On the Simulation- Supported
Design of Logic Controllers for Known Dynamic Systems

J. Medanic and A. Pfleger
Dept. General Engineering
University of Illinois at Urbana-Champaign
Email: jmedanic@uiuc.edu

Abstract

The paper presents a methodological procedure for systematically exploiting the information on the dynamics of the response as observed through simulations to expand the region of attraction of a nominal design. A particular novelty is that an integral state is added to state feedback in each region of the partition to provide a mechanism so that the state is driven towards the desired equilibrium point at the origin.

1. Introduction

Recent development of logic control systems, represent an attempt to fill the need of providing off-the-shelf nonlinear controls for nonlinear systems. Although practical applications has for long time resisted proposals for introducing nonlinear controls, the acceptance of fuzzy logic controls by industry clearly implies that such a need exists. Problems typically arising in practice rarely required to attain more that convergence to the origin from a pre-specified operating region (and additional design goals within this region such as tracking, regulation, disturbance rejection). Thus the Practical Asymptotic Stability (PAS) problem is: Given a bounded regions of initial conditions, S, develop a control u(x) that will ensure that S belong to the region of attraction $\mathcal{A}(u(x))$ of the equilibrium point at the origin. The available analytical tools are not well suited to solve stability problems under such constraints. Yet, it is reasonable to assume that the PAS problem should be solvable for a wider class of systems than the Global Asymptotic Stability (GAS) problem. The PAS problem is typically tackled by estimating the region of attraction of a candidate control obtained by applying some version of energy function approach (Lyapunov, iterative Lyapunov etc.) [3] or by linearizing the system about the origin to find a candidate linear control (see e.g. Khalil [2], or by attempting to insure a semi GAS solution. But, if for any such candidate control $S \not\subset \mathcal{A}$ there is no systematic procedure for modifying the control to enlarge $\mathcal{A}$ and achieve the design goal.

The development of fuzzy logic controllers has introduced, or re-enforced, a number of features into the design of nonlinear control for nonlinear systems: it and broadened the classes of controllers now incorporated into analytical design procedures, it introduced the use of overlapping partition (thus providing smooth controllers), it promoted the idea of adding rules to existing controller structure to improve system behavior in specific regions of the state space , it has promulgated the use of off-the-shelf- controllers with a given structure and free parameters, and, by its appeal, it has expanded the class of systems for which nonlinear controls are now actively pursued in practice. Clearly the positive features of logic controllers should be incorporated in future developments, and combined with advances that allow the dynamic characteristics of the considered system to influence the structure of the logic controller.

A peculiar, aspect of current developments in fuzzy logic controls, however, is the somewhat myopic focus on the control poorly defined plants. Thus, even when the system is fully described by a reliable nonlinear model there is no attempt, and no obvious procedure, to exploit the dynamic features of the system in developing the structure of the logic controller. Commercially available fuzzy logic controllers partition the input space and the output space into overlapping hypercubes and define the control by using simple membership functions. More involved partitions, and more complex membership functions could be used, but under the "poorly modeled plant" assumption that justifies the use of fuzzy logic controls, this would be self-contradictory. And so it appears reasonable to stay with a standard partition and a few types of membership functions. The consequence is that the overlapping partition and the imbedded membership functions and free parameters are unrelated to the dynamic characteristics of the system.

These underlying assumptions do not hold when the system dynamics is fully known and the difficulty is to determine a control that will stabilize the system. To the present, this important distinction has not influenced the structure of (fuzzy) logic controllers. The fact that significant information exists for well defined nonlinear systems, therefore, offers no particular advantage in the use of logic controllers, since the information is not utilized in defining controller structure, nor is it obvious that the presently used structures are appropriate in this case.

Iterative redesign based or results of simulations is an often overlooked, alternative. Simulation results, with an existing controller, and correction of the control action only in certain regions of the state space may be a necessary and valuable tool in solving specific nonlinear control design problems. Furthermore, this alternative is well suited to the development of logic controllers, and if anything, it is simulation that should replaces the omniscient "expert" implied, but never identified, in the synthesis of logic controllers. This is particularly appropriate in tackling new applications.

In this paper we propose the rudiments of a methodological approach to the design of logic controls that attempts to adapt the controller structure to the dynamic characteristics of the system using simulation results to guide the development of the controller structure and its parameters. In the simplest realization this partition divides the state space into K+1 overlapping regions $R_0, R_1, \ldots, R_K$, where R_0 is a spherical region about the origin, and the $R_1, \ldots, R_K$ are polar regions defined via scalar products as will be explained in the sequel. A nominal point x_{ci}, $i = 1,..,K$ is selected in each polar region, and the origin is used in R_0, to linearize the system and determine an appropriate control to use within the region . The novelty is that in regions R_i, $i = 1,\ldots,K$ an integral state is added to the linearized model, and a control is determined to guide the state towards the origin, and not to the point of linearization.

