced to the form (1) using a similarity transformation to phase variable canonical form (e.g. [13]) where G and H will then have no particular form, and a compensating control can be applied to make $A_c = A_o$, a chain of integrators form. The question of interest is the same as before: Given a nominal set of eigenvalues $\Lambda_0 = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and the corresponding controller gains $\{k_1, k_2, \ldots, k_n\}$, let the transfer function of the system be

$$G(s) = H(sI - A_c)^{-1} G \qquad (27)$$

Determine the system transfer function after a radial spectral shift is applied resulting in the new closed-loop poles related to the initial poles by $\bar{\lambda}_q = r\lambda_q, \ q = 1,2,\ldots,n$.

Let, as before, $G_{ij}(s)$ be the transfer function of the simple system associated with the pair {i,j}, for which the expression was derived in Section 4. It is a simple consequence of Theorem 1 that the transfer function $\bar{G}(s)$, after a radial shift, is a linear combination of shifted transfer functions of simple systems, and so

$$\bar{G}(s) = H(sI - \bar{A}_c)^{-1} G = \sum_{i=1}^{n} \sum_{j=1}^{n} r^{i-j-1} h_i g_j \bar{G}_{ij}(s) \qquad (28)$$

The expression provides insight into the dependence of $\bar{G}(s)$, on the shift ratio r, and in particular of the behavior of $\bar{G}(s)$ when $r \to 0$, or $r \to \infty$. Expression (28) is also instrumental in applying a one-dimensional search over $0 < r < \infty$ to improve an existing design with respect to the induced L_2 norm.

Introducing the *structural distance* d = i – j –1 expression (30) can be written in the form

$$\bar{G}(s) = \sum_{d<0} \sum r^{i-j-1} h_i g_j \bar{G}_{ij}(s) + \sum_{d\geq 0} \sum r^{i-j-1} h_i g_j \bar{G}_{ij}(s) \qquad (29)$$

and the following then becomes evident:

Theorem 5. If both double sums in (29) are not empty there is a finite $r \in (0,\infty)$ at which $\bar{G}(s)$ has a global minimum. If there is no factor in the second sum for which d = 0, and the first sum is empty, the H_∞ norm can be reduced as low as desired by allowing $r \to 0$. When the second sum is empty, the H_∞ norm can be reduced as low as desired by allowing $r \to \infty$.

Expression (29) also leads to a useful bound on the induced L_2 norm. Let $\|G_{ij}(s)\|_\infty$, i,j, = 1,2,…,n, be the H_∞ norms associated with all simple system before application of the shift r. From (33) and norm properties:

$$\| \bar{G}(s) \| \leq B(r) = \sum_{i=1}^{n} \sum_{j=1}^{n} r^{i-j-1} |h_i| |g_j| \| G_{ij}(s) \|_\infty$$

This bound is a useful tool in narrowing the search for the optimal shift using simple computational algorithms. This is based on the fact that the bound can also written in the form

$$B(r) = \sum_{d<0} \sum r^{i-j-1} | h_i || g_j ||| G_{ij}(s) \|_\infty + \sum_{d \geq 0} \sum r^{i-j-1} | h_i || g_j ||| G_{ij}(s) \|_\infty \quad (31)$$

from which easily follows:

Theorem 6. The function B(r) is a strictly convex function of r over $0 < r < \infty$.

7. Applications to general SISO system - Bounds on the Induced L_1 norm

Consider now the impulse response of (3), and the consequences of a radial shift on the L_1 norm of the impulse responses. Let $g_{ij}(t)$ be the impulse response of the simple system associated with the pair {i,j} before the application of the spectral shift.

Theorem 7. The impulse response $\bar{g}(t)$ of the system (3) after the application of a spectral shift with the shift ratio r is given by

$$g(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} h_i g_j \bar{g}_{ij}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} r^{i-j} h_i g_j g_{ij}(rt) \quad (32)$$

This expression (32) can be used to perform a one-dimensional search over $0 < r < \infty$ and improve an existing nominal design with respect to the L_1 norm of the impulse response (induced L_∞ norm). It also leads to a useful bound on the L_1 norm of the impulse response. Let $\|g_{ij}(t)\|_1$ is the L_1 norm associated with the pair {i,j} before the shift. Then, from the above expression and standard bounding arguments:

$$\| \bar{g}(t) \|_1 \leq N(r) \quad (33)$$

where

$$N(r) = \sum_{i=1}^{n} \sum_{j=1}^{n} r^{i-j-1} | h_i || g_j | \| g_{ij}(t) \|_1 \quad (34)$$

