

Stabilisation of a Class of Nonlinear Time-Delay Systems Using Fuzzy Models

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ABSTRACT

This paper examines the problem of stabilising a class of nonlinear time delay systems using fuzzy models. The class of nonlinear time delay systems under investigation is described by the Takagi-Sugeno (T-S) or Takagi-Sugeno-Kang (TSK) fuzzy model. Based on a predictor-based controller and a well-known *Lyapunov functional* approach, we develop a technique for designing a fuzzy state feedback control law which globally stabilises this class of nonlinear time delay systems. A numerical simulation example is presented to illustrate the theory development.

1. INTRODUCTION

Before designing a model-based controller for a given process, a mathematical model which captures all the relevant characteristics of the process has to be derived. However, in many practical situations a "good" mathematical model that describes the dynamics of processes is very difficult, if not impossible to obtain. Recently, fuzzy systems have attracted considerable amount of attentions from both the academic and industrial communities [1]-[11]. It has been shown that the qualitative knowledge of a system can also be represented in a nonlinear functional form. Based on this fact, fuzzy models based fuzzy control system design methods have appeared in the literatures. These design methods are conceptually simple and straightforward. An example of fuzzy models is the Takagi-Sugeno or Takagi-Sugeno-Kang type fuzzy model. The continuous-time Takagi-Sugeno fuzzy dynamic model is described by fuzzy IF-THEN rules, which locally represent linear input-output relations of nonlinear systems. The i th rule of the fuzzy model is of the following form:

Plant Rule i :

IF $x_1(t)$ is F_{i1} and \dots and $x_n(t)$ is F_{in}

THEN $\dot{x}(t) = A_i x(t) + B_i u(t)$, $i = 1, 2, \dots, r$

where F_{ij} ($j = 1, 2, \dots, n$) are fuzzy sets, $x(t) \in \mathfrak{R}^n$ is the state vector, $u(t) \in \mathfrak{R}^m$ is the input, the matrices A_i and B_i are of appropriate dimensions and r is the number of IF-THEN rules.

Given a pair $[x(t) \ u(t)]$, by using a singleton fuzzifier, product fuzzy inference and weighted average defuzzifier, the final state of the fuzzy system is inferred as follows:

$$\begin{aligned} \dot{x}(t) &= \frac{\sum_{i=1}^r w_i(x(t)) [A_i x(t) + B_i u(t)]}{\sum_{i=1}^r w_i(x(t))} \\ &= \sum_{i=1}^r \mu_i(x(t)) [A_i x(t) + B_i u(t)] \end{aligned} \quad (1.1)$$

where

$$w_i(x(t)) = \prod_{j=1}^n F_{ij}(x_j(t)), \quad \mu_i(x(t)) = \frac{w_i(x(t))}{\sum_{i=1}^r w_i(x(t))}$$

$F_{ij}(x_j(t))$ is the grade of membership of $x_j(t)$ in F_{ij} . It is assumed in this paper that

$$w_i(x(t)) \geq 0, \quad i = 1, 2, \dots, n; \quad \sum_{i=1}^r w_i(x(t)) > 0$$

for all t . Therefore

$$\mu_i(x(t)) \geq 0, \quad i = 1, 2, \dots, n; \quad \sum_{i=1}^r \mu_i(x(t)) = 1$$

for all t . For the convenience of notations, let $w_i = w_i(x(t))$ and $\mu_i = \mu_i(x(t))$; then the final state of the

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fuzzy system can be represented as

$$\dot{x}(t) = \sum_{i=1}^r \mu_i A_i x(t) + \sum_{i=1}^r \mu_i B_i u(t) \quad (1..2)$$

Recently, robust stabilization of systems with time delays has been of a challenging problem. As we know, in general, the existence of time delays degrades the control performance and sometime makes the closed-loop stabilization difficult, especially for nonlinear systems. The motivation of the paper stems from the fact that all the aforementioned results assumed that the systems are free of time-delays. What we intend to do in this paper, is to propose a procedure for designing a fuzzy stabilizing controller for a class of time-delay nonlinear system described by the Takagi-Sugeno (T-S) or Takagi-Sugeno-Kang (TSK) fuzzy model. Our approach is based on the predictor based controller and the constructive use of appropriate *Lyapunov functionals*, see [12, 13]. The predictor-based controller includes a predictor to compensate for the time delay. Under the predictor-based controller, a time-delay system can be transformed into a delay free system.