2. Problem formulation

Consider the nonlinear system

$$\dot{x} = F(x,u) = f(x) + B(x)u \qquad (1)$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $f(x)$, $B(x)$ are continuous function of x, with $f(0) = 0$ and rank $B(x) = m$ for $x \in M \subset R^n$; the set M is sufficiently large, and in particular large enough so that $S \subset M$, with S is the admissible region of initial condition of (1). The goal, in the PAS problem is: Determine a control strategy $u = u(x)$ defined over M so that for the closed loop system $S \subset \mathscr{A}$ where $\mathscr{A}$ is the region of attraction of the equilibrium point at the origin of. Furthermore, it is required that $x(t) \in M$ for all t.

We concentrate here on single input systems although the approach extends directly to multi input systems. The standard approach is to solve the GAS, or semi GAS problem (and thus by default the PAS problem). Instead, we directly consider how to determine a u(x) to guarantee that $S \subset \mathscr{A}$ by applying the following approach: We start by linearizing the system about the origin (or by applying some other local approach) to obtain a stabilizing control $u_0(x)$ with an initial region of attraction $\mathscr{A}_0 = \mathscr{A}_0(u_0(x))$. Subsequently, we systematically modify the control to expand the region of attraction, obtaining in sequence $\mathscr{A}_0 \subset \mathscr{A}_1 \subset \mathscr{A}_2 \ \ldots \subset \mathscr{A}_q$ until $S \subset \mathscr{A}_q$ is satisfied.

3. Approach

The initial control for (1) is obtained by linearizing the system about the origin, resulting in

$$\dot{x} = A_0 x + B_0 u \qquad (2)$$

and finding a control designed to achieve desired behavior about the origin. For the purpose here, the control will be obtained by pole-placement, to guarantee a certain speed of response in the vicinity of the origin. Given a desired closed-loop spectrum Λ_0, let $u = - K_0 x$ be the associated state feedback control. Furthermore, let $\mathscr{A}_0$ be an estimate of the region of attraction. Suppose $S \not\subset \mathscr{A}_0$. To expand the region of attraction we propose the following mechanism.

Suppose, based on simulation, that trajectories of the closed-loop system system with control diverge towards infinity by approaching well defined asymptotic **escape directions**. This is often the case, and the direction will be easily determined by simulation. Moreover, this is the more difficult case then if trajectories spiral off to infinity. Let these and additional selected directions, form a totality of n+1 directions characterized by direction vectors $\pi_1, \pi_2, \ldots, \pi_n, \pi_{n+1}$, and let the tips of these direction vectors. For simplicity, form the vertices of a equidistant simplex [1],[4] (see Section 5). Define the region R_i to be the polar region

$$R_i = \left\{ x : \max\{ \frac{\pi_i^T x}{\| x \|} - \omega, 0 \} \right\}, \quad i = 1,2,\ldots,n+1 \qquad (3)$$

It has been shown that if the parameter ω, referred to here as the **aperture** of the region R_i, satisfies the condition $\omega < 1/n$, the sets R_i, $i = 1,\ldots,n+1$ form an overlapping partition that covers R^n. We refer to this as a polar partition of R^n (See Section 5).

Next, in each R_i choose a point x_{ci} that lies on the ray in the direction π_i, and outside $\mathscr{A}_0$ but within S and linearize the system about x_{ci} resulting in the collection of linearized models

$$\delta\dot{x}_i = A_i \delta x_i + B_i \delta u_i + f_{ni}, \quad i = 1,\ldots,n+1 \qquad (4)$$

where $f_{ni} = F(x_{ci}, u_{ni})$ is the zero-th order term in the linearization of (1) since the point $\{x_{ci}, u_{ni}\}$ will not necessarily be an equilibrium point of (1). Now, the model will certainly be valid in a subregion of R_i about x_{ci}.

