

# Optimal pricing and replenishment policies when consumer demand depends on the in-store inventory level\*

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**Abstract.** The paper studies an optimal control problem of pricing and inventory replenishment in a system with sequential inventories. Consumer demand for a specific product depends on price as well as the in-store stock of the product. The hypothesis is that for some consumer products, a large volume of displayed goods leads consumers to buy more than if the stock is small. In addition to the displayed stock, there is an inventory of the product in a central warehouse. We consider a setup in which management of the two stocks is decentralized such that pricing decisions are made by the store manager who also decides on the level of in-store inventory. The manager of the warehouse makes replenishment decisions about the stock in the warehouse. We also study the problem where decisions are centralized and investigate the policy impacts of having a stock-dependent demand. Phase diagrams and a synthesizing procedure are used to derive optimal inventory, replenishment, and retail price policies.

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## 1. Introduction

The paper studies a problem of pricing and inventory replenishment for a specific product. Our hypothesis is that consumer demand at a retail outlet depends on price as well as the in-store stock of the product. In particular, demand decreases with price but increases with the displayed inventory of the product. The idea is that a large stock leads consumers to buy more. This phenomenon is not a general one but may pertain to certain product categories, e.g., used automobiles sold from an open yard of a car dealer, or displays of large quantities

of soft drinks, detergents, and canned food sometimes found in supermarkets. (The argument can be reversed. For other products, the existence of a large stock may decrease demand if the customers believe that the product is a bad one).

The idea of letting the demand rate depend on the level of the inventory was introduced in Wolfe (1968), Schary and Becker (1972); the problem of determining optimal ordering and inventory policies under stock-dependent demand was addressed by, e.g., Baker and Urban (1988), Goh (1992), Urban (1995), Gerchak and Wang (1994). These contributions use an EOQ (Economic Order Quantity) approach and determine an optimal lot size. A common, but implicit, assumption of this literature is that the whole stock of the product is on display and hence is capable of affecting demand. This paper studies a different situation and supposes that there are *two stocks* of the product. One is displayed in the retail store and the size of this stock affects sales (as in the above literature). Another stock is stored in a central warehouse/distribution center. From this stock, the retail store's inventory is replenished, but the size of the stock in the warehouse does not affect consumer demand. One may think of a distribution center from which retail outlets are supplied. For simplicity, we assume that all goods are put on display, upon arrival at a retail store.

With respect to the management of the inventories, we consider a setup in which management is *decentralized*. The store manager decides how his inventory should be replenished and sets the consumer price. The manager of the central warehouse makes the replenishment decisions concerning that stock. Each

decision maker has his own objective function which he seeks to optimize. We also consider an organizational structure in which decisions are *centralized*. Here, a single decision maker (e.g., the headquarters) makes all replenishment and pricing decisions and seeks to optimize the total system payoff.

As regards the organization of the inventory system, our paper departs from the EOQ literature which considers a single inventory and hence cannot address the organizational problem. There are other differences. In contrast to the EOQ models, we use review policies where inventory levels are monitored continuously and a manager may place an order at any instant of time. We do not allow the EOQ-type of ordering where the stock is replenished instantaneously at discrete instants of time. Moreover, our model includes price as a second decision variable that influences consumer demand, where the EOQ literature often assumes a constant sales price. One reason is that marketing studies have found that the impact on sales of a large inventory may be enhanced by price cuts (Berman (1996)).

Research on production and inventory planning using continuous-time modeling includes Holt et al. (1960) and Pekelman (1974). The latter introduced price as a decision variable in a production-inventory model. Later work, using optimal control theory or differential games, includes Thompson et al. (1984), Feichtinger and Hartl (1986), Jørgensen (1986), Abad (1987), and Jørgensen et al. (1999).

The contributions of the present paper are the following. EOQ models with inventory-dependent demand are extended to a serial, two-stock setting in which one of the inventories does not affect demand, and where price is an additional decision variable. Optimal control models of inventory and pricing are extended with an additional determinant of demand, viz., the level of displayed inventory. In the setup of the retail distribution system, we assess the policy implications of having a stock-dependent demand and identify the negative cost effects of decentralized decision making.

In Section 2 we present an optimal control model of pricing, ordering, and inventory management. Section 3 solves the decentralized problem and Section 4 gives some concluding remarks. Due to the page

limitations, we had to omit the derivation of the centralized policy, comparisons of the outcomes of decentralized and centralized decision making, as well as the assessment of the policy impacts of having a stock-dependent demand. However, this material is contained in the full-length paper which can be requested from the authors.

