

Some Applications of the Polytope Problem

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Abstract

In this paper, a solution for the existence of a fixed order stabilizing controller for any SISO plant (continuous domain or discrete domain) is proposed. Also the question of the existence of a constant output feedback to stabilize a SIMO or a MISO plant is addressed. These applications are based on the algorithm for determining the existence of a stable polynomial in a polytope, which has recently been proposed by the author. Three numerical examples are included to illustrate various applications.

Introduction

Among the issues related to the synthesis of controllers, it is always desirable to work with a controller of as low order as possible to stabilize a plant. In this paper, necessary and sufficient conditions for the existence of a fixed order stabilizing controller for any single-input single-output plant (both continuous and discrete) are proposed. The proposed algorithm yields infinitely many stabilizing controllers of a fixed order, if one such controller exists. If possible, from this family of stabilizing controllers, the designer could pick a controller which meets the specifications. This aspect, however, is not covered in this paper. In the same way, the question of the existence of a constant output feedback to stabilize a SIMO or MISO plant is also addressed.

It turns out that the issue of the existence of a fixed order stabilizing controller for SISO plants and the question of the existence of a constant output feedback to stabilize a SIMO or a MISO plant is very closely related to the well known Polytope Problem. More precisely, once the designer has determined the order of a fixed order controller, the next important decision is to assign bounds to the parameters involved in the controller. These will be called the design parameters. Guided by trial and error, an insight into pole-zero structure of the plant and any implementation issues, the designer fixes the bounds of the controller parameters. With these fixed, the closed loop characteristic polynomial generates a polytope. From here on, it becomes a Polytope Problem. Using the recent

solution of the Polytope Problem [10] proposed by the author, it is checked whether the resulting polytope contains a stable polynomial. If it does, then the algorithm always yields infinitely many stabilizing controllers of the same fixed order. A numerical example to illustrate the procedure is given.

Similar comments apply in the case of the existence of a stabilizing constant output feedback for a SIMO or a MISO plant. An illustrative numerical example is included.

As regards the discrete version of the fixed order controller problem, it is handled by converting a given discrete plant into the continuous domain by applying a stability preserving bilinear transformation. A controller of a fixed order is synthesized in the s-domain. Then, by applying the inverse bilinear transformation, a fixed order stabilizing controller for the given discrete plant is determined. An illustrative numerical example, is given.

Since these applications very heavily depend on the solution of the Polytope Problem proposed by the author [10], for the sake of completeness, some important relevant results from the Polytope Problem algorithm are included [10].

Some Results from the Polytope Problem [10]

Definition 1. A polynomial of the type

$f(s, \underline{q}) = a_0(\underline{q})s^n + a_1(\underline{q})s^{n-1} + \dots + a_{n-1}(\underline{q})s + a_n(\underline{q})$ (1)
is called an affine coefficient polynomial where \underline{q} is the vector of uncertainties and each $a_i(\underline{q})$ is an affine function of \underline{q} . Suppose the uncertainty vector \underline{q} has m -components and lies in an m -dimensional hyperrectangle Q characterized by

$$q_i^- \leq q_i \leq q_i^+, \quad 1 \leq i \leq m$$

where q_1, \dots, q_m are components of a generic \underline{q} in Q , and,

$$a_i(\underline{q}) = \sum_{r=1}^m \alpha_{ir} q_r + c_i, \quad 0 \leq i \leq n$$

where each α_{ir} and each c_i is a real number. Without loss of generality, we assume that

$$a_i(\underline{q}) > 0$$

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for each i , $0 \leq i \leq n$ and each \mathbf{g} in Q .