In complete analogy with (29) we can write (34) as a sum of two terms and show that the bound N(r) is also a strictly convex function of r in $0 < r < \infty$. And so, if both sums in (35) are not empty there is a unique $r \in (0,\infty)$ at which N(r) has a global minimum. If the first sum is empty, associated with terms where $d \leq 0$, the L_1 norm can be reduced to a constant by allowing $r \to 0$, and to an arbitrarily small value if there are no terms with d = 0. If the second sum, associated with terms where d > 0, is empty, the L_1 norm can be reduced as low as desired by allowing $r \to \infty$.

Example 3. For 3rd order systems the structural distance, d = i-j-1 can take on any integer value from –3 to 1, and so this class of systems can illustrate a diverse set of possibilities. Considering the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} w, \qquad y = [1 \;\; 1 \;\; 1] x$$

and selecting, arbitrarily, the nominal controller gains to be $k_0 = 2$, $k_1 = 1.5$, $k_2 = 2$, results in the closed-loop eigenvalues $\lambda_1 = -0.1065 + 1.0526i$, $\lambda_2 = -0.1065 - 1.0526i$, $\lambda_3 = -1.7869$.

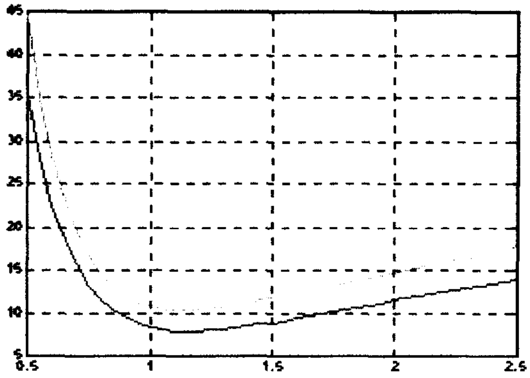


Figure 2. H_∞ and L_1 norms in funciton of r

The results in Figure 2, show that in this case the minima are achieved, for both norms, for r = 1.15 , the minimal value of the H_∞ norm being J_{inf} * =7.7685 and the minimal value of the L_1 norm being $J_1^* = 10.3263$.

To demonstrate the effect of the radial shift, the responses y(t) for a typical persistent disturbance, shown in Figure 3 (a combination of a component with a uniform distribution and a component with a gaussian distribution), are shown in Figures 4, 5 and 6 for shift ratios r = 2, 1 and 0.5, respectively. While this is not the worst disturbance the results show the better amplitude bounding of the response with r = 1, which is close to the optimal value, than with the other two values. Also shown is the common input w(t) used in all three simulations.

For the considered 3rd order system, with the defined control gains, the minimizing r remains close to r = 1 for both the H_∞ norm and the L_1 norm as long as both sums, containing terms d > 0 and d < 0, are not empty. This is by no means typical. When the nominal gains are $k_0 = 1$, $k_1 = 3$, $k_2 = 2$ with G = [1 1 1], H = [1 10 1], the dependence of the Hoo norm and the L_1 norm on r are displayed in Figure 4, and show that J_{inf}^* = 8.4962, and occurs for r = 1.775 while $J_1^* = 10.7916$ and occurs for r = 9.51. The effect on the shape of the impulse response is shown in Figure 5 which displays the impulse responses for r = 1, 1.75 and 9.51 (nominal, H_∞ norm optimal and L_1 norm optimal shifts, respectively).