## 2. The dynamic model

We confine our interest to a *single-product, single-retail store* problem. A central warehouse orders the product from an outside supplier. Upon arrival at the warehouse, goods are stocked until being ordered by the retail store. All goods shipped from the warehouse to the store are put on display. We consider a fixed planning period  $[0, T]$  and the problem is to determine optimal policies for the retail price, the ordering rate of the warehouse from the manufacturer, the ordering rate of the retail store from the warehouse, and the inventories of the store and the warehouse, respectively.

Let  $I_D(t)$  ( $D$  for “displayed”) represent the inventory level in the retail store by time  $t \in [0, T]$ . The stock in the warehouse is denoted by  $I_N(t)$  ( $N$  for “not” displayed). The stock  $I_D$  is reduced according to the demand rate, but can be replenished from the stock  $I_N$ . Denote by  $q_D(t)$  the replenishment rate of the in-store inventory. The stock in the warehouse is replenished by ordering from the supplier; we denote this ordering rate by  $q_N(t)$ . Letting  $x(t)$  represent the rate of consumer demand at time  $t$ , and assuming zero lead times in the ordering processes, the inventory dynamics constitute a serial two-stage system:

$$\dot{I}_D(t) = q_D(t) - x(t), I_D(0) = I_{D0} > 0 \quad (1a)$$

$$\dot{I}_N(t) = q_N(t) - q_D(t), I_N(0) = I_{N0} > 0 \quad (1b)$$

The demand rate depends on  $p(t)$ , the consumer price at time  $t$ , and the level of displayed stock. We have chosen a linear specification

$$x(t) = f(p(t), I_D(t)) = [\alpha - \beta p(t)] k I_D(t) \quad (2)$$

in which  $\alpha$ ,  $\beta$ , and  $k$  are positive constants. The demand function in (2) has the following properties

$$f(p, I_D) \geq 0, f(p, 0) = 0$$

$$\partial f / \partial p < 0, \partial f / \partial I_D > 0, \partial^2 f / \partial p \partial I_D < 0.$$

The second assumption should be noted: if the displayed inventory is zero, the demand rate is zero (“no stock, no sale”). The demand rate increases linearly with the displayed stock level but note that many of the EOQ studies suppose that demand depends on inventory in a concave (increasing) way. Finally, negativity of the mixed second order partial derivative means that the marginal impact of the displayed stock on demand increases when price decreases. This hypothesis reflects a conjecture of marketing literature stating that the impact on demand of a large inventory may be enhanced by a low price (Berman (1996)).

Admissible decisions  $q_D(t), q_N(t), p(t)$  must satisfy

$$q_j(t) \geq 0, j \in \{D, N\}, 0 \leq p(t) \leq \alpha / \beta. \quad (3)$$

and the state of the system at any time  $t$  is described by the stock levels  $I_N(t), I_D(t)$ . Backlogging is not allowed; This constraint is necessary to be consistent with the “no stock, no sale” assumption. In, for example, Urban (1995) stockouts are allowed (at a cost) and even during a stockout period, some demand occurs. Using (1a) yields

$$\dot{I}_D(t)|_{I_D(t)=0} = q_D(t) - f(p(t), 0) \geq 0. \quad (4)$$

Then the “no stock, no sale” assumption, and the fact that  $I_{D0} > 0, q_D(t) \geq 0$ , shows that  $I_D(t)$  is positive for all  $t$ . Satisfaction of the other nonnegativity constraint

$$I_N(t) \geq 0 \quad \forall t \in [0, T] \quad (5)$$

is not guaranteed.

The cost of replenishing the stock  $I_N$ , by ordering from the supplier, is given by

$$C_N(q_N) = c_N q_N \quad \text{where } c_N = \text{const.} > 0$$

assuming no quantity discounts or fixed ordering costs.