2. SYSTEM DESCRIPTION AND DEFINITION

The class of nonlinear time delay systems under consideration is of the form

Plant Rule i :

IF $x_1(t)$ is F_{i1} and \dots and $x_n(t)$ is F_{in} THEN

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_i u(t - h_i) + f_i(x(t)), \\ &\quad i = 1, 2, \dots, r \quad (2..1) \\ x(t) &= \xi(t), \quad t \in [-\max\{h_1, h_2, \dots, h_r, 0\}], \end{aligned}$$

where F_{ij} ($j = 1, 2, \dots, n$) are fuzzy sets, $x(t) \in \mathfrak{R}^n$ is the state vector, $u(t) \in \mathfrak{R}^m$ is the input, the matrices A_i and B_i are of appropriate dimensions, h_i are nonnegative constants that represent the time delays in control, r is the number of IF-THEN rules. $\xi(t) \in C^n[-\tau, 0]$ and $\tau = \max\{h_1, h_2, \dots, h_r\}$.

We consider a predictor of the form:

Predictor Rule i :

IF $x_1(t)$ is F_{i1} and \dots and $x_n(t)$ is F_{in} THEN

$$\bar{x}(t) = e^{A_i h_i} x(t) + \int_{-h_i}^0 e^{-A_i \theta} B_i u(t + \theta) d\theta, \quad (2..2) \\ i = 1, 2, \dots, r,$$

where $\bar{x}(t)$ is the predictive state. Using (2.2), the fuzzy model (2.1) can be re-expressed as follows:

Plant Rule i :

IF $x_1(t)$ is F_{i1} and \dots and $x_n(t)$ is F_{in} THEN

$$\dot{\bar{x}}(t) = A_i \bar{x}(t) + B_i u(t) + e^{A_i h_i} f_i(x(t)), \quad (2..3) \\ i = 1, 2, \dots, r$$

Using a singleton fuzzifier, product fuzzy inference and weighted average defuzzifier, the transformed fuzzy system is given as follows:

$$\begin{aligned} \dot{\bar{x}}(t) &= \frac{\sum_{i=1}^r w_i(x(t)) [A_i \bar{x}(t) + B_i u(t) + e^{A_i h_i} f_i(x(t))]}{\sum_{i=1}^r w_i(x(t))} \\ &= \frac{\sum_{i=1}^r \mu_i(x(t)) [A_i \bar{x}(t) + B_i u(t) + e^{A_i h_i} f_i(x(t))]}{\sum_{i=1}^r \mu_i(x(t))} \quad (2..4) \end{aligned}$$

where

$$w_i(x(t)) = \prod_{j=1}^n F_{ij}(x_j(t)), \quad \mu_i(x(t)) = \frac{w_i(x(t))}{\sum_{i=1}^r w_i(x(t))}$$

$F_{ij}(x_j(t))$ is the grade of membership of $x_j(t)$ in F_{ij} . The final state of the fuzzy system is given as follows:

$$\dot{\bar{x}}(t) = \frac{\sum_{i=1}^r \mu_i A_i \bar{x}(t) + \sum_{i=1}^r \mu_i B_i u(t)}{\sum_{i=1}^r \mu_i e^{A_i h_i} f_i(x(t))} \quad (2..5)$$

The matrix functions $f_i(x(t))$ are assumed to satisfy the following matching condition:

Assumption 1. For a given matrix M , there exists a positive function $\rho(x(t))$ such that

$$\left\| \sum_{i=1}^r \mu_i M e^{A_i h_i} f_i(x(t)) \right\| \leq \left\| \sum_{i=1}^r \mu_i M B_i \right\| \sum_{i=1}^r \rho_i(x(t)).$$

Remark 1. Note that the matching condition given in (1) is different from the well known matching condition, ie with the term $e^{A_i h_i}$. The presence of the term $e^{A_i h_i}$ in (1) is due to the delay in the control input. Generally speaking, the delay in the control input will mismatch the nonlinear uncertainty from the input matrix.

Remark 2. Two special cases of system (2.5) have been well studied. The case when $h_i = 0$ and $f_i(x(t)) = 0$ has been considered in [8, 10, 7, 5], and when $h_i = 0$ and $f_i(x(t)) = f(x(t))$ has been examined in [9]. What we intend to do in this paper is to develop a unified method to treat the general case.

