

Design of Linear Dynamic Controllers for a Class of Nonlinear Systems

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Abstract

Integral transformations are applied to a fourth order nonlinear system with two nonlinearities. The stabilizing control is developed based on an asymptotic analysis that defines the appropriate controller structure, one in which all variables remain bounded and approach well defined limits. The results are generalized to higher order systems with a similar structure, and are the basis for the design of a dynamic controller to stabilize a related class of nonlinear systems.

1. Introduction

This paper considers the problem of stabilizing the class of nonlinear systems having the structure

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(x_4) \\ \dot{x}_2 &= x_3 + \varphi_2(x_4) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{aligned} \quad (1)$$

This is an open problem because existing iterative methods (backstepping [2], forwarding [3], saturation methods [4]) do not apply for arbitrary $\varphi_i(x_4)$, $i = 1, 2$. The approach developed generalizes to similar systems of higher order (the exact structure of which will be defined at the appropriate time). We make only one assumption on both nonlinearities (in addition to the standard assumption that $\varphi_1(0) = 0$, $\varphi_2(0) = 0$).

Definition 1. A nonlinear function of a scalar variable is *exponentially integrable* if

$$\int_0^{\infty} \phi(\sigma(\tau)) d\tau < \infty \quad (2)$$

for all exponentially decaying functions of time $\sigma(t)$.

This property is shared by many functions including all polynomial functions of finite order, and all functions bounded by a polynomial function of bounded order.

Assumption 1. The functions $\varphi_1(x_4)$, $\varphi_2(x_4)$ are exponentially integrable.

The feature that distinguishes (1) from the class of 4th order systems considered in [1], where the integral transformation approach was introduced, is that the nonlinear terms occur in two state equation. It will be shown that the sequential application of two integral transformations produces a linear system, which is easily stabilized. The effect of the integral

transformations, and specifically the effect controls designed for the transformed system have on the original system is the main issue in the analysis of the problem. Because the original system contains controller states that grow unbounded the approach concentrates on determining the appropriate "controller type" by illuminating the nature of the asymptotic behavior (i.e. the steady state behavior) of (1) when specific controller structures are employed. The analysis of the asymptotic behavior when additional integral components are added to the controller and the development of a final tenable controller, in which all signals remain bounded, is the main result of this paper. A stability analysis is then conducted for the resulting closed-loop system and it identifies conditions that determine controller gain relationships that produce semi globally asymptotically stable (semi GAS) behavior. Two examples, with one, or both, sign definite nonlinearities are presented to illustrate gain relationships that result in stable response and the nature of the response. The approach is also employed to develop a class of dynamic controllers for a related class of nonlinear systems.

2. Integral transformation of the nonlinear system

Consider the approach to the problem of stabilizing (1) by reducing it first to a linear system by applying two integral transformations. Suppose the regular integral transformation

$$\begin{aligned} y_1 &= x_1 - \int_0^t \varphi_1(x_4(\tau)) d\tau \\ y_2 &= x_2 - \int_0^t \varphi_2(x_4(\tau)) d\tau \end{aligned} \quad (3)$$

$$\begin{aligned} y_3 &= x_3 \\ y_4 &= x_4 \end{aligned}$$

is applied to (1). The result is the system

$$\begin{aligned} \dot{y}_1 &= y_2 + \int_0^t \varphi_2(y_4(\tau)) d\tau \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= y_4 \\ \dot{y}_4 &= u \end{aligned} \quad (4)$$

which is structurally the same as that treated in (Medanic 1998). A second regular integral transformation

$$\begin{aligned} z_1 &= y_1 - \int_0^t \psi(\tau) d\tau, \quad \psi(\tau) = \int_0^{\tau} \varphi_2(y_4(\lambda)) d\lambda \\ z_2 &= y_2 \\ z_3 &= y_3 \\ z_4 &= y_4 \end{aligned} \quad (5)$$

reduces (4) to a chain of four integrators:

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \quad i = 1, 2, 3 \\ \dot{z}_4 &= u \end{aligned} \quad (6)$$

From this it is easy to deduce that the control

$$u = -k_1 z_1 - k_2 z_2 - k_3 z_3 - k_4 z_4, \quad (7)$$

with the proper choice of the control gains, produces globally asymptotically stable (GAS) closed-loop behavior of (6).

Definition 2. A collection of gains $\{k_1, k_2, \dots, k_n\}$ associated with a linear n -th order system will be referred to as a *set of stabilizing gains*, if $\phi(s) = s^n + k_n s^{n-1} + \dots + k_2 s + k_1$ is a stable polynomial (all eigenvalues in the left half of the complex plane).

