OPTIMAL CONTROL OF DISCRETE-TIME NONLINEAR STOCHASTIC SYSTEMS WITH GENERAL CRITERIA

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Abstract. A general formulation is presented for finite horizon suboptimal control of a class of discrete-time, nonautonomous, uncertain, nonlinear stochastic systems. Full state information is assumed to be available in controller derivation and optimization is carried out for a variety of performance criteria within a common framework. These performance criteria include H_2, H_∞, and various dissipative control objectives for such stochastic systems. It is shown that the results obtained in this paper include the previous ones in the literature as special cases.

1. Introduction

This work is on the problem of state feedback control of a class of discrete-time uncertain nonlinear stochastic systems. Several criteria ranging from H_2 to stochastic passivity are used in designing these controllers in a unified framework. In the system model used, the uncertainty involves the powers of white noise sequences which may be unknown nonlinear functions of the state and the applied control. Such nonlinear models are introduced in [1], their system theoretic properties are investigated using the LMI approach in [2], mean-square (m.s.) state estimation of such models are considered in [3], and [4] and [5] are respectively on the infinite horizon and moving finite horizon control of these systems. In this work, we generalize the previous H_2 control results to the case where we have a variety of performance criteria including H_2 optimality.

2. Main Results

Let us consider the following discrete-time nonlinear stochastic system:

$$x_{k+1} = A_k x_k + B_k u_k + f_k \tag{1}$$

where the state $x_k \in \mathbf{R}^n$ and the control input $u_k \in \mathbf{R}^m$. The nonlinear function $f_k = f(x_k, v_k)$ where v_k is a zero-mean white noise is defined by its statistical properties as follows:

$$E\{f_k {f_j}^T\} = 0, \quad k \neq j, \qquad E_{x_k}\{f_k\} = 0 \tag{2}$$

$$E_{x_k}\{f_k {f_k}^T\} \leq \sum_{i=1}^{r} {T_{fk}}^i ({x_k}^T {M_{fk}}^i x_k + {u_k}^T {W_{fk}}^i u_k) \tag{3}$$

where the matrices on the right hand side of (3) are all symmetric and positive semidefinite since they signify upper bounds to noise covariances. However, note that the exact form of the nonlinear function does not have to be known. This description is quite general as explained in [1] - [5].

Let us consider the performance output

$$z_k = C_k x_k + D_k u_k + h_k \tag{4}$$

where h_k has properties similar to those of f_k:

$$E\{h_k {h_j}^T\} = 0, \; k \neq j, \; E_{x_k}\{h_k\} = 0, \; E\{f_k {h_k}^T\} = 0 \tag{5}$$

$$E_{x_k}\{h_k {h_k}^T\} \leq \sum_{i=1}^{r} {T_{h_k}}^i ({x_k}^T {M_{h_k}}^i x_k + {u_k}^T {W_{h_k}}^i u_k) \tag{6}$$

Let us consider the optimal control problem of minimizing

$$J_N = E\{\| x_N \|^2 + \sum_{k=0}^{N-1} ({z_k}^T \quad {u_k}^T) \begin{pmatrix} Q_k & S_k \\ {S_k}^T & R_k \end{pmatrix} \begin{pmatrix} z_k \\ u_k \end{pmatrix}\} \tag{7}$$

over $u_k, 0 \leq k \leq N-1$, subject to system (1) - (3) and output (4)-(6). Note that minimization of J_N amounts to maximizing the "dissipation rate" given by

$$-E\{({z_k}^T \quad {u_k}^T) \begin{pmatrix} Q_k & S_k \\ {S_k}^T & R_k \end{pmatrix} \begin{pmatrix} z_k \\ u_k \end{pmatrix}\}$$

of the "system energy" defined by a quadratic storage function V_k.Define

$$\Gamma_k \doteq {B_k}^T P_{k+1} A_k + (Q_k D_k + S_k)^T C_k \tag{8}$$

$$\Phi_k \doteq {A_k}^T P_{k+1} A_k + {C_k}^T Q_k C_k + \sum_{i=1}^{r} \mathrm{tr}[{T_{fk}}^i