We now determine for each R_i a control that will make the system locally stable in each R_i and steer the system, while in this region, not to approach x_{ci}, the point of linearization, but to approach the origin of the state space. To this end we introduce an integral state, so that the model of the system becomes

$$\begin{aligned} \delta\dot{x}_{i0} &= C_i x \\ \delta\dot{x}_i &= A_i \delta x_i + B_i \delta u_i + f_{ni} \end{aligned} \qquad (5)$$

where the signal $y_i = C_i x$ will be determined to achieve this goal (see Section 5). The linearized model in R_i reduces to

$$\begin{bmatrix} \delta\dot{x}_{i0} \\ \delta\dot{x}_i \end{bmatrix} = \begin{bmatrix} 0 & C_i \\ 0 & Ai \end{bmatrix} \begin{bmatrix} \delta x_{i0} \\ \delta x_i \end{bmatrix} + \begin{bmatrix} 0 \\ B_i \end{bmatrix} \delta u + \begin{bmatrix} C_i x_{ci} \\ f_{ni} \end{bmatrix} \qquad (6)$$

A pole placement control $\delta u = -K_{i0} x_{i0} - K_i \delta x_i$ is then determined that guarantees a desired speed of the linearized system in R_i via a desired closed-loop spectrum Λ_{ie}.

As long as $x(t) \in R_i$, but $x(t) \notin Rj$, for $j \neq i$, the control applied to the system is the linear control with a bias term:

$$u(x) = u_i(x) = u_{in} + \delta u = -K_{i0} x_{i0} - K_i \delta x_i + u_{in} \qquad (7)$$

When x(t) belongs to a subregion where two or more sets R_i, $i = 1,\ldots,n+1$, overlap, the control becomes

$$u(x) = \frac{\sum\limits_{i \in I} \mu_i(x) u_i(x)}{\sum\limits_{i \in I} \mu_i(x)} \qquad (8)$$

where I is the subset of $\{1,2,\ldots,n+1\}$ defining the active regions that overlap at state x, and

$$\mu_i(x) = \max\left\{ \frac{\pi_i^T x}{\| x \|} - \omega, 0 \right\}, \qquad i \in I \qquad (9)$$

are continuous membership functions. This insures a smooth transition from one region to another and a continuous control. Expression (8) may then be replaced by

$$u(x) = \frac{\sum\limits_{i=1}^{n+1} \mu_i(x) u_i(x)}{\sum\limits_{i=1}^{n+1} \mu_i(x)} \qquad (10)$$

where $u_P(x)$ designates a continuous control defined over a polar partition of the state space.

The intent in proposing this control is to drive the state towards the origin of the state space from the vicinity of any of the selected points x_{ci}. In reality, and sufficient for our purposes, the expectation is that the control $u_P(x)$ will drive the state towards a neighborhood of the origin, and therefore into the region of attraction established by the linear control, which will drive the state to the origin. To achieve the combine effect the actual control applied to the system is proposed to have the form

$$u(x)=\frac{\|x_D\|_2^N}{\|x_D\|_2^N+\|x\|_2^N}u_L(x)+\frac{\|x\|_2^N}{\|x_D\|_2^N+\|x\|_2^N}u_L(x) \tag{11}$$

where $\|x_D\|_2{}^N$ is a fixed bound that depends on the region of attraction achieved by the linear control. The first term will dominate in the vicinity of the origin, while the second term will dominate for $x \in S$ but away from the origin.

3. Polar partitions of the State Space

Consider an m-dimensional space associated with the $x \in R^n$. The equidistant simplex of R^n, the polar region and polar partition of R^n are defined as follows [1], [4]:

Definition 1. A collection of n+1 unit vectors in R^n is called an **equidistant simplex** if the vectors $\{\pi_1, \pi_2, \ldots, \pi_{n+1}\}$ satisfy the following conditions:

(a) $\pi_i^T\pi_i = 1, i = 1,\ldots,n+1$ (unit length) (12a)

(b) $\pi_i^T\pi_j = \eta$, for all $i \neq j$ (equal angles) (12b)

(c) $\|\pi_i - \pi_j\| = d,\ i \neq j$ (equal distance) (12c)

(d) $\pi_1 + \pi_2 + \ldots + \pi_{m+1} = 0$ (symmetric) (12d)

For the properties of the simplex partition and simplex controls the reader is referred to [5,6].

Definition 2. The set $R_i \subset R^n$ is called a **polar region** in the direction π, $\|\pi\|_2 = 1$, with aperture ω if

$$R = \{x : \frac{\pi^T x}{\|x\|} \geq \omega\} \tag{13}$$

Definition 3. Given the equidistant partition vectors $\{\pi_1, \pi_2, \ldots, \pi_{m+1}\}$ the sets $R_1,\ldots, R_{n+1}$ form an **overlapping polar partition** of R^n if

$$R_i = \{x: \frac{\pi_i^T x}{\|x\|} \geq \omega,\ 0 \leq \omega \leq \frac{1}{n}\},\ i=1,2,\ldots,n+1 \tag{14}$$

Polar controls, directly associated with the overlapping polar partition were proposed in [1] and applied to the two-link [4] and three-link[5] excavator control problem. The reason for initially basing the structure of the logic controller proposed here with an overlapping polar partition is that it is the simplest partition that covers the entire state space when $\omega < 1/n$.