Note 1. In this paper, the words 'stable' and 'Hurwitz' will be interchangeably used. For $n = 2l + 1$, the family of polynomials given by (1) is written as

$$f(s, \mathbf{g}) = (a_{2l+1}(\mathbf{g}) + a_{2l-1}(\mathbf{g})s^2 + \dots + a_1(\mathbf{g})s^{2l}) + s(a_{2l}(\mathbf{g}) + a_{2l-2}(\mathbf{g})s^2 + \dots + a_2(\mathbf{g})s^{2l-2} + a_0(\mathbf{g})s^{2l}) \quad (2)$$

We write

$$f^e(s, \mathbf{g}) = a_{2l+1}(\mathbf{g}) + a_{2l-1}(\mathbf{g})s^2 + \dots + a_1(\mathbf{g})s^{2l} \quad (3)$$

and

$$f^o(s, \mathbf{g}) = a_{2l}(\mathbf{g}) + a_{2l-2}(\mathbf{g})s^2 + \dots + a_2(\mathbf{g})s^{2l-2} + a_0(\mathbf{g})s^{2l} \quad (4)$$

Notation 1: The polytope generated by (2) will be denoted by P_{2l+1} . The 'subpolytope' generated by (3) will be denoted by P_1^e and the one generated by (4) by P_1^o .

For any positive integer r , define

$$I_r = \{1, 2, \dots, r\}$$

$$I_{ro} = \{\text{the subset of odd integers in } I_r\}$$

$$I_{re} = \{\text{the subset of even integers in } I_r\}$$

As usual, if A and B are any two sets, $A-B$ denotes the difference set; i.e., those elements of A which are not in B .

In order to be able to state and prove the main theorem, we introduce a very important concept of 'frequency set'. Suppose $V_1(s), V_2(s), \dots, V_p(s)$ are the vertices of the polytope P_1^e generated by the polynomial (4).

Definition 2. A subset W_o of the positive real axis defined as

$W_o = \{\omega: \omega > 0, V_k(j\omega) V_r(j\omega) \leq 0; \text{ for some } k, r, 1 \leq k, r \leq p\}$ is called a frequency set for P_1^o .

The following simple theorem justifies the name 'frequency set'.

Theorem 1 [10] A number ω_o is in W_o if and only if there exists a polynomial $g^o(s)$ in P_1^o such that $g^o(j\omega) = 0$.

Suppose, O_1, O_2, \dots, O_l are distinct points in the frequency set W_o . We can assume

$$0 < O_1 < O_2 < O_3 < \dots < O_l$$

A set of l -points in W_o satisfying the above condition will be called an l -string in W_o .

Theorem 2 [10] Suppose the polytope P_1^o is generated by even degree polynomial of degree $2l$. Suppose O_1, O_2, \dots, O_l form an l -string in W_o . Suppose

$$f^o(s^2) = \prod_{i=1}^l (s^2 + O_i^2) = s^{2l} + \sum_{i=1}^l b_{2r} s^{2l-2r} \quad (5)$$

Then

if P_1^o is a subpolytope generated by (5), then there exists $b_o > 0$ such that $b_o f^o(s^2)$ is in P_1^o if and only if

$$b_o = \sum_{i=1}^m \alpha_i q_i + c_o, \text{ for some } q_1, q_2, \dots, q_m \text{ in } Q \text{ satisfying}$$

$$b_i b_o = \sum_{r=1}^m \alpha_r q_r + c_i, \quad i = 2, 4, 6, \dots, 2l$$

For a general uncertainty vector \mathbf{g} in Q and the corresponding even polynomial $f^e(s, \mathbf{g})$ in P_1^e , define

$$B_i(\mathbf{g}) = f^e(jO_i, \mathbf{g}), \quad 1 \leq i \leq l.$$

Since the coefficients of each polynomial in (3) are affine in \mathbf{g} , it follows that

$$B_i(\mathbf{g}) = 0, \quad 1 \leq i \leq l.$$

represents a hyperplane in the parameter space Q . Define

$$S = \{\mathbf{g} \in Q: B_i(\mathbf{g}) < 0 \text{ if } i \in I_{lo} \text{ and } B_i(\mathbf{g}) > 0 \text{ if } i \in I_{le}\}$$

It is clear that each element of S generates a stable polynomial in the polytope P_{2l+1} such that its even part is $f^e(s, \mathbf{g})$ and the odd part is

$$b_o s(s^2 + O_1^2)(s^2 + O_2^2) \dots (s^2 + O_l^2) \text{ for some suitable } b_o > 0.$$