Applying (5) to (7) results in the control

$$u = -k_1 \left[y_1 - \int_0^t \int_0^\lambda \varphi_2(y_4(\tau)) d\tau \right] - k_2 y_2 - k_3 y_3 - k_4 y_4 \quad (8)$$

which, in view of the properties of integral transform [1], produces a closed-loop system (4) with a bounded solution where asymptotically $y_2, y_3, y_4 \rightarrow 0$ and $y_1 \rightarrow c_1$, with c_1 depending on the initial conditions. Finally, applying the transformation (3) to (8) results in the control

$$\begin{aligned} u = & -k_1 \left[x_1 - \int_0^t \varphi_1(x_4) d\tau - \int_0^t \int_0^\lambda \varphi_2(x_4(\tau)) d\tau \right] \\ & - k_2 \left[x_2 - \int_0^t \varphi_2(x_4) d\tau \right] - k_3 x_3 - k_4 x_4 \end{aligned} \quad (9)$$

which will produce a bounded solution of (1) for any finite t . (These properties hold for any collection of stabilizing gains $\{k_1, k_2, k_3, k_4\}$.) Analysis of the closed-loop system defined by (1) and (9), however, shows that $|x_1(t)| \rightarrow \infty$ as $t \rightarrow \infty$. To see this recall, from (5), that the limiting values of the vectors z , and y are $z = \{0, 0, 0, 0\}$, $y = \{c_1, 0, 0, 0\}$, and so, asymptotically (recall (2)),

$$\dot{x}_1 = c_1 - \int_0^t \varphi_2(x_4(\tau)) d\tau, \quad x_1(t) \rightarrow c_1 t - c_0 \quad (10)$$

Hence, the trajectories of (1) when the control (9) is applied do not converge to the origin. The initial controller, defined by (9), does not satisfy the specifications for the desired asymptotic behavior of the system. Because the system with a single nonlinearity, studied in [1], could be made semi GAS by introducing an integral component in the controller, the same idea will be pursued here.

3. Addition of a single integral state

Adding an integral state, and a term proportional to the integral state to the control (10), results in the closed-loop system

$$\begin{aligned} \dot{x}_0 &= x_1 \\ \dot{x}_1 &= x_2 + \varphi_1(x_4) \\ \dot{x}_2 &= x_3 + \varphi_2(x_4) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{aligned} \quad (11)$$

with the control given by (9) replaced by

$$\begin{aligned} u = & -k_0 x_0 - k_1 \left[x_1 - \int_0^t \varphi_1(x_4) d\tau - \int_0^t \int_0^\lambda \varphi_2(x_4(\tau)) d\tau \right] \\ & - k_2 \left[x_2 - \int_0^t \varphi_2(x_4) d\tau \right] - k_3 x_3 - k_4 x_4 \end{aligned} \quad (12)$$

It goes without saying that the set $\{k_0, k_1, k_2, k_3, k_4\}$ is now chosen to be a set of stabilizing gains as defined previously.

Since for $k_0 = 0$ there is no finite time blow-up, assume based on continuity that there is no finite time blow-up for sufficiently small k_0 . The effect of the control (12) on (11) is then: (i) either a steady state is reached where states $\{x_1, x_2, x_3, x_4\}$ converge to zero steady state values, while the integral state x_0 converges to a finite value, or (ii) the state x_1 does converge to a finite value, but x_0 continues to increase in magnitude proportional to time.

We claim that (i) cannot satisfy the steady state conditions, and that in fact case (ii) occurs. To show this consider an asymptotic analysis of (i). That is, assume that asymptotically, for sufficiently large t , $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ and $u = 0$, while $x_1 = c$, so that asymptotically $x_0 = a_0 + tc$. The condition $u = 0$ implies

$$-k_0 x_0 - k_1 c + k_1 \int_0^t \varphi_1(x_4) d\tau + k_1 \int_0^t \psi d\lambda + k_2 \int_0^t \varphi_2(x_4) d\tau \approx 0$$

where, by Assumption 1, three of the five terms in the above expression converge to constants, and the integral involving ψ grows unbounded. This condition will, therefore, hold asymptotically only if in the limit

$$-k_0 x_0 + k_1 \int_0^t \psi d\lambda \approx 0$$

which indicates that x_0 cannot reach a finite steady state value because in the limit $\psi \rightarrow \text{const}$, and the integral grows unbounded. Hence case (ii) above occurs. Furthermore, since $u = 0$ asymptotically, the first derivative is also zero asymptotically, leading to the condition