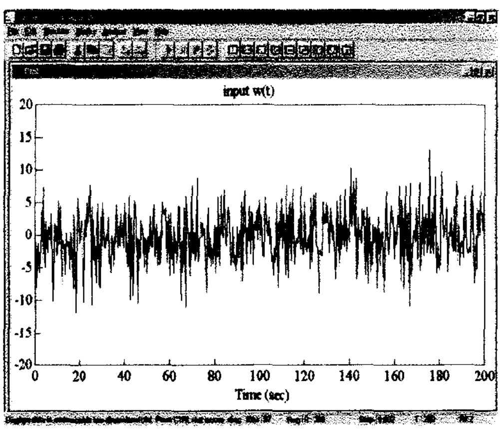


Figure 3. Input disturbance

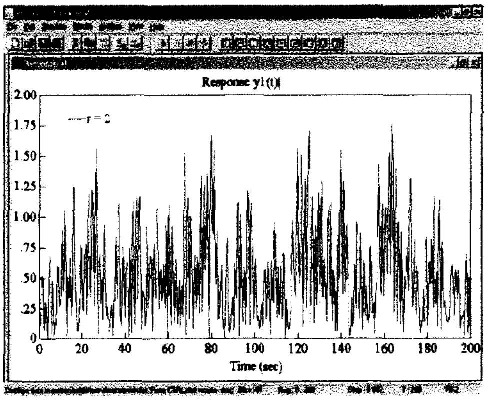


Figure 4. Shift r = 2 (greater than optimal)

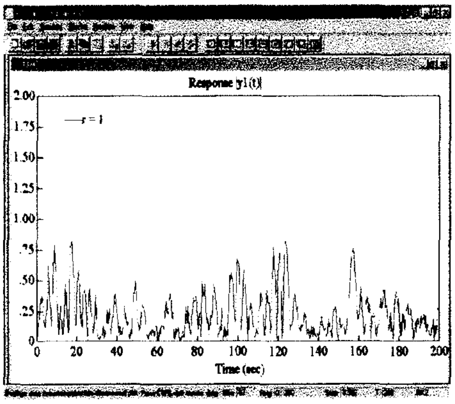


Figure 5. Shift r = 1 (close to optimal)

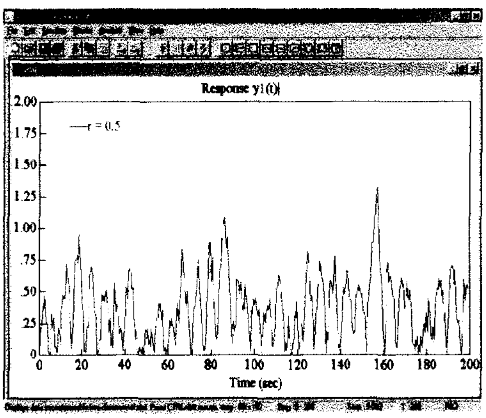


Figure 5. Shift r = 0.5 (smaller than optimal)

Remark. Observe that the above results are independent of the nominal control vector K. That is, whether the H_∞ norm of the system, and the L_1 norm of the impulse response, can be reduced to an arbitrary low value is a structural property of linear systems, and depends only on the pair {i,j} in case of simple linear systems, and on the set $S = \{i,j: h_i g_j \neq 0, i,j,=1,2,\ldots,n\}$ of all the pairs of active indices for systems of the form (3). As captured in Theorem 5, if for all active pairs in S it holds that i-j-1 > 0 then both norms can be reduced to an arbitrarily low value by using a shift $r \to 0$, and if for all pairs i-j-1 < 0 then the same effect can be achieved using a shift $r \to \infty$. If both types of pairs exist the minimal value of the norms is achieved for a finite r. Similarly, in view of the results in Corollary 1, expression (25), if for all active pairs in S it holds that i-j > 0 then the L_∞ norm can be reduced to an arbitrarily low value by using a shift $r \to 0$, and if for all pairs i-j < 0 then the same effect can be achieved using a shift $r \to \infty$.