In this application we accept polar regions as the regions more nature for the partition of the state space in considering stability problems, but the orientation and number of regions will be dictated by problem features, as discussed subsequently.

5. Design of linear controls for Regions within the partition

Consider the nonlinear system (1) linearized about a nominal point x_n and suppose a controller has been developed as described in Section 3, and given by expressions (6) to (8). Dropping the index i, the state of the closed-loop system will reach the steady state

$$\begin{aligned}\delta x_{0_{ss}} &= -\frac{1}{CF^{-1}BK_0}[Cx_n + CF^{-1}f_n]\\ \delta x_{ss} &= -\frac{1}{CF^{-1}BK_0}[K_0F^{-1}BCx_n + K_0(CF^{-1}B - F^{-1}BC)F^{-1}f_n]\end{aligned} \tag{15}$$

where F = A-BK. Ideally, one would like $\delta x_{ss} = -x_n$ since then, if the system remains linear throughout, this would bring the system into the origin. But in general, point regulation is not possible even for linear systems. Instead, the idea is to steer the state towards the origin. To that end define the error $e_n = \delta x_{ss} - \delta x_d$ where $\delta x_d = -x_n$ and consider the problem of choosing C to minimize this error. Since $e_n = \delta x_{ss} - \delta x_d = \delta x_{ss} + x_n$ it follows from (15) that

$$e_n = (I - \frac{1}{CF^{-1}B}F^{-1}BC)(x_n - F^{-1}f_n) \tag{16}$$

Thus, it is reasonable to choose C so that $\|e\|_2$ is minimized. However, minimizing $\|e\|_2$ with respect to C under the constraint that the closed-loop system has a prescribed spectrum is currently an open-problem, and so we propose an alternative criterion based on the following rationale. From (16)

$$\|e_n\|_2 \leq \|(I - \frac{1}{CF^{-1}B}F^{-1}BC)\|_2 \|(x_n - F^{-1}f_n)\|_2 \tag{17}$$

The second term in the inequality does depend on C via K, but we propose to choose C so that

$$J = \|I - \frac{1}{CF^{-1}B}F^{-1}BC\|_2 \tag{18}$$

is minimized which, without loss of generality, produces the condition

$$C = (F^{-1}B)^T = B^TF^{-T} \tag{19}$$

A fixed point iteration is used to determine a separate C for each region. An initial C_0 is selected and K_{e0} is determined using pole placement, which will be denoted as $K_{e0} = P\{A_{e0}, B_e, \Lambda_e\}$ (A_{ej}, B_j are defined below). The next iterate, C_{j+1}, is determined from the following sequence of steps:

$$\begin{aligned}&A_{ej} = \begin{bmatrix}0 & C_j\\ 0 & A\end{bmatrix},\quad B_e = \begin{bmatrix}0\\ B\end{bmatrix}\\ &K_{ej} = \mathscr{P}\{A_{ej}, B_e, \Lambda\},\quad K_{ej} = [K_{0j} \quad K_j]\\ &F_j = A - BK_j\\ &C_{j+1} = (F_j^{-1}B)^T\end{aligned} \tag{20}$$

The procedure terminates in a small number of iterations (three or less). Once C is determined the distance from the origin is bounded by $\|e_n\|_2{}^* \leq \|(x_n - F^{-1}f_n)\|_2$ and the steady state point to which the trajectory is driven is

$$\delta x_{ss} = -F^{-1}f_n - \frac{UU^T}{U^TU}(x_n - F^{-1}f_n),\quad U = F^{-1}B \tag{21}$$

6. Examples of Design of nonlinear controls

Example 1. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1(1 + x_2^2) + x_3\\ \dot{x}_2 &= -x_1 - (1 + x_3^2)x_2\\ \dot{x}_3 &= -x_3 + u\end{aligned} \tag{22}$$

This system can be stabilized by backstepping, and so can be made GAS. A particular stabilizing control is

$$\begin{aligned}u = &-x_1[6 + 2x_2^2 + x_2^4 - 2x_2{}^2x_3^2]\\ &- x_2[x_3^2 - 2x_1{}^2] - x_3(2 + x_2^2)\end{aligned} \tag{23}$$

Thus, a Lyapunov type control can be compared with a logic control. The control (23) guarantees GAS behavior, and in this case has a relatively simple form of state feedback with state - dependent gains. Its transient amplitudes are, however, significant, and in fact much greater then obtained with a logic controller. Thus, if it suffices to solve the Practical stability problem so that the region S = {x: $\|x\|_\infty \le 20$} belong to the region of attraction of the origin, a polar partition-based logic controller can de developed that exhibits better transient performance.