$$\dot{u} = -k_0 \dot{x}_1 - k_1 \left[x_2 - \int_0^t \varphi_2(x_4(\tau)) d\tau \right] - k_2 x_3 - k_3 \dot{x}_4 - k_4 \dot{x}_4 \approx 0 \quad (13)$$

which reduces to

$$-k_0 c + k_1 \int_0^\lambda \varphi_2(x_4(\tau)) d\tau = 0,$$

and defines c and the steady state value of x_1 . That is, as $t \rightarrow \infty$,

$$x_0(t) = \frac{k_1}{k_0} \int_0^t \psi d\lambda \rightarrow \infty \quad (14)$$

$$c = \frac{k_1}{k_0} \int_0^\infty \varphi_2(x_4(\tau)) d\tau \quad (15)$$

And so, concerning asymptotic behavior, there are two drawbacks associated with the control (12). First, x_1 does not converge to zero and so the system state does not converge to the origin. Second $x_0(t)$ grows unbounded and so the controller is untenable.

4. Addition of two integrator states

Consider now the effect of the control

$$\begin{aligned} u = & -k_{00} x_{00} - k_0 x_0 \\ & - k_1 \left[x_1 - \int_0^t \varphi_1(x_4) d\tau - \int_0^t \int_0^\lambda \varphi_2(x_4(\tau)) d\tau \right] \\ & - k_2 \left[x_2 - \int_0^t \varphi_2(x_4) d\tau \right] - k_3 x_3 - k_4 x_4 \end{aligned} \quad (16)$$

where and additional integral state has been added to the controller, resulting in the closed-loop system:

$$\begin{aligned}
 \dot{x}_{00} &= x_0 \\
 \dot{x}_0 &= x_1 \\
 \dot{x}_1 &= x_2 + \varphi_1(x_4) \\
 \dot{x}_2 &= x_3 + \varphi_2(x_4) \\
 \dot{x}_3 &= x_4 \\
 \dot{x}_4 &= -k_{00}x_{00} - k_0x_0 - k_1 \left[x_1 - \int_0^t \varphi_1(x_4) d\tau - \int_0^{\lambda} \int_0^{\lambda} \varphi_2(x_4(\tau)) d\lambda \right] \\
 &\quad - k_2 \left[x_2 - \int_0^t \varphi_2(x_4) d\tau \right] - k_3x_3 - k_4x_4
 \end{aligned} \tag{17}$$

Assume the control (16) can provide a bounded response for sufficiently small k_{00} and consider the asymptotic analysis. It will be shown that the asymptotic behavior of this system is characterized by $x_i = 0$, $i = 1, \dots, 4$, $x_0 = \text{const}$, with the state $x_{00}(t)$ asymptotically approaching a linear function of t . So, assume, based on (17), that in the steady state $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ and $u = 0$. From the fact that asymptotically $u \rightarrow 0$, $du/du \rightarrow 0$, and $d^2u/dt^2 \rightarrow 0$ it is possible to determine the steady state conditions for x_1 , x_0 and x_{00} . From $u = 0$ follows the condition

$$-k_{00}x_{00} - k_0x_0 - k_1 \left[x_1 - \int_0^t \varphi_1(x_4) d\tau - \int_0^{\lambda} \int_0^{\lambda} \varphi_2(x_4) d\lambda \right] + k_2 \int_0^t \varphi_2(x_4) d\tau = 0 \tag{18}$$

Since

$$u = -k_{00}x_{00} - k_0x_1 - k_1 \left[x_2 - \int_0^t \varphi_2(x_4) d\tau \right] - k_2x_3 - k_3x_4 - k_4x_4 \tag{19}$$

$$\ddot{u} = -k_{00}\dot{x}_1 - k_0[x_2 + \varphi_1(x_4)] - k_1\dot{x}_3 - k_2\dot{x}_4 - k_3\dot{x}_4 - k_4\dot{x}_4 \tag{20}$$

the conditions $\dot{u} = 0$, $\ddot{u} = 0$ reduce, respectively, to

$$-k_{00}x_0 - k_0x_1 + k_1 \left[\int_0^t \varphi_2(x_4) d\tau \right] = 0 \tag{21}$$

$$-k_{00}\dot{x}_1 = 0 \tag{22}$$

Therefore, $\dot{x}_1 = 0$, and from (18) and (21), asymptotically

$$x_0 = \frac{k_1}{k_{00}} \left[\int_0^{\infty} \varphi_2(x_4) d\tau \right] \rightarrow \text{const} = c_1 \tag{23}$$

$$\begin{aligned}
 x_{00}(t) &= -\frac{k_0}{k_{00}}x_0 + \frac{k_1}{k_{00}} \left[\int_0^t \varphi_1(x_4) d\tau - \int_0^{\lambda} \int_0^{\lambda} \varphi_2(x_4) d\lambda \right] \\
 &\quad + \frac{k_2}{k_{00}} \int_0^t \varphi_2(x_4) d\tau \rightarrow c_1t + c_0
 \end{aligned} \tag{24}$$

Hence, the asymptotic behavior indicates that the first drawback has been eliminated since now $x_1(t) \rightarrow 0$. But, the second drawback has not, since there is now the signal $x_{00}(t)$ in the controller that grows unbounded. This controller is, therefore, also an untenable solution.