Definition 1. The system (2.5) (with $u(t) = 0$) is said to be exponentially stable if there exist a positive definite

matrix P and positive constants α, ϵ and β such that the derivative of the Lyapunov function

$$V(\bar{x}(t)) = \bar{x}^T(t)P\bar{x}(t) \quad (2..6)$$

with respect to time t satisfies

$$\dot{V}(\bar{x}(t)) \leq -\alpha\|\bar{x}(t)\|^2 + \epsilon e^{-\beta t}, \quad (2..7)$$

for all $\bar{x}(t) \in \mathfrak{R}^n$.

In this paper, the local state feedback controllers are designed as follows:

Controller Rule i :

IF $x_1(t)$ is F_{i1} and \dots $x_n(t)$ is F_{in}

THEN $u(t) = -\phi_{c_i}(x(t), \bar{x}(t))$, $i = 1, 2, \dots, r$

The final fuzzy controller is

$$u(t) = -\sum_{i=1}^r \mu_i \phi_{c_i}(x(t), \bar{x}(t)) \quad (2..8)$$

where we use the same weight μ_i as the weight of i th rule of the fuzzy system (2..5).

Definition 2. The nonlinear time delay system (2..5) is said to be exponentially stabilisable if there exist a state feedback control law (2..8), a positive definite symmetric matrix $P \in \mathfrak{R}^{n \times n}$ and constants $\alpha > 0$, ϵ and β such that the time derivative of the Lyapunov function

$$V(\bar{x}(t)) = \bar{x}^T(t)P\bar{x}(t) \quad (2..9)$$

along the system (2..5) with (2..8) satisfies

$$\dot{V}(\bar{x}(t)) \leq -\alpha\|\bar{x}(t)\|^2 + \epsilon e^{-\beta t} \quad (2..10)$$

for all $\bar{x}(t) \in \mathfrak{R}^{n \times n}$. Then, the closed-loop system (2..5) is said to be exponentially stable.

3. MAIN RESULT

We now state our main result:

Theorem 1. *The system (2..5) satisfying Assumption 1 is globally exponentially stabilizable via a nonlinear state feedback controller of the form (2..8) if there exist positive definite symmetric matrices $Q_{ij} \in \mathfrak{R}^{n \times n}$, such that the following algebraic Riccati equations*

$$A_i^T P + P A_i - 2P B_i B_i^T P + Q_{ii} = 0, \quad \forall i = 1, 2, \dots, r \quad (3..1)$$

and

$$(A_i + A_j)^T P + P(A_i + A_j) - 2P B_i B_j^T P - 2P B_j B_i^T P + Q_{ij} = 0 \quad \forall i < j \leq r. \quad (3..2)$$

have a positive definite symmetric solution P . If this is the case, then a suitable stabilizing control law is given by

$$u(t) = -\sum_{i=1}^r \mu_i(\bar{x}(t)) B_i^T P \bar{x}(t) [I + \phi(x(t))], \quad (3..3)$$

with

$$\phi(x(t)) = \frac{\sum_{i=1}^r \rho_i^2(x(t))}{\|\bar{x}^T(t) \sum_{i=1}^r \mu_i P B_i\| \sum_{i=1}^r \rho_i(x(t)) + \epsilon^{-\beta t}} \quad (3..4)$$

where β and ϵ are any positive constants.

Proof: Suppose there exists a P such that (3..1)-(3..2) hold. Then a suitable choice of Lyapunov candidate for system (2..5) with (3..3) is

$$V(\bar{x}(t)) = \bar{x}^T(t)P\bar{x}(t). \quad (3..5)$$

Differentiating (3..5) along (2..5) with (3..3) yields:

$$\begin{aligned} \dot{V}(\bar{x}(t)) &= 2\bar{x}^T(t) \sum_{i=1}^r \mu_i P [A_i - B_i \sum_{j=1}^r \mu_j B_j^T P] \bar{x}(t) \\ &\quad + 2\bar{x}^T(t) \sum_{i=1}^r \mu_i P e^{A_i h_i} f_i(x(t)) \\ &\quad - 2\bar{x}^T(t) \sum_{i=1}^r \mu_i P B_i \sum_{j=1}^r \mu_j B_j^T P \bar{x}(t) \phi(x(t)) \\ &= 2\bar{x}^T(t) \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j P [A_i - B_i B_j^T P] \bar{x}(t) \\ &\quad + 2\bar{x}^T(t) \sum_{i=1}^r \mu_i P e^{A_i h_i} f_i(x(t)) \\ &\quad - 2\bar{x}^T(t) \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j P B_i B_j^T P \bar{x}(t) \phi(x(t)) \end{aligned} \quad (3..6)$$