Choosing points of linearization to form an equidistant simplex with results in

$$x_{c1}=\begin{bmatrix}4.55\\7\\5.4\end{bmatrix}, x_{c2}=\begin{bmatrix}5\\-1.5\\-8.47\end{bmatrix}, x_{c3}=\begin{bmatrix}-0.32\\-8.5\\5.25\end{bmatrix}, x_{c4}=\begin{bmatrix}-9.31\\2.93\\-2.17\end{bmatrix}$$

The desired spectrum was {-0.5,-2,-2±j4} for all polar regions, with an integral state added to the controller and the vector C defining the input to the integral obtained by iterative solution of (19), as described by (20). In all cases C was determined by the iterative procedure described by (20), and converged in at most 2 iterations (a third iteration being carried out to confirm the fixed point was found). The linear control that stabilizes the system about the origin was obtained in the same way except an integral state in this case is not appended to the control and the desired spectrum was shifted to {-1,-2,-3}.

The results in Figure 1 show that an arbitrarily selected Lyapunov function based control need not have better performance than the Logic control. The polar control, however, has a finite region of attraction. The control applied to the system has the form

$$\mu_{tot}=\frac{\mu}{\mu+\mu_d}u_P(x)+\frac{\mu_d}{\mu+\mu_d}u_L(s) \tag{24}$$

where

$$\mu=\|x\|^K,\ \mu_d=\|xd\|^K \tag{25}$$

Trajectories shown are from the initial condition {5,-5,5} using the nonlinear control (23) and the logic polar with K = 1, $\|x_d\| = 50$. Similar results were found to hold for an entire region of initial conditions. Furthermore, the results hold for a wide range of the weighting factor μ_d. Figure 2, displays trajectories when μ_d = 20, 40, 60, 80 and 100.

Example 2. Consider the nonlinear system

$$\dot{x}_1=(1+x_2^2)x_1+x_2x_3+u$$

$$\dot{x}_2=-(1+x_3^2)x_1+x_2$$

$$\dot{x}_3=-x_3+u$$

The goal is to design a control and ensue that S_d = {x : $\|x\|_\infty \le 4$} belong to the region of attraction to the origin. The problem is difficult due to the fact that the system linearized about any point in state space is locally unstable, most often having two unstable eigenvalues. For example, the open-loop trajectory from the point x_0 = {-4-,4,4} exhibit a finite-time blow-up at T = 0.1353 sec. Linearization about the origin results in the linear model

$$\dot{x}=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&-1\end{bmatrix}x+\begin{bmatrix}1\\0\\1\end{bmatrix}u$$