It is easy to verify that S is a convex subset of Q . Denote by \bar{S} the topological closure of S in the hyperrectangle Q endowed with the Euclidean metric. Since \bar{S} is convex, closed and bounded, it has extreme points. Let $\mathbf{g}_1^e, \mathbf{g}_2^e, \dots, \mathbf{g}_p^e$ be the set of extreme points of \bar{S} . Write

$$E = \{\mathbf{g}_1^e, \mathbf{g}_2^e, \dots, \mathbf{g}_p^e\} \subset Q$$

In view of the definition of S given above, the following theorem provides a computational technique to compute the extreme points of \bar{S} . Since each extreme point in E has m -components, m linearly independent equations are needed to uniquely determine the m -components. The following theorem provides such equations.

Theorem 3 [10] A member \mathbf{g}^e of Q is an extreme point of \bar{S} ; i.e., is an element of E if and only if

$$B_r(\mathbf{g}^e) \leq 0, \text{ for every } r \in I_{lo} \quad (6)$$

$$B_k(\mathbf{g}^e) \geq 0, \text{ for every } k \in I_{le} \quad (7)$$

and

$$q_\mu = q_\mu^+ \text{ or } q_\mu^-, \quad \mu \in I_m' \subset I_m; \quad \mathbf{g}^e = (q_1, \dots, q_m) \quad (8)$$

where I_m' is a suitable subset of I_m such that the number of linearly independent equations resulting from (6), (7) and (8) is m . (For example, for $m = 5$, one possible set of five linearly independent equations may be $B_1(\mathbf{g}) = 0, B_2(\mathbf{g}) = 0, B_4(\mathbf{g}) = 0, q_2 = q_2^-$ and $q_5 = q_5^+.$)

Theorem 4 [10] A polytope P_{2l+1} generated by an odd degree polynomial (2) contains a Hurwitz polynomial if and only if there exist O_1, O_2, \dots, O_l (in the frequency set W_o of P_1^o , $0 < O_1 < O_2 < \dots < O_l$) such that

i) $f^o(s) = b_o \prod_{i=1}^l (s^2 + O_i^2)$ belongs to the family (4) for a suitable b_o .

ii) There exist uncertainty vectors $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_l$ in Q such that

a) For each $r \in I_{lo}$ there exists $\mathbf{q}_r, 1 \leq r \leq l$ such

that

$$B_r(\mathbf{q}_r) < 0$$

$$B_\mu(\mathbf{q}_r) \leq 0, \text{ for each } \mu \in I_{lo} - \{r\}$$

$$B_v(\mathbf{q}_r) \geq 0, \text{ for each } v \in I_{le}$$

and

b) For each $k \in I_{le}$, there exists $\mathbf{q}_k, 1 \leq k \leq l$

such that

$$B_k(\mathbf{q}_k) > 0$$

$$B_\mu(\mathbf{q}_k) \geq 0, \text{ for each } \mu \in I_{le} - \{k\}$$

$$B_v(\mathbf{q}_k) \leq 0, \text{ for each } v \in I_{lo}$$

Similar to the $B_i(\mathbf{g})$ functions, $D_i(\mathbf{g})$ functions for P_1^o are defined. Define

$$D_i(q) = f^{\circ}(j\theta_i, q) \quad 1 \leq i \leq l$$

where $\theta_1, \dots, \theta_l$ is an l -string in the frequency set for P_1° .

Theorem 5 [10] The polytope P_{2l} generated by an even degree polynomial contains a stable polynomial if and only if there exist $(\theta_i)_{i=1}^l$, $0 < \theta_1 < \theta_2 < \dots < \theta_l$ in the frequency set W_{θ} of P_1° such that