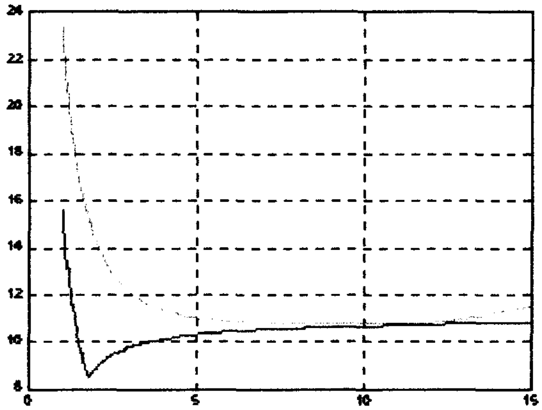


Figure 7. H_∞ and L_1 norms in funciton of the shift ratio r

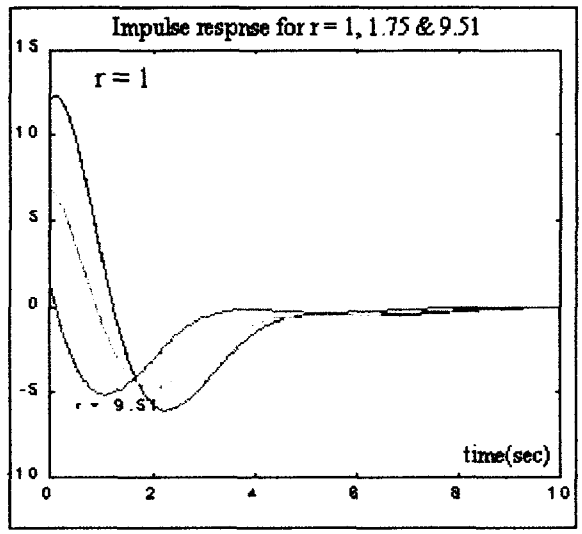


Figure 8. Impluse response for = 1, 1.75 and 9.51

Remark. In [6] the author considered the conditions under which the response of the system will exhibit a growth in peak amplitude as the eigenalues of the system recede from the origin toward infinity in function of the initial condition. Since the Laplace transform of the zero input response of the system is given by

$$y_{zi}(s)=H(sI-A_c)^{-1}x_0 \tag{36}$$

The expression for $y_{zi}(s)$ may be considered as the transfer function of a fictitious linear system defined by the triple { A_c,x_0 H}. The results obtained above hold here verbatim, and so whenever $H = H_i$, where H_i is defined by (2), and x_0 has nonzero components only for indices j < i, then i-j > 0 for all terms in the sum in (36). The L_∞ norm of the response $y_{zi}(t)=He^{A_c t}x_0$ (the peak amplitude) will approach infinity as $r \to \infty$, i.e. as the eigenvalues recede towards infinity. In [6] this is illustrated on a second order system where y = [0 1]x and so i = 2, while the initial condition is x_0 = {-1, 0}, and so j = 1. The results presented here encompass this result, and also enable us to treat simultaneously the effect of the radial shifts on both the zero input response , and the zero state response to disturbance inputs.

Remark. One application of the obtained results and insight is in the domain of finding improved state feedback control laws to reduce the induced L_∞ norm of linear system. It is well known that the H_∞ norm is a lower bound on the L_1 norm[1], and it has been shown here that for all simple systems a radial shift that reduces further the H_∞ norm of the system will also reduce the L_1 norm in same proportion. Hence, the nominal gain K be obtained by solving the H_∞ problem, that is by reducing the H_∞ norm, from the disturbance input to the system output, as much as feasible. The standard solution to the H_∞ problem will also reduce the H_∞ norm from the disturbance input to the system output, and this may serve as a procedure to determine a nominal design. Equipped with a nominal design there are two possible ways to proceed: (a) Compute the *optimal radial shift*: For the nominal set of controller parameters determine all transfer functions of all active simple systems $G_{ij}(s)$, i,j, = 1,2,...,n, and then do a one dimensional search over r to determine the minimal value of the L_1 norm (34). This involves computing for each considered value of r the impulse response g(t) and its L_1 norm , or (b) *Compute the radial shift which minimizes the norm bound*: Compute for all active simple systems the L_1 norms $\|g_{ij}\|_1$, i,j, = 1,2,...,n for the nominal controller parameters and minimize the L_1 norm bound (34), a convex minimization problem. Once the optimal shift is determined the final gains are obtained from (6).

Remark. The results are useful in minimizing, or bounding the L_∞-norm of the output when both the zero input component and a zero state component are present for two cases: (a) the system has a known non-zero initial conditions, and (b) the non-zero initial condition belongs to a defined set. Considering just the first case, one can derive a bound $\| y \|_\infty \leq Y$ where Y takes the form

$$Y = \| g_0 \|_\infty + \| g \|_1 \| w \|_\infty$$

with

$$\| g_0 \|_\infty = \max_{\tau \in [0,\infty)} \{ | He^{A_c t} x_0 | \}$$

Hence after an initial design, finding the optimal shift ratio reduces to minimizing a sum of an L_∞ norm of the fictive system defined by the triples {H,Ac,x0}, and the L1 norm of the system defined by the triplet {H,Ac,G}.