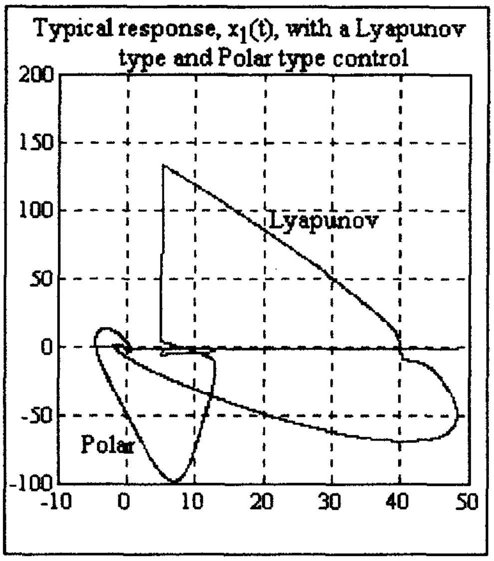


Figure 1. Comparison of typical trajectories

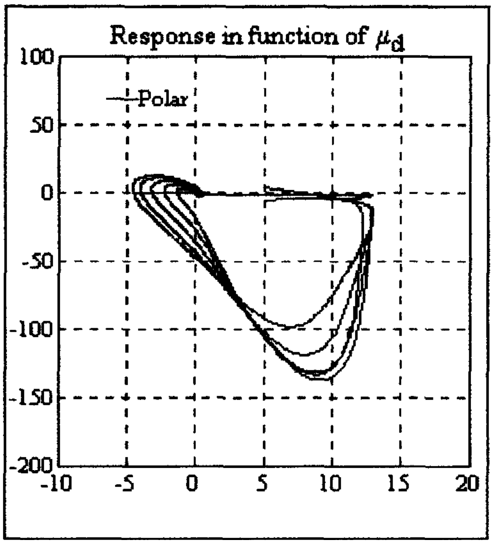


Figure 2. Effect of μ_d

Table 1. Gains and output matrices

$K_{el}=[-0.0423\ \ 24.2\ \ 101.23\ \ 10.685]$ $C_1=[-3.3859\ \ 0.9149\ \ -0.9991]$
$K_{e2}=[-0.0145\ \ 43.11\ \ -4.24\ \ 4.53]$ $C_2=[0.27\ \ 6.03\ \ 2.34]$
$K_{el}=[-0.0034\ \ 35.24\ \ 14.85\ \ 6.9]$ $C_1=[3.09\ \ -0.79\ \ -12.43]$
$K_{el}=[-0.0097\ \ 108.46\ \ -228.6\ -51.84]$ $C_1=[6.988\ \ 3.156\ \ 0.7366]$

which is locally controllable about the origin. A stabilizing control that places the poles at {-1, -1.5,-2} is $u_L = -5.5x_1 - 7x_2$, and was used throughout the analysis. (As will be seen, simulation results indicate that a "faster" spectrum would probably be a better choice for this system, but this was not pursued .)

This design did not meet the design goal and could not achieve more that $S_1 \in \mathscr{A}(u_L)$, where $S_1 = \{x : \|x\|_\infty \leq 2.5\}$. Of course one may consider if there is a better linear control, but we will at this point imagine that such an analysis might have been attempted and that the current candidate is the best alternative. The purpose is to illustrate how one would proceed if a best linear alternative has been found, and the design goals are still not met. To compare with the open-loop behavior consider the same initial point $\{-4,-4,4\} \in S_d$. Trajectory still exhibits a finite-time blow-up for at T = 0.1322 (worse than open-loop).

To proceed with the development of the logic control, the first direction vector π_1 was determined based as one of the escape directions of the state vector when the linear control is used. The other three vectors π_2, π_3, π_4 were determined to form an equidistant simplex resulting in the following for points of linearization (all multiples of the corresponding π_i):

$$x_{c1} = \begin{bmatrix} -8.78 \\ -4.78 \\ 0.258 \end{bmatrix}, x_{c2} = \begin{bmatrix} 0.31 \\ 5.97 \\ -8.02 \end{bmatrix}, x_{c3} = \begin{bmatrix} 1.05 \\ 5.49 \\ 8.29 \end{bmatrix}, x_{c4} = \begin{bmatrix} 7.422 \\ -6.68 \\ -0.53 \end{bmatrix}$$

with $\|x_{ci}\|_2 = 10$.The magnitude of x_{ci} was chosen based on the extend of the set S.

In each case linearization was made about the point $\{x_{ci}, u_{ni}\}$ with u_{ni} selected so that the zero-th order term in the third state equation was zero ($u_{ni} = x_{3i}$), and controller gains were determined to achieve the desired spectrum, $\Lambda_e = \{-2,-3,-4\pm j4\}$.

Table 2. Data for additional regions

$K_{e1} = [-0.005 \quad -7.98 \quad 17.5 \quad 35.98]$ $C_1 = [-7.91 \quad 5.583 \quad -4.376]$
$K_{e1} = [-0.0037 \quad -596.45 \quad 635 \quad 624.45]$ $C_1 = [10.2832 \quad 4.4122 \quad 5.3251]$

Shown in Table 1 are the gain matrix and the output matrix C computed as described. It is noted that in each case the open-loop linearized systems were unstable. Furthermore, to illustrate the point, in each case C was obtained in essentially 2 iterations, as before. For example, starting with $C_0 = [\ 1\ 0\ 1]$ (used as a starting point in all regions), in R_1 produced the sequence $C_1 = [-3.3293 \quad 0.8996 \quad -0.9824]$, and $C_2 = [-3.3859 \quad 0.9149 \ -0.9991]$. If for example one chooses another initial guess, such as $C_0 = [10 \ -5 \quad 4]$, the iteration produce the sequence $C_1 = [\ -3.3766 \quad 0.9124 \quad -0.993]$ and $C_2 = [-3.3859 \quad 0.9149 \ -0.999]$.