5. Modified Controller Structure

The final controller is developed by correcting the drawbacks characterizing the previous candidate controller. Observe from (18) that if the control is to asymptotically

approach zero, the two terms that grow with time must balance each other out. This raises the possibility that signals in the controller that grow unbounded may be eliminated by omitting the two offensive terms: the term involving the integral state x_{00} , and the term involving ψ . The suggests that the candidate controller have the structure

$$\begin{aligned}
 u &= -k_0x_0 - k_1 \left[x_1 - \int_0^t \varphi_1(x_4) d\tau \right] \\
 &\quad - k_2 \left[x_2 - \int_0^t \varphi_2(x_4) d\tau \right] - k_3x_3 - k_4x_4
 \end{aligned} \tag{25}$$

with the system given by (11), and the state x_{00} discarded since it has no function in the resulting system.

Consider first the steady state. Obviously if the steady state is established (recall $\varphi_1(0) = 0$, $\varphi_2(0) = 0$), then $x_1, x_2, x_3, x_4 = 0$, and the fifth condition reduces to

$$-k_0x_0 + k_1 \left[\int_0^t \varphi_1(x_4) d\tau \right] + k_2 \left[\int_0^t \varphi_2(x_4) d\tau \right] = 0 \tag{27}$$

and provides a finite steady state value for x_0 . Hence, in the steady state, both drawbacks of previous control structures have been eliminated.

Since (25) is a viable controller in terms of steady state behavior consider now the stability issue. The goal is to identify subregions of stabilizing gains that should provide semi GAS behavior of the nonlinear system. A differential equation for x_4 is obtained by successive differentiation:

$$\begin{aligned}
 x_4^{(5)} + k_4x_4^{(4)} + k_3\ddot{x}_4 \\
 + \left[k_2 + k_1 \frac{\partial \varphi_2(x_4)}{\partial x_4} + k_0 \frac{\partial \varphi_1(x_4)}{\partial x_4} \right] \dot{x}_4 \\
 + k_1 \left(1 + \frac{d}{dt} \left[\frac{\partial^2 \varphi_2(x_4)}{\partial x_4^2} \right] \right) \dot{x}_4 \\
 + k_0 \left(\frac{d}{dt} \left[\frac{\partial^2 \varphi_1(x_4)}{\partial x_4^2} \right] + \frac{\partial \varphi_2(x_4)}{\partial x_4} \right) \dot{x}_4 + k_0x_4 = 0
 \end{aligned} \tag{28}$$

Clearly, conditions

$$\begin{aligned}
 \left| k_1 \frac{\partial \varphi_2(x_4)}{\partial x_4} + k_0 \frac{\partial \varphi_1(x_4)}{\partial x_4} \right| \ll |k_2| \\
 \left| k_0 \left(\frac{d}{dt} \left[\frac{\partial^2 \varphi_1(x_4)}{\partial x_4^2} \right] + \frac{\partial \varphi_2(x_4)}{\partial x_4} \right) \right| \ll \left| k_1 \left(1 + \frac{d}{dt} \left[\frac{\partial^2 \varphi_2(x_4)}{\partial x_4^2} \right] \right) \right|
 \end{aligned} \tag{29}$$

need to be satisfied in order to make the effect of some nonlinear terms negligible. These conditions can be stated in the form of a single condition: $k_0 \ll k_1 \ll k_2$ where the magnitudes of k_0, k_1, k_2 will depend on $\varphi_1(x_4)$ and its derivatives within the region of interest, X . If this condition is satisfied expression (28) may then be approximated by

$$\begin{aligned}
 x_4^{(5)} + k_4x_4^{(4)} + k_3\ddot{x}_4 + k_2\dot{x}_4 \\
 + k_1 \left[1 + \frac{d}{dt} \left[\frac{\partial^2 \varphi_2(x_4)}{\partial x_4^2} \right] \right] \dot{x}_4 + k_0x_4 = 0
 \end{aligned} \tag{30}$$