- i) $f^{\circ}(s) = b_0 \prod_{i=1}^l (s^2 + \theta_i^2)$ for some suitable $b_0 > 0$, is a member of the even part of the polytope.
- ii) There exist uncertainty vectors $\underline{q}_{i_1}, \underline{q}_{i_2}, \dots, \underline{q}_{i_l}$ in Q such that
- a) For each fixed $r \in I_{l_0}$, there exists \underline{q}_{i_r} , such that

$$\left. \begin{aligned} D_r(\underline{q}_{i_r}) &> 0, \\ D_{\mu}(\underline{q}_{i_r}) &\geq 0, \quad \mu \in I_{l_0} - \{r\} \\ D_v(\underline{q}_{i_r}) &\leq 0, \quad v \in I_{l_0} \end{aligned} \right\}$$

and

- b) For each fixed $k \in I_{l_0}$, there exists \underline{q}_{i_k} , such that

$$\left. \begin{aligned} D_k(\underline{q}_{i_k}) &< 0, \\ D_{\mu}(\underline{q}_{i_k}) &\leq 0, \quad \mu \in I_{l_0} - \{k\} \\ D_v(\underline{q}_{i_k}) &\geq 0, \quad v \in I_{l_0} \end{aligned} \right\}$$

Having proved Theorem 4 and Theorem 5 where the existence question of a Hurwitz polynomial is addressed, it is now proposed to give a very straightforward algorithm to search for a stable polynomial in a polytope. We will discuss the algorithm only for an odd degree polynomial family and a similar algorithm for an even degree polynomial can be similarly discussed.

An Algorithm for the Polytope Problem for an Odd Degree Polynomial

Suppose a polynomial of odd degree with affine coefficients in the uncertainty vector q is given by (2). Denote this subpolytope by P_1° . Find the vertices $V_1(s), V_2(s), \dots$ of the odd part of the polytope generated by the given polynomial.

1. Find the positive roots of each vertex polynomial $V_i(j\omega)$. Suppose $\omega_1, \omega_2, \omega_3, \dots, \omega_p$ are the distinct positive roots of all the vertices. We can assume $0 < \omega_1 < \omega_2 < \dots < \omega_p$.

It is easy to see that a subinterval $[\omega_k, \omega_r]$ is a part of the frequency set W_{θ} if and only if there exists vertices $V_{\mu}(s)$ and $V_v(s)$ of P_1° such that

$$V_{\mu} \left(\frac{j(\omega_1 + \omega_2)}{2} \right) V_v \left(\frac{j(\omega_1 + \omega_2)}{2} \right) \leq 0.$$

The union of all such subintervals which satisfy the above condition is the frequency set W_{θ} of P_1° .

2. Pick a gridding factor and do a gridding of the frequency set W_{θ} and generate $\Lambda = \{O_1, O_2, \dots, O_k\}$ where O_1, O_2, \dots are the grid points. Assume $0 < O_1 < O_2 < O_3 < \dots < O_k$
3. l different members of Λ are defined to form a string if they are arranged in increasing magnitude. Pick a string $O_{i_1}, O_{i_2}, \dots, O_{i_l}$ from Λ .

4. Form the polynomial $f^{\circ}(s) = \prod_{r=1}^l (s^2 + O_{i_r}^2)$

By using Theorem 2, check if the polynomial $b_0 f^{\circ}(s)$ for a suitable b_0 is a member of the odd part of the subpolytope P_1° for a suitable b_0 . If $b_0 f^{\circ}(s)$ is not a member of the odd part of P_1° , then go to a next string of l members of Λ and repeat step 4. Otherwise go to the next step.

5. If $b_0 f^{\circ}(s)$ is a member of the odd part, then by using the technique of Theorem 4, find the extreme point vectors $\underline{q}_{i_1}^{\circ}, \underline{q}_{i_2}^{\circ}, \dots, \underline{q}_{i_l}^{\circ}$, which satisfy the conditions (ii) of Theorem 4.
6. If $\underline{q}_{i_1}^{\circ}, \underline{q}_{i_2}^{\circ}, \dots, \underline{q}_{i_l}^{\circ}$ exist, then a stable polynomial exists and can be constructed as explained in Theorem 4.
7. If such l vectors in Q don't exist, then keep going until all strings of l -components of Λ are exhausted.

Some Applications of the Polytope Problem

With the preliminaries out of the way, we are now ready to present some applications of the Polytope Problem.

The Fixed Order Controller Problem for SISO Plants

The fixed order controller problem for SISO plants can be stated as follows, "What are the necessary and sufficient conditions for the existence of a fixed order controller to stabilize a SISO plant?"