Applying the triangle inequality and Assumption 1 and using (3..4) on the second and the third terms, we obtain the following inequality

$$\begin{aligned} &2\bar{x}^T(t) \sum_{i=1}^r \mu_i P e^{A_i h_i} f_i(x(t)) \\ &- 2\bar{x}^T(t) \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j P B_i B_j^T P \bar{x}(t) \end{aligned}$$

$$\begin{aligned}
& \times \frac{\sum_{i=1}^r \rho_i^2(x(t))}{\|\bar{x}^T(t) \sum_{i=1}^r \mu_i P B_i\| \sum_{i=1}^r \rho_i(x(t)) + \epsilon e^{-\beta t}} \\
& \leq 2 \left\| \bar{x}^T(t) \sum_{i=1}^r \mu_i P B_i \right\| \sum_{i=1}^r \rho_i(x(t)) \\
& - \frac{2 \|\bar{x}^T(t) \sum_{i=1}^r \mu_i P B_i\|^2 \sum_{i=1}^r \rho_i^2(x(t))}{\|\bar{x}^T(t) \sum_{i=1}^r \mu_i P B_i\| \sum_{i=1}^r \rho_i(x(t)) + \epsilon e^{-\beta t}} \\
& \leq \frac{2 \|\bar{x}^T(t) \sum_{i=1}^r \mu_i P B_i\| \sum_{i=1}^r \rho_i(x(t)) \epsilon e^{-\beta t}}{\|\bar{x}^T(t) \sum_{i=1}^r \mu_i P B_i\| \sum_{i=1}^r \rho_i(x(t)) + \epsilon e^{-\beta t}} \\
& \leq 2\epsilon e^{-\beta t}
\end{aligned} \tag{3.7}$$

Using (3.7), we have

$$\begin{aligned}
\dot{V}(\bar{x}(t)) & \leq \bar{x}^T(t) \sum_{i=1}^r \sum_{j=1}^r [A_i^T P + P A_i - 2P B_i B_j^T P] \bar{x}(t) \\
& + 2\epsilon e^{-\beta t}
\end{aligned} \tag{3.8}$$

Then, using (3.1) and (3.2), we have

$$\begin{aligned}
\dot{V}(\bar{x}(t)) & \leq -\bar{x}^T(t) \sum_{i=1}^r \mu_i^2 Q_{ii} \bar{x}(t) - \\
& \sum_{i=1}^r \sum_{j>i}^r \mu_i \mu_j \bar{x}^T(t) Q_{ji} \bar{x}(t) + 2\epsilon e^{-\beta t} \\
& \leq -\alpha \|\bar{x}(t)\|^2 + 2\epsilon e^{-\beta t}
\end{aligned} \tag{3.9}$$

Therefore, (2.5) is globally exponentially stabilized, according to Definition 1. $\nabla\nabla\nabla$

Remark 3. Following from the transformation (2.3) and the fact that the transformed state, $\bar{x}(t)$ is exponentially stable, we can conclude that the fuzzy system (2.1) is asymptotically stable.

Remark 4. When $h_i = 0$ and $f_i(x(t)) = 0$ our result boils down to the results given in [7, 5], and when $h_i = 0$ and $f_i(x(t)) = f(x(t))$ our result reduces to the result presented in [9].

Remark 5. The algebraic Riccati equations given in (3.1) and (3.2) can be solved efficiently through a linear matrix inequality (LMI) framework.