Remark. The results may also be of interest in considered stability problems in certain classes of nonlinear systems, which is currently under investigation. A class of typical systems to which the approach directly applies was defined in [8] where the role of the H_∞ norm was highlighted. However, it is the L_1 norm that is relevant for most unbounded nonlinearities, and the results reported here are expected to contribute to the better understanding of the stability issues in these nonlinear problems, and the ability to expand the region of attraction of a nominal design.

8. Conclusions

The effects of radial shifts on the system transfer function and impulse response provide simple ways of designing state feedback controls that provide improved response to persistent L_∞ disturbances, by reducing the induced L_∞ norm of the system. It is noted that the L_1 problem has been solved for linear systems [4],[5]. However, the controller is infinite dimensional and the associated computational burden is quite considerable. For that reason, the discrete version has received much more attention, and has been studied, particularly as part of a multi-objective problem formulations[7],[8],[9]. The controllers proposed here are suboptimal but have the advantage that they have the simple static state feedback structure, and are easy to compute. Specifically, given a nominal control law one seeks the *conditionally optimal design* that corresponds to the optimal radial shift of the nominal eigenalues obtained by a nominal state feedback design. One-dimensional search algorithms are proposed for this conditional optimization. It is believed that the approach will efficiently generate L_1-norm suboptimal state feedback designs by using the H_∞ suboptimal designs [12] to generate the nominal control law. This is based on the simple relationship between the H_∞ norm and the L_1 norm under radial shifts developed in Section 4.

9. References

1. C.A. Desoer and M. Vidyasagar, "Feedback Systems: Input-Output Properties, Academic Press, New York, 1975.
2. S.P. Boyd and C.H. Barratt, *Linear Controller Design, Limits of Performance*, Prentice-Hall Inc., Englewood Cliffs, N.J. 1991.
3. G. Zames, "Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms and approximate inverses," *IEEE Trans. On Automatic Control*, Vol. AC-26, pp. 301-320, April, 1981.
4. M. Vidyasagar, "Optimal Rejection of Persistent Bounded disturbances" *IEEE Trans. On Automatic Control*, Vol. AC-31, pp. 5234, June, 1986.
5. M.A. Dahleh and J.B. Pearson, Jr., "L^1-Optimal Compensators for continuous-time Systems," *IEEE Trans. On Automatic Control*, Vol. AC-32, pp. 889-895, Oct., 1987.
6. T. Mita, "On Zeros and Responses of Linear Regulators and Linear Observers," *IEEE transaction on Automatic Control*, Vol. AC-22, No. 3, June, 1977, pp.423-427.
7. M.V. Salapaka, M. Dahaleh and P.G. Voulgaris, "MIMO Optimal control Design: The interplay between the H2 norm and the l_1 norms," IEEE Transactions on Automatic Control, Vol. 40, 1995, pp. 1881-1888.
8. P. G. Voulgaris, "Optimal H_2/l_1 control via duality theory"", IEEE Transactions on Automatic Control, Vol. 43, No. 10, 1998, pp. 1374-1388.
9. M. Snazier and F. Blanchini, "Mixed $H\infty/L_\infty$ suboptimal controllers for single input single output continuous-time systems"", Proc. CDC, San Antonio, TX, Dec., 1993, Vol. 3, pp. 34-38.
10. R. Sepulchre, "Slow Peaking and Low-Gain Design for Global Stabilization of Nonlinear Systems", 37th IEEE Conf. On Decision and Control, Dec. 16-18, Tampa, FL, 1998.
11. B.D.O. Anderson and J.B. Moore, *Optimal Control: Linear Quadratic Methods*, Prentice-Hall Inc., Englewood Cliffs, N.J. 1989.
12. J. Medanic, "Integral Control for Nonlinear Systems with a Single Nonlinearity" submitted for publication
13. J.C. Doyle, K. Glover, P.P. Khargonekar and B.A. Francis, "State-space solutions to standard H_2 and H_∞ control problems", *IEEE Trans. On Automatic Control*, Vol. AC-34, No. 8, pp. 831-84.