The cost of replenishing the retail store stock  $I_D$  from the warehouse is quadratic

$$C_D(q_D) = \frac{c_D}{2} q_D^2 \quad \text{where } c_D = \text{const.} > 0$$

and includes expenses incurred by shipping goods from the warehouse to the retail shop, and processing goods before being put on display. Supposing that there are no warehousing constraints, we assume linear inventory holding costs

$$S_j(I_j) = s_j I_j \quad \text{where } s_j = \text{const.} > 0, j \in \{D, N\}$$

The overall profit function of the system is given by

$$J = \int_0^T \{p(t)f(p(t), I_D(t)) - [c_N q_N(t) + \frac{c_D}{2} q_D(t)^2 + s_D I_D(t) + s_N I_N(t)]\} dt \quad (6a)$$

In a decentralized system, we represent the store manager by superscript  $D$  and the warehouse manager by  $N$ . Manager  $D$  solves an optimal control problem, to maximize  $J_D$ , whereas  $N$  solves an optimal control problem to maximize  $J_N$ . These objective functions are given by

$$J_D = \int_0^T [p(t)f(p(t), I_D(t)) - \frac{c_D}{2} q_D(t)^2 - s_D I_D(t)] dt \quad (6b)$$

$$J_N = - \int_0^T [c_N q_N(t) + s_N I_N(t)] dt$$

$$\text{and } J_D + J_N = J.$$

### 3. Optimal policies in a decentralized system

Here, the store manager maximizes  $J_D$  in (6b), subject to (1) and the constraints on  $p$  and  $q_D$  stated in (3). The warehouse manager maximizes  $J_N$  in (6b), subject to (1) and the constraints on  $q_N$  and  $I_N$  in (3) and (5). In this decentralized setup it is reasonable to make the warehouse manager responsible for the satisfaction of the no-backlogging constraint (5).

### 3.1. Solution of Problem N

Let  $S(t)$  represent accumulated orders from the store to the warehouse by time  $t$ , and let  $\tau$  be given by

$$S(\tau) = \int_0^{\tau} q_D(t) dt = I_{N0}.$$

where  $\tau$  is the instant of time at which the initial stock  $I_{N0}$  will be exhausted if  $q_N(t) = 0$  for  $t \in [0, \tau]$ . The solution of problem N is stated in Propositions 1 and 2. All proofs are omitted; they can be obtained from the authors.

**Proposition 1.** *If in Problem N it holds that*

$$I_{N0} \geq S(T) = \int_0^T q_D(t) dt,$$

*then the optimal replenishment rate is given by  $q_N^*(t) = 0$  for  $t \in [0, T]$ .*

Thus, if the initial stock is sufficient to satisfy the accumulated orders from the store for the entire planning period, no replenishment of the warehouse inventory should be made as this would only imply unnecessary procurement and holding costs. The warehouse inventory is positive, except possibly at the horizon date.

**Proposition 2.** *If  $I_{N0} < S(T)$ , the optimal replenishment rate of problem N is given by*

$$q_N^*(t) = 0 ; I_N(t) = I_{N0} - S(t) \quad \text{for } t \in [0, \tau]$$

$$q_N^*(t) = q_D(t) ; I_N(t) = 0 \quad \text{for } t \in (\tau, T].$$

To minimize holding costs, the warehouse manager starts out by decreasing his stock as fast as possible. This is done by having a zero replenishment rate until the initial stock is exhausted. Thus, the manager escapes the holding costs as soon as possible and avoids any replenishment cost. Thereafter he replenishes at the “demand rate”  $q_D(t)$  for the rest of the planning period and incurs no holding cost. Note that the optimal policy satisfies  $I_N(t)q_N(t) = 0$  which is well-known from standard inventory models. Also note that the warehouse

manager only needs to react to the ordering policy of the store manager,  $q_D(t)$ . The pricing policy has no direct influence on the warehouse manager’s policy.

### 3.2. Solution of Problem D

It is useful to distinguish four regimes:

$$\text{Regime 1 (R1): } \lambda_D \leq -\alpha/\beta, p^* = q_D^* = 0$$

$$\text{Regime 2 (R2): } -\alpha/\beta < \lambda_D \leq 0$$

$$p^* = [\alpha + \beta \lambda_D]/2\beta > 0, q_D^* = 0$$

$$\text{Regime 3 (R3): } 0 < \lambda_D < \alpha/\beta$$

$$p^* = [\alpha + \beta \lambda_D]/2\beta > 0, q_D^* = \lambda_D/c_D > 0$$

$$\text{Regime 4 (R4): } \lambda_D \geq \alpha/\beta$$

$$p^* = \alpha/\beta, q_D^* = \lambda_D/c_D > 0.$$

The solution of Problem D is stated in

**Proposition 3.** *The optimal policy of Problem D is as follows*

$$\text{Case A: } k\alpha^2/4\beta < s_D < k\alpha^2/\beta$$

$$R2 \text{ for all } t \in [0, T]$$

$$\text{Case B: } s_D > k\alpha^2/\beta$$

$$R1 \text{ for } t \in [0, \bar{t}), R2 \text{ for } t \in [\bar{t}, T]$$

$$\text{Case C: } s_D < k\alpha^2/4\beta:$$

$$R3 \text{ for } t \in [0, T), R2 \text{ for } t = T.$$

Regime 4 is not used in an optimal solution. This result is driven by the assumption of zero salvage value, but recall that Regime 4 is an extreme one; price is chosen to make demand vanish. In our context, where inventory stimulates demand, it may be less plausible to choose such a price. The proposition shows that the size of the