Simulations show that s the region of attraction with the use of the resulting logic controller now includes $S_2 = \{\ x : \|x\|_\infty \leq 3.25\}$, and that further increase of the region of attraction is hampered by trajectories now escaping in two directions characterized by direction vectors $\pi_5 = [4\ 4\ -4]'$ and $\pi_6 = [4\ -4\ -4]'$. Furthermore, it was found that region R_4 was not crucial , and that it was in turn covered by the other regions and by the two new regions R_5 and R_6 formed around the identified escape directions, and so R_4 was dropped. The controls in these two new regions were characterized by the control gains and output matrices shown in Table 2.

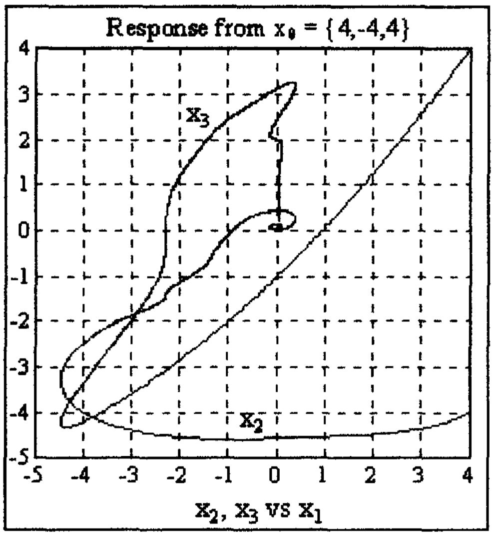


Figure 3a. Typical response

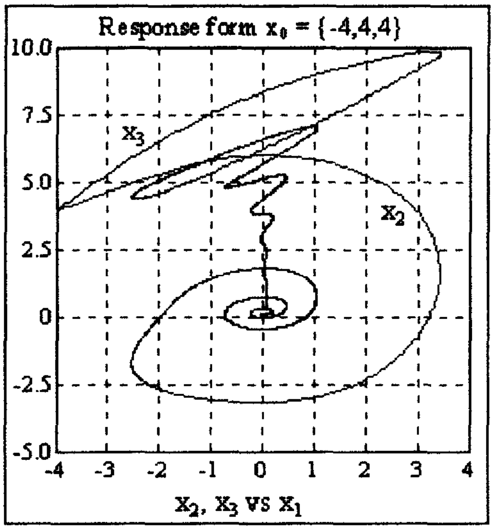


Figure 3. Typical trajectories in state space

Finally, it was realized that the desired spectra were slow for effective control of (1) due to its significant open loop rate of exponential growth. However, instead of re-computing controls for each region based on new desired spectra, the overall gain of the control in the regions R_1,. R_6 was modified (from a nominal value of 1) to higher values to achieve a similar effect. Thus the controls in all the regions were of the form $u_{if}(x) = K_i u_i(x)$ where $u(x)$ still given by (8) with gains specified above. This further increased the region of attraction which now included $S_2 = \{\ x : \|x\|_\infty \leq 3.6\}$.

Further progress was hampered by the inability to improve the response when the state enters region R_6 because a high gain was demanded when $x_2 > 0$, while a low gain was demanded when $x_2 < 0$. With this insight, the linear control in region R_6 was modified to a nonlinear control, by having the gain be a

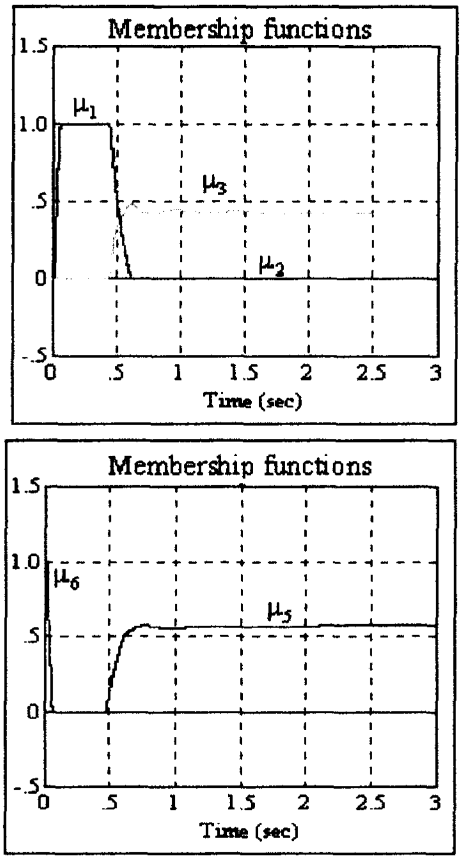
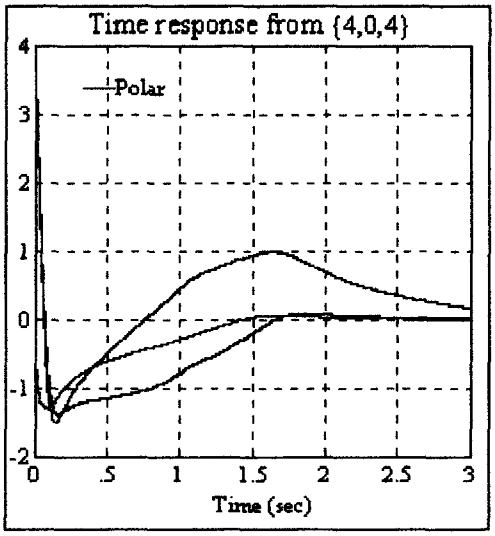