4. NUMERIAL EXAMPLE

To illustrate the main result, we give a numerical example. Consider the system (2.5) with $h_1 = 1$ second and $h_2 = 2$ seconds:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}; B_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}; A_2 = \begin{bmatrix} 1 & 2 \\ 3 & -3 \end{bmatrix}; \tag{4.1}$$

$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $f_1(x(t)) = \begin{bmatrix} 23.88 \\ 13.88 \end{bmatrix} x^2(t)$ and $f_2(x(t)) = \begin{bmatrix} 6.43 \\ 2 \end{bmatrix} x^2(t)$. The corresponding fuzzy system model is

$$\begin{aligned}
\dot{x}(t) & = (\mu_1 A_1 + \mu_2 A_2)x(t) + \mu_1 B_1 u(t - h_1) \\
& + \mu_2 B_2 u(t - h_2) + \mu_1 f_1(x(t)) + \mu_2 f_2(x(t))
\end{aligned} \tag{4.2}$$

where $x(t) = (x_1(t), x_2(t))$ is the state variables, the membership function $\mu_1(x(t))$ and $\mu_2(x(t))$ are defined as follows:

$$\mu_1(x(t)) = e^{-\sqrt{x_1(t)^2 + x_2(t)^2}} \tag{4.3}$$

$$\mu_2(x(t)) = 1 - \mu_1(x(t)) \tag{4.4}$$

A predictor is given as follows:

$$\begin{aligned}
\bar{x}(t) & = \mu_1(x(t)) \left[e^{A_1 d_1} x(t) + \int_{-h_1}^0 e^{-A_1 \theta} B_1 u(t + \theta) d\theta \right] \\
& + \mu_2(x(t)) \left[e^{A_2 d_2} x(t) + \int_{-h_2}^0 e^{-A_2 \theta} B_2 u(t + \theta) d\theta \right].
\end{aligned} \tag{4.5}$$

Applying Theorem 1, we have

$$P = \begin{bmatrix} 24.3163 & 9.0584 \\ 9.0584 & 3.5756 \end{bmatrix}; Q_{11} = \begin{bmatrix} 27942 & 10644 \\ 10644 & 4056 \end{bmatrix};$$

$$Q_{22} = \begin{bmatrix} 53.7416 & 20.6214 \\ 20.6214 & 9.6399 \end{bmatrix}$$

and

$$Q_{12} = \begin{bmatrix} 109450 & 42140 \\ 42140 & 16229 \end{bmatrix}.$$

With the above P , the stabilising is

$$\begin{aligned}
u(t) & = -(\mu_1(x(t))[118.2411 \ 45.0534] + \\
& \mu_2(x(t))[9.0584 \ 3.5756])\bar{x}(t)(1 + \phi(x(t))).
\end{aligned} \tag{4.6}$$

where

$$\phi(x(t)) = \frac{4x^4}{2\|\bar{x}^T(t) \sum_{i=1}^2 \mu_i P B_i\| x^2 + 0.01e^{-100t}}$$

This fuzzy controller will globally asymptotically stabilise the system (4.2). Typical responses are shown in Figure 1.

5. CONCLUSION

The problem of stabilising a class of nonlinear time delay systems using fuzzy models has been examined in this

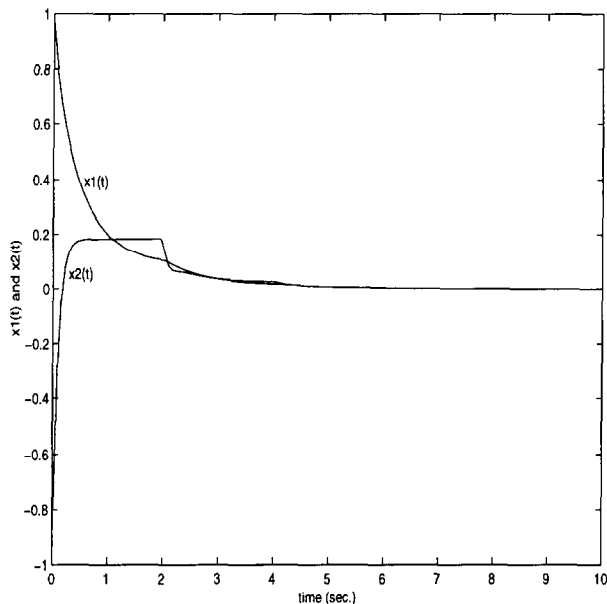


Figure 1. Plots of $x_1(t)$ and $x_2(t)$ versus time with $x_1(\eta) = 1$ and $x_2(\eta) = -1$.

paper. The class of nonlinear time delay systems under investigation is described by the Takagi-Sugeno (T-S) or Takagi-Sugeno-Kang (TSK) fuzzy model. Based on a well-known Riccati approach, a technique for designing a fuzzy state feedback control law which globally quadratically stabilises this class of nonlinear time delay systems has been derived. A numerical simulation example has been used to illustrate the theory development.

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