It is clear from (30) that to insure asymptotic stability one must insure that the effect of the term

$$\Delta(x_4) = \frac{d}{dt} \left[\frac{\partial^2 \varphi_2(x_4)}{\partial x_4^2} \right] \tag{31}$$

over time is small. If one considers the closed-loop system (26) it is obvious, then, that the control must first of all drive the magnitude of x_4 to zero. This may be achieved by setting k_4 large, since then the term $-k_4 x_4$ will overpower all other terms in

$$\begin{aligned} \dot{x}_4 = & -k_0 x_0 - k_1 \left[x_1 - \int_0^t \varphi_1(x_4) d\tau \right] \\ & - k_2 \left[x_2 - \int_0^t \varphi_2(x_4) d\tau \right] - k_3 x_3 - k_4 x_4 \end{aligned}$$

the last state equation of (26), and reduce the magnitude of x_4 . Once this is accomplished the stabilizing gains $\{k_0, k_1, k_2, k_3, k_4\}$ will then insure that the state is driven to the origin. Since this can be achieved for an arbitrary region X in the state space by appropriately choosing k_0, k_1, k_2 to satisfy (29), then choosing k_4 large enough to insure $|x_4| \rightarrow 0$, and finally choosing k_3 to produce a set of stabilizing gains, the control (25) will achieve semi GAS behavior (the gains depending on the bounds on $\varphi_1(x_4)$ and its derivatives within X).

It is noted that the control (25) cannot be viewed as stemming from the application of a particular integral transformation to (1). The two sequential integral transformation (3), (5) are equivalent to the single integral transformation

$$\begin{aligned} z_1 &= x_1 - \int_0^t \varphi_1(x_4(\tau)) d\tau - \int_0^t \int_0^\tau \varphi_2(x_4(\lambda)) d\lambda d\tau \\ z_2 &= x_2 - \int_0^t \varphi_2(x_4(\tau)) d\tau \\ z_3 &= y_3 \\ z_4 &= y_4 \end{aligned}$$

While it is tempting to simply neglect the last term in the expression for z_1 to obtain the integral transformation

$$\begin{aligned} z_1 &= x_1 - \int_0^t \varphi_1(x_4(\tau)) d\tau \\ z_2 &= x_2 - \int_0^t \varphi_2(x_4(\tau)) d\tau \\ z_3 &= y_3 \\ z_4 &= y_4 \end{aligned} \quad (32)$$

which formally leads to the control (25), the application of (32) to (1) does not lead to a chain of integrators that would justify the formal development of (25).

6. Examples

Two examples will be presented to illustrate the performance, and the characteristics, of the controller (i.e. the sets of stabilizing gains) and the nature of the response. The systems are characterized by the model of the form

$$\begin{aligned} \dot{x}_1 &= x_2 + 4.5x_4^2 \\ \dot{x}_2 &= x_3 + \varphi_2(x_4) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{aligned} \quad (33)$$

where in the first model $\varphi_2(x_4) = x_4^3$, and in the second model $\varphi_2(x_4) = x_4^2$. In the first case one of the nonlinearities is sign definite, in the second both nonlinearities are sign definite.

In the first example the stabilizing gains $\{k_0, k_1, k_2, k_3, k_4\} = \{0.2, 2, 10, 20, 48\}$, insure asymptotic stability from the box with

vertices $X = \{\pm 60, \pm 10, \pm 10, \pm 40\}$. The spectrum of the characteristic polynomial of the linear system is $\{-37.4734, -0.1257 \pm j0.3845, -0.1376 \pm j0.1170\}$, and displays a fast eigenvalue that may be associated with forcing x_4 to converge to zero fast, and the four small eigenvalues associated with insuring convergence of the entire state x to the origin. Responses from the eight vertices of the box, associated with the initial conditions $\{\pm 60, \pm 10, -10, \pm 40\}$ are shown in Figure 1. One can systematically develop controls that stabilize the system for larger regions by increasing k_4 (and adjusting the other gains if necessary to preserve a set of stabilizing gains, as well as good performance around the origin) but at the expense of a slower rate of convergence after the initial transient.

In the second example the stabilizing gains $\{k_0, k_1, k_2, k_3, k_4\} = \{0.2, 2, 18, 48, 80\}$, insure asymptotic stability from the box $\{\pm 10, \pm 10, \pm 10, \pm 10\}$. The spectrum of the characteristic polynomial of the linear system is $\{-79.3983, -0.2456 \pm j0.3093, -0.0552 \pm j0.1145\}$, and clearly shows the fast eigenvalue associated with forcing x_4 to converge to zero fast, and the four small eigenvalues associated with insuring convergence of the entire state x to the origin. Responses from the vertices $\{0, \pm 10, \pm 10, -10, -10\}$ are shown in Figure 2.