We will show that the above problem is equivalent to a certain Polytope Problem. Consider a SISO feedback control system in the standard block diagram given below.

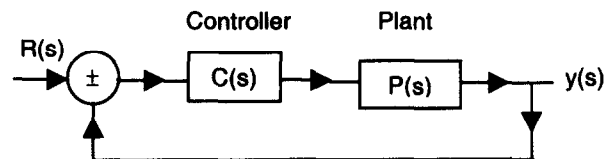


Figure 1. Feedback Control system

Suppose the plant $P(s)$ is given by

$$P(s) = \frac{\sum_{i=0}^m a_i s^i}{\sum_{i=0}^n b_i s^i}, \quad n \geq m. \quad (9)$$

Suppose it is desired to synthesize a fixed order controller of order l to stabilize $P(s)$. Suppose

$$C(s) = \frac{\sum_{i=0}^k c_i s^i}{s^l + \sum_{i=0}^{l-1} d_i s^i} \quad (10)$$

Clearly, the characteristic polynomial of the closed loop system is given by

$$\left(\sum_{i=0}^n b_i s^i \right) \left(s^l + \sum_{i=0}^{l-1} d_i s^i \right) + \left(\sum_{i=0}^k c_i s^i \right) \left(\sum_{i=0}^m a_i s^i \right) \quad (11)$$

If the parameter bounds are assigned, then (11) generates a polytope of real polynomials. Thus, the existence of a stabilizing controller $C(s)$ is equivalent to the existence of a stable polynomial in the polytope generated by (11).

Following is the proposed algorithm for the synthesis of a fixed order controller for the stabilization of a given SISO plant.

- Check if the transfer function $P(s)$ of the plant satisfies the parity interlacing (Pip) property [11]. If $P(s)$ satisfies Pip, this would imply that a stable stabilizing controller exists [11]. This information has significant implication in terms of implementation of the controller as also in the choice of parameter bounds, i.e., no parameter in the denominator of the controller need be negative.
 - Now the important question of selecting the bounds of the parameters, is addressed. These bounds will be classified as design parameters. In this context, several aspects are worth mentioning which should be taken into consideration. For example, the bounds suggested by the implementation aspects of the controller, the nature of poles and zeros of $P(s)$ and a lot of trial and error are involved in deciding the bounds of the controller parameters.
 - Once the bounds for the controller parameters have been fixed, the closed loop characteristic polynomial generates a polytope. Thus, there exists a stabilizing controller $C(s)$ (of the prescribed order) if and only if the resulting polytope contains a stable polynomial. Thus, the algorithm for solving the Polytope Problem can be directly applied for this situation.
 - It should be pointed out that the algorithm for the Polytope Problem [10] yields infinitely many stabilizing controllers of a fixed order if one such controller exists.
 - If the polytope generated by (11) does not contain any stable polynomial, then this would imply that a stabilizing controller of the fixed order under consideration does not exist for the selected parameter bounds. In this case, the designer has to pick either another set of parameter bounds or pick a controller of a higher order and start all over again.
- Next, an illustrative numerical example is presented.

Example 1.

Consider a SISO plant whose transfer function is given by

$$P(s) = \frac{3s + 1}{s^4 + s^3 + 2s^2 + 5s + 3} \quad (12)$$

Note that the plant is unstable with poles at $-1, -1$ and $0.5 \pm 1.68j$. Also it has a zero at $-1/3$ and three zeros at infinity. It is desired to design a stabilizing controller of as low order as possible. First, it is noticed that a stable stabilizing controller exists as the plant $P(s)$ satisfies the Pip property. We investigate the existence of a second order stabilizing controller. Suppose

$$C(s) = \frac{q_3 s^2 + q_4 s + q_5}{s^2 + q_1 s + q_2} \quad (13)$$

Next, one has to pick the bounds for the controller parameters, i.e., the design parameters. By taking into consideration the pole-zero configuration of $P(s)$ and, by trial and error, the following bounds were selected:

$$\begin{aligned} 10 \leq q_1 \leq 16; \quad 40 \leq q_2 \leq 50, \quad 30 \leq q_3 \leq 40, \\ 30 \leq q_4 \leq 40, \quad 40 \leq q_5 \leq 50 \end{aligned} \quad (14)$$