Figure 4. Membership function for trajectory from x_{01}



function of the state x_2: $K_6 = 5(1+1.05\text{sign}(x_2))$.With this modification the region of attraction now contained $S_3 = \{ x : \|x\|_\infty \le 4\}$. Figure 3 show the trajectories in the $\{x_1,x_2\}$ $\{x_1,x_3\}$space for the initial conditions $x_{01} = \{4, -4, 4\}$ and $x_{02} = \{-4, 4, 4\}$, respectively. Figure 4 shows the membership functions for the trajectory initiating at x_{10}. Figure 5 shown the time responses from two additional initial conditions, $x_{03} = \{4, 0, 4\}$ and $x_{04} = \{4, 4, 0\}$.

The results not only confirm the viability of the approach, but in particular the wealth of information that can be obtained from simulations and used to expand the region of attraction by expanding the structure of the polar controller, and modifying the control in specific regions of the state space. In the process, the use of membership function plots, as shown in Figures 4 and 6 provided the information needed to focus on the regions of state space where the adjustment of the control was required.

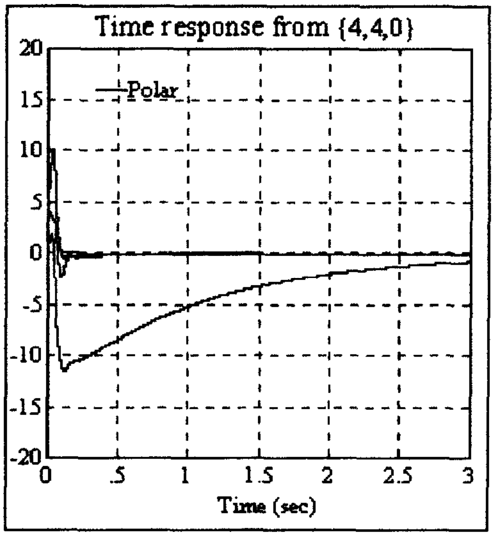


Figure 5. Selected time responses

The traces of membership functions with the final controller do confirm that a polar partition is often a natural partition in the control of dynamic systems because the terminal portions of all the responses approached the origin from distinctly defined directions, as can be seen by the fairly constant values of membership functions.

7. Conclusions

The development presented here must be considered as preliminary but it does provide encouraging results. The idea of multiple point linearization with the introduction of an integral state to direct the trajectory towards the origin appears promising. The results also confirm that a polar partition may be a more viable partition on which to base the logic controller structure than a partition based on overlapping hypercubes. Third, feedback from results of simulation is a valuable tool and systematic ways of concisely representing, and using this information should be developed. While the basic procedure, and its obvious generalizations, are fairly well defined, proofs of stability are yet to be developed.

8. References

1. J.V. Medanic, "Logic controller design for multivariable nonlinear systems," *13th World Congress of IFAC*, San Francisco, July 1996.
2. H. Khalil, *Nonlinear Systems*, Prentice-Hall Publ. Co. , Second Ed., 1996.
3. Sepulchre, R., Jankovic ., M. and Kokotovic, P., 1997, Constructive Nonlinear Control (Springer-Verlag, London).
4. Medanic, M. Yuan and B. Medanic "Robust Multivariable Nonlinear Control of a Two Link Excavator: Part I", Proc. 1997 IEEE CDC, San Diego, Dec. 10-12.
5. Control of Excavators Using Advanced Variable Structure and Logic Control Methodology: Phase II – Three Link Excavator Control," Medanic Consulting, Final Report to Caterpillar Inc. on Contract No. 340-1109-A, Dec. 15, 1999.