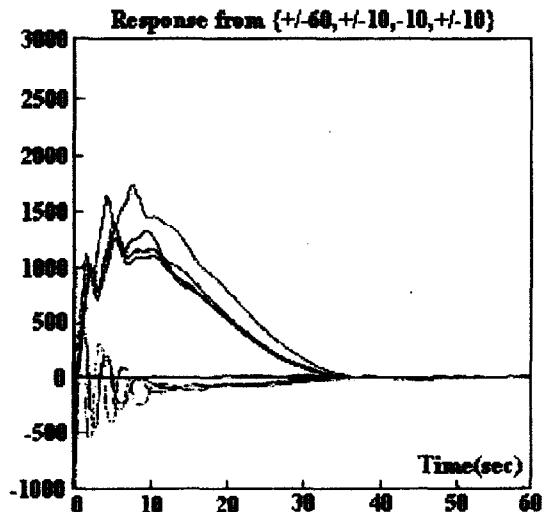


Figure 1. System with one sign-definite nonlinearity

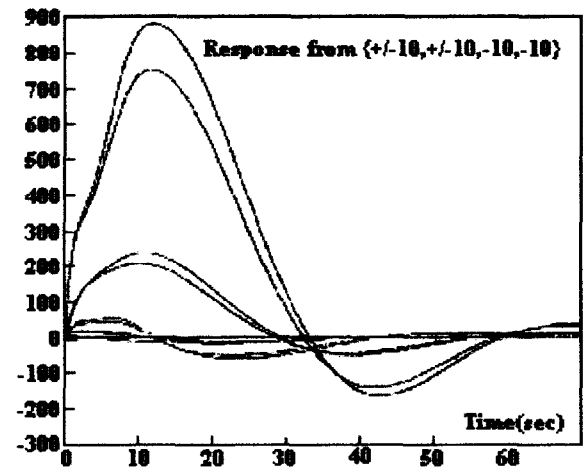


Figure 2. System with two sign-definite nonlinearities

The responses from the vertices $\{0, \pm 10, \pm 10, -10, 10\}$ are quite similar but are not shown. Here, again, one can systematically develop stabilizing controls for larger regions of

attraction as in the previous example, again at the expense of a slower rate of convergence. It is noted that the gains k_0, k_1, k_2 differ by an order of magnitude in accordance with condition (30). It is also observed that as the size of the box of initial conditions was increased, it demanded an increase primarily in the gain k_4 , to arrest the growth of the state x_1 by decreasing the effect of the nonlinear term (32). It is finally noted that the purely linear control law, obtained from (26) by omitting the integral terms, cannot stabilize this system from any initial condition with $x_2(0) > 0$. This does not imply that there is no linear control law that can stabilize (35) from $x_2(0) > 0$. However, there are no guidelines how to obtain it, and there are clear indication that there will exist a better nonlinear control law of the form (26).

7. Generalization to higher order systems

The controller structure developed in Section 5 generalizes to higher order systems of similar structure. To illustrate the point consider the 5th order system

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(x_5) \\ \dot{x}_2 &= x_3 + \varphi_2(x_5) \\ \dot{x}_3 &= x_4 + \varphi_3(x_5) \\ \dot{x}_4 &= x_5 \\ \dot{x}_5 &= u \end{aligned} \quad (34)$$

where the $\varphi_i(x_5)$ are exponentially integrable and $\varphi_i(0) = 0$, and where the control analogous to (26) has the form

$$\begin{aligned} u = & -k_0 x_0 - k_1 \left[x_1 - \int_0^t \varphi_1(x_5) dt \right] - k_2 \left[x_2 - \int_0^t \varphi_2(x_5) dt \right] \\ & - k_3 \left[x_3 - \int_0^t \varphi_3(x_5) dt \right] - k_4 x_4 - k_5 x_5 \end{aligned} \quad (35)$$

with x_0 defined by (16). The control (35) will secure a semi GAS behavior of the closed-loop system for appropriate choice of the controller gains. To show this consider both the steady state and the asymptotic stability issues.