The closed loop characteristic polynomial turns out to be $s^6 + (q_1+1)s^5 + (q_1+q_2+2)s^4 + (2q_1+q_2+5)s^3 + (5q_1+q_2+3q_3+3)s^2 + (3q_1+5q_2+q_3+3q_4)s + (3q_2+q_4)$ (15)

By using the Polytope Problem algorithm [10], the frequency set for the even part of (15) subject to the restrictions (14) was determined to be

$$W = I_1 \cup I_2 \cup I_3 \quad (16)$$

where

$$\begin{aligned} I_1 &= [0.7409, 0.9609]; \\ I_2 &= [1.9443, 2.3934], \\ I_3 &= [6.7572, 7.9536] \end{aligned} \quad (17)$$

In order to form strings of three elements from W , the frequency interval I_1 was gridded into five equally spaced points; in the same way, I_2 and I_3 were gridded into ten points each. The Polytope Problem [10] program yielded the following output:

For a string of three grid points from W consisting of $e_1 = 0.8281, e_2 = 2.12398, e_3 = 7.3554$, The following output was obtained.

q_1	q_2	q_3	q_4	q_5
15.2986	42.0019	30.847	30	41.694
D_1	D_2	D_3		
301.8	-24.9	38912.7		

(18)

Thus, the corresponding 2nd order stabilizing controller $C(s)$ is given

$$\begin{aligned} C(s) &= \frac{q_3 s^2 + q_4 s + q_5}{s^2 + q_1 s + q_2} \quad \text{or} \\ C(s) &= \frac{30.847s^2 + 30s + 41.694}{s^2 + 15.2986s + 42.0019} \end{aligned} \quad (19)$$

which is stable.

The corresponding closed loop characteristics polynomial is given by $s^6 + 16.2986s^5 + 59.3s^4 + 170.14s^3 + 284.3438s^2 + 410.9872s + 167.6997$

which is stable.

Thus, a second order stable stabilizing controller exists and is known.

The Output Feedback Problem for SIMO or MISO Plants

The Output Feedback Problem for SIMO or MISO Plants can be stated as follows: "What are the necessary and sufficient conditions for the existence of a constant output feedback to stabilize a SIMO or MISO plant?" We will show that the above problem is equivalent to a certain Polytope Problem. We will discuss the case of a SIMO plant as the case of a MISO plant is similar. Consider a SIMO constant output feedback control system in the standard block diagram.

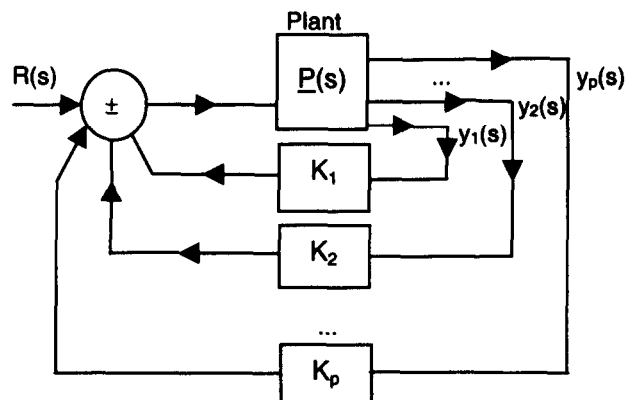


Figure 2. SIMO Closed Loop System.

Suppose the transfer function matrix of the plant is given by

$$\underline{P}(s) = \begin{pmatrix} \frac{n_1(s)}{d(s)} & \frac{n_2(s)}{d(s)} & \dots & \frac{n_p(s)}{d(s)} \end{pmatrix} \quad (20)$$

It can be easily checked that the characteristic polynomial of the closed loop system given by Figure 2 is

$$d(s) + \sum_{i=1}^p K_i n_i(s) \quad (21)$$

It is quite clear that the polynomial represented by (21) is an affine coefficient polynomial. Thus, the problem is to find K_1, K_2, \dots, K_p so that (21) yields a stable polynomial. As in the case of previous application, by using the pole-zero information of the plant and trial and error, a set of bounds for the 'design parameters' K_1, \dots, K_p are assigned. Once this is done, then the characteristic polynomial (21) represents a polytope. Thus, the question of the existence of K_1, K_2, \dots, K_p such that (21) is stable is equivalent to the question whether the polytope represented by (21) contains a stable polynomial. Thus, the algorithm for solving the Polytope Problem can be used to answer this question. It is worth pointing out if one set of values of $(K_i)_{i=1}^p$ exists, then there are infinitely many such solutions.