holding cost  $s_D$ , relative to parameters  $k$ ,  $\alpha$ , and  $\beta$ , determines the optimal ordering and pricing policies. Recall that the larger the parameter  $k$ , the larger the demand rate (given any level of the in-store inventory).

Cases *A* and *B* are qualitatively the same. In *B*, the holding cost,  $s_D$ , is “large” and the demand-stimulating effect,  $k$ , is “small”. The implication is that replenishment of the inventory is not worthwhile at all. To reduce the inventory as quickly as possible, the manager selects the minimal price. Price increases over time which tends to reinforce the negative effect of the decreasing inventory on demand. This policy confirms the marketing conjecture that the sales impact of a large inventory may be enhanced by price cuts.

Case *C* can occur if the holding cost  $s_D$  is “small” and the efficiency parameter  $k$  is “large”. It means that it is not costly to have a large inventory, and the inventory level has a considerable impact on demand. This makes it worthwhile to have a positive replenishment rate. The inventory level is steadily decreasing (if large initially), or it first increases and then decreases (if small initially). This result may seem counterintuitive as one might expect an increasing inventory when holding cost  $s_D$  is small and efficiency  $k$  is large. The primary explanation for the result is the zero salvage value of the inventory, which induces the overall decrease of the shadow price. Since the inventory becomes less valuable over time, it makes sense to reduce it even if it carries a low holding cost and has a sizeable impact on demand. In this case we see that to take advantage of a large value of  $k$ , price is steadily decreased over time. This tends to counterbalance the effect on demand of a decreasing inventory. It also confirms the marketing conjecture reported in Berman (1996).

### 3.3. Combining the solutions of Problems *N* and *D*

Insert the policies  $q_D^*(t)$  from Proposition 3 into Problem *N* to obtain the following. In Cases *A* and *B* we have  $q_D^*(t) \equiv 0$  and are in the situation dealt with in Proposition 1. The implication is  $q_N^*(t) \equiv 0$  which makes the warehouse inventory constant over time. In Case *C*, Regime 2 only applies at  $t = T$  where we have  $q_D^*(T) = 0$ . In Regime 3 it holds that  $q_D^*(t) = \lambda_D(t)/c_D$  where  $\lambda_D(t)$  is the costate

variable, given as the solution of a Riccati equation.

To see if we are in the situation of Proposition 1, or in that of Proposition 2, calculate

$$S(T) = \int_0^T q_D^*(t) dt = \frac{1}{c_D} \int_0^T \lambda_D(t) dt$$

and then compare  $S(T)$  with  $I_{NO}$ . However, as long as the parameter values are unspecified, this does not provide a definitive answer to the problem of finding a best balance between a central and a local stock.

## 4. Concluding remarks

The paper extends the EOQ literature with inventory-dependent demand to a serial, two-stock setting. Optimal control models of inventory and pricing with continuous review policies have been extended with an additional determinant of demand, viz., the level of displayed inventory. Optimal solutions are obtained for three setups: (1) decentralized and (2) centralized decision making under stock-dependent demand, and (3) decentralized decision making under stock-independent demand. Our results for the first two setups are new; our result for the third one is a straightforward modification of results found in Pekelman (1974), Feichtinger and Hartl (1986).

The paper raised the following questions. What are the policy impacts of having a stock-dependent demand, and are there disadvantages of letting the two managers make their decisions independently? The impacts of a stock-dependent demand are that replenishment rates are decreasing rather than increasing over time, and the in-store inventory is never exhausted. Indeed, it increases over time on an initial interval if the initial stock is small.

In a setup where decision making is centralized, the single manager takes explicitly into account the costs that are incurred to replenish the warehouse inventory. In the decentralized setup, the store manager is not concerned about the inventory in the warehouse, neither its cost nor its shadow price. The implication is that - ceteris paribus - the period during which the warehouse inventory needs replenishment will be longer in the decentralized case. Furthermore, when the warehouse inventory is large, centralized decision making will imply larger orders of goods from the warehouse to

the store, in order to reduce warehouse holding costs.

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