Steady state conditions produce $\dot{x} = 0$ with the steady state value of x_0 given by

$$\begin{aligned} k_0 x_0 = & k_1 \left[x_1 - \int_0^\infty \varphi_1(x_5) dt \right] + k_2 \left[x_2 - \int_0^\infty \varphi_2(x_5) dt \right] \\ & + k_3 \left[x_3 - \int_0^\infty \varphi_3(x_5) dt \right] \end{aligned}$$

Stability analysis in this case hinges on developing the differential equation that governs the dynamics of x_5 , which ultimately produces

$$\begin{aligned} x_5^{(6)} + k_5 x_5^{(5)} + k_4 x_5^{(4)} + [k_3 + \theta_3(x_5, k)] x_5^{(3)} \\ + \left[k_2 + 2k_2 \frac{d}{dt} \left\{ \frac{\partial \varphi_3}{\partial x_5} \right\} + \theta_2(x_5, k) \right] x_5^{(2)} \\ + \left[k_1 + k_1 \frac{d}{dt} \left\{ \frac{\partial \varphi_3}{\partial x_5} \right\} + k_1 \frac{d^2}{dt^2} \left\{ \frac{\partial \varphi_3}{\partial x_5} \right\} + \theta_1(x_5, k) \right] x_5 + k_0 x_5 = 0 \end{aligned}$$

where

$$\theta_3(x_5, k) = k_2 \frac{\partial \varphi_3}{\partial x_5} + k_1 \frac{\partial \varphi_2}{\partial x_5} + k_0 \frac{\partial \varphi_1}{\partial x_5}$$

$$\theta_2(x_5, k) = k_1 \frac{\partial \varphi_3}{\partial x_5} + 2k_1 \frac{d}{dt} \left\{ \frac{\partial \varphi_2}{\partial x_5} \right\} + 2k_0 \frac{d}{dt} \left\{ \frac{\partial \varphi_1}{\partial x_5} \right\} + k_0 \frac{\partial \varphi_2}{\partial x_5}$$

$$\theta_1(x_5, k) = k_0 \frac{d^2}{dt^2} \left\{ \frac{\partial \varphi_1}{\partial x_5} \right\} + k_0 \frac{d}{dt} \left\{ \frac{\partial \varphi_2}{\partial x_5} \right\} + k_0 \frac{\partial \varphi_2}{\partial x_5}$$

In this case, we demand that the set of stabilizing gains satisfy the additional requirement $k_0 \ll k_1 \ll k_2 \ll k_3$ which produces the approximation

$$\begin{aligned} x_5^{(6)} + k_5 x_5^{(5)} + k_4 x_5^{(4)} + k_3 x_5^{(3)} + \left[k_2 + 2k_2 \frac{d}{dt} \left\{ \frac{\partial \varphi_3}{\partial x_5} \right\} \right] x_5^{(2)} \\ + \left[k_1 + k_1 \frac{d}{dt} \left\{ \frac{\partial \varphi_3}{\partial x_5} \right\} + k_1 \frac{d^2}{dt^2} \left\{ \frac{\partial \varphi_3}{\partial x_5} \right\} \right] x_5 + k_0 x_5 = 0 \end{aligned} \quad (36)$$

Clearly, if in addition to satisfying (36) one reduces x_5 fast to the vicinity of zero (and recall that $\varphi_i(0) = 0$ holds for $i = 1, 2, 3$) the effect of the remaining nonlinear terms will be small, and the set of stabilizing gains will insure that the state goes to the origin. Hence k_5 should be large, and k_4 should then be selected so that $\{k_0, k_1, k_2, k_3, k_4, k_5\}$ form a set of stabilizing gains.

8. Dynamic controllers for nonlinear systems

The basic result in Section 6 allows the solution of the stabilization problem for a number of related nonlinear systems. Consider the problem of stabilizing the system

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(v) \\ \dot{x}_2 &= x_3 + \varphi_2(v) \\ \dot{x}_3 &= v \end{aligned} \quad (37)$$

where v is a control input. It may be possible, but impractical, to employ forwarding techniques to obtain a stabilizing control. However, an application of the results from Section 5 provides an immediate solution in the form of a dynamic controller.

Suppose the system is augmented by an additional state so that the system becomes

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(v) \\ \dot{x}_2 &= x_3 + \varphi_2(v) \\ \dot{x}_3 &= v \\ \dot{v} &= u \end{aligned} \quad (38)$$

where u is considered as the control input. This system is structurally identical to (1), and therefore the "control" u defined by

$$\begin{aligned} u = & -k_0 x_0 - k_1 \left[x_1 - \int_0^t \varphi_1(v) dt \right] \\ & - k_2 \left[x_2 - \int_0^t \varphi_2(v) dt \right] - k_3 x_3 - k_4 v, \end{aligned} \quad (39)$$

with x_0 an integral state as before, will secure semi GAS

behavior. Because $u = \dot{v}$ it follows after differentiation of (39) that the following expression holds:

$$\ddot{v} = -k_0 x_1 - k_1 x_2 - k_2 x_3 - k_3 v - k_4 \dot{v} \quad (40)$$

This last expression represents a second order linear dynamic controller with two interesting features. The controller may be represented as $v(s) = G(s)u(s)$ where

$$G(s) = \frac{1}{s^2 + k_4s + k_3} \quad (41)$$

and $u = -k_0x_1 - k_1x_2 - k_2x_3$. Thus, the controller is a 2nd order stable controller driven by the control signal that can be chosen to stabilize the linear part of (34).