Next, an illustrative example is presented.

Example 2.

Consider a SIMO plant whose transfer matrix is given by

$$\begin{pmatrix} \frac{15.5s^2 + 17.1s - 12.4}{s^3 + s^2 + 2s + 4} \\ \frac{13.9s^2 + 14.2s - 6.5}{s^3 + s^2 + 2s + 4} \\ \frac{-7.5s^2 + 34.6s + 25}{s^3 + s^2 + 1} \\ \frac{20.7s^2 + 30.5s + 4.4}{s^3 + s^2 + 1} \end{pmatrix} \quad (22)$$

It can be checked that, with feedback vector (K_1, K_2, K_3, K_4) as shown in Figure 2 (for $p = 4$), the closed loop characteristics polynomial turns out to be

$$\begin{aligned} & s^6 + (15.5k_1 + 13.9k_2 - 7.5k_3 + k_4 + 2)s^5 \\ & + (32.6k_1 + 28.1k_2 + 27.1k_3 + 3.429k_4 + 3)s^4 \\ & + (4.67k_1 + 7.7k_2 + 44.6k_3 + 6.5k_4 + 7)s^3 \\ & + (3.1k_1 + 7.4k_2 + 64.2k_3 + 9.4k_4 + 5)s^2 \\ & + (17.1k_1 + 14.2k_2 + 188.4k_3 + 6.6k_4 + 2)s \\ & + (12.42k_1 - 6.5k_2 + 100k_3 + 0.85k_4 + 4) \end{aligned} \quad (23)$$

Now the bounds of the 'design parameters' k_1, k_2, k_3, k_4 are to be selected. By 'trial and error', the bounds are selected as

$$\begin{aligned} 0.1 & \leq K_i \leq 1, \quad i = 1, 2, 3 \\ 5 & \leq K_4 \leq 20 \end{aligned} \quad (24)$$

With the bounds (K_i) fixed, (23) represents a polytope. By using the Polytope Problem algorithm [10], the frequency set for the even part of (23) comes out to be

$$W = I_1 \cup I_2 \quad (25)$$

where

$$I_1 = [0.2446, 1.5548]; \quad I_2 = [5.1693, 12.5573] \quad (26)$$

Each of the frequency subintervals I_1 and I_2 were gridded into ten equally spaced points. With all possible strings of three points from the grid points of W , the Polytope Prob-

lem algorithm was applied. For a string of grid points given by

$$e_1 = 0.5067; \quad e_2 = 1.4238; \quad e_3 = 6.647 \quad (27)$$

the algorithm yielded the following information:

K_1	K_2	K_3	K_4
0.1065	0.1745	0.1367	9.1524
D_1	D_2	D_3	
74.29	-0.0487	24531.44	

(28)

The above data shows that the K -values obtained above yield a stable polynomial in the polytope (23). In fact, the stable polynomial is given by

$$\begin{aligned} & s^6 + 14.2042s^5 + 46.4661s^4 + 74.4303s^3 \\ & + 101.4327s^2 + 92.4652s + 22.9947 \end{aligned} \quad (29)$$

Thus, we have determined a constant output feedback for the given SIMO plant which will stabilize it.

The Discrete Version of the Fixed Order Controller Problem

Next, we discuss the discrete version of the fixed order controller problem. More precisely, "What are the necessary and sufficient conditions for the existence of a fixed order controller in the discrete domain which will stabilize a given SISO discrete plant?"

We will show that this is equivalent to the existence of a stable polynomial in a certain polytope in the continuous domain. This is accomplished by applying a stabilizing bilinear transformation between the s -domain and the z -domain. This simple procedure is illustrated with the help of a numerical example.