The results generalize to higher order systems. Consider the nonlinear system which, when augmented by the auxiliary control takes the form

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(v) \\ \dot{x}_2 &= x_3 + \varphi_2(v) \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \varphi_3(v) \\ \dot{x}_n &= v \\ \dot{v} &= u \end{aligned} \quad (42)$$

The closed-loop system will become semi GAS using the control

$$u = -k_0x_0 - \sum_{i=1}^{n-1} k_i \left[x_i - \int_0^t \varphi_i(v) dt \right] - k_n x_n - k_{n+1} v \quad (47)$$

for appropriate choice of the stabilizing gains. Expression (47) is equivalent to

$$\dot{v} = -k_0x_0 - \sum_{i=1}^{n-1} k_i \left[x_i - \int_0^t \varphi_i(v) dt \right] - k_n x_n - k_{n+1} v \quad (48)$$

and differentiation produces the expression

$$\ddot{v} = -\sum_{i=1}^{n-1} k_i x_{i+1} - k_n v - k_{n+1} \dot{v} \quad (49)$$

This is equivalent to the statement $v(s) = G(s)u(s)$ where the controller input is

$$u = -\sum_{i=1}^{n-1} k_i x_{i+1} \quad (50)$$

and the controller transfer function is

$$G(s) = \frac{1}{s^2 + k_{n+1}s + k_n} \quad (51)$$

And so in the general case the controller is a second order stable controller driven by the control signal that can be selected to stabilize the linear part of (45).

9. Example

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + 2 \text{sat}_1(10v) \\ \dot{x}_2 &= x_3 + 8 \text{sat}_2(4v^3) \\ \dot{x}_3 &= x_4 + 4v^2 \\ \dot{x}_4 &= v \end{aligned}$$

with

$$\text{sat}_i(v) = \begin{cases} -0.5, & v \leq -0.5 \\ v, & -0.5 < v < 0.5, \quad i = 1, 2 \\ 0.5, & v \geq 0.5 \end{cases}$$

Shown in Figure 4 are typical responses from the initial conditions $\{+/-5, +/-5, +/-5.2\}$ with controller gains $\{k_0, k_1, k_2, k_3, k_4, k_5\} = \{0.02, 1, 6, 60, 60, 120\}$ which corresponds to the spectrum $\{-119.5, -0.2 +/-j0.63, -0.0364 +/-j0.125, -0.022\}$

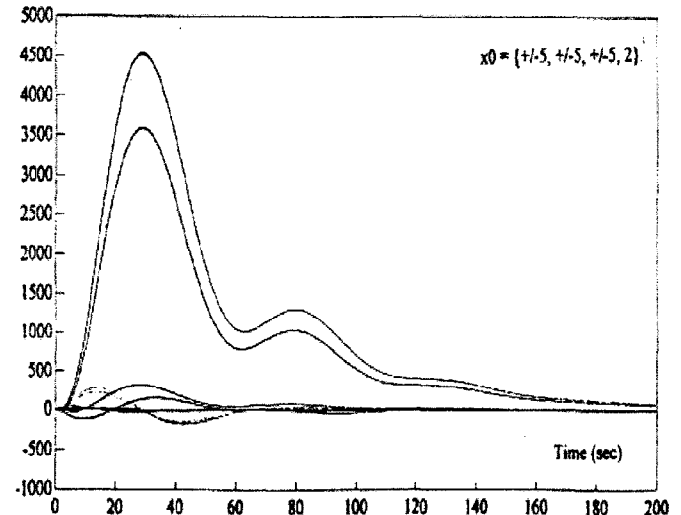


Figure 4. Fourth order system with a dynamic controller

10. Conclusions

The controller structures were analyzed based on the steady state properties of the closed-loop system and the controller assuming that the closed-loop system would be stable. While control structures employing one, or two integral states were discarded because their contained signals that must grow unbounded to insure convergence of the system states, the analysis of their asymptotic performance provided guidelines for the final controller structure. The final controller exhibited desired steady-state properties and provided clear indications as to the sets of stabilizing controller gains that resulted in large regions of attraction, and in fact semi GAS behavior.

11. References

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