Example 3.

Consider a typical feedback control system for a SISO system given by the following block diagram:

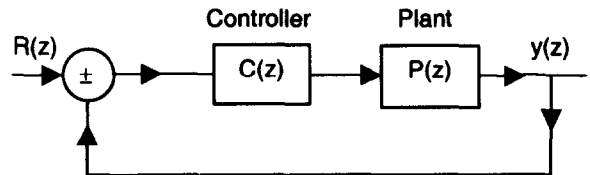


Figure 3. Discrete Control System.

Suppose

$$P(z) = \frac{4z + 1}{z(z - 1.5)(z + 0.5)} \quad (30)$$

Note that $P(z)$ is unstable. It is desired to find a first order controller $C(z)$ such that the closed loop system is stable. We transform this synthesis problem into s -domain by applying the stability preserving transformation given by

$$z = \frac{1+s}{1-s} \quad (31)$$

Under this transformation, $P(z)$ is transformed into $P(s)$ and is given by

$$P(s) = \frac{4 \left(\frac{1+s}{1-s} \right) + 1}{\left(\frac{1+s}{1-s} \right) \left[\left(\frac{1+s}{1-s} \right)^2 - \frac{1+s}{1-s} - \frac{3}{4} \right]} = \frac{(3s+5)(s-1)^2}{(s+1)(5s^2+14s-3)} \quad (32)$$

Since a first order controller in a z -domain corresponds to a first order controller in an s -domain, we find a first order controller $C(s)$ of the type

$$C(s) = \frac{q_2 s + q_3}{s + q_1} \quad (33)$$

which will stabilize $P(s)$ under the standard unity feedback configuration. Next, the bounds on the 'design parameters' q_1, q_2, q_3 are to be selected. Once again, by trial and error and also guided by the pole-zero configuration of $P(s)$, the bounds are selected as follows:

$$0.5 \leq q_1 \leq 0.8 \quad 1 \leq q_2 \leq 2.5 \quad 0.5 \leq q_3 \leq 1 \quad (34)$$

The closed loop characteristic polynomial turns out to be

$$(3q_2+5)s^4 + (5q_1-q_2+3q_3+19)s^3 + (19q_1-7q_2-q_3+11)s^2 + (11q_1+5q_2-7q_3-3)s + (-3q_1+5q_3) \quad (35)$$

With the bounds of q_1, q_2, q_3 fixed, the characteristic polynomial (35) represents a polytope and the Polytope Problem algorithm [10] is applied to check if (35) contains a stable polynomial. By applying the algorithm, the frequency set for the even part was found to be

$$W = [0.0732, 1.5271] \quad (36)$$

The frequency set W was gridded into ten equally spaced grid points. For the following string of points from the frequency set,

$$e_1 = 0.364, \quad e_2 = 0.8002 \quad (37)$$

The Polytope algorithm yielded the following data

q_1	q_2	q_3	D_1	D_2
0.6692	2.0668	0.5916	7.6325	-3.5660

(38)

The above information shows that for the above parameter values, (27) will contain a stable polynomial given by

$$11.2005s^4 + 22.0538s^3 + 8.6554s^2 + 10.5545s + 0.9502 \quad (39)$$

Thus, the controller $C(s)$ is

$$C(s) = \frac{2.0668s + 0.5916}{s + 0.6692} \quad (40)$$

Then the corresponding controller in the z -domain is given by

$$C(z) = \frac{2.0668 \left(\frac{z-1}{z+1} \right) + 0.5916}{\frac{z-1}{z+1} + 0.6692} = \frac{2.6584z - 1.4752}{1.6692z - 0.3308} \quad (41)$$

Thus, the existence of a first-order stabilizing controller in the z -domain is established.

Summary

In this paper, the solution to three well-known open problems is proposed. These three problems are: (1) the question of the existence of a fixed order controller to stabilize a SISO plant, (2) the question of the existence of a constant output feedback to stabilize a SIMO or a MISO plant and (3) the question of the existence of a fixed order controller to stabilize a SISO plant in the discrete domain. Three numerical examples are given to illustrate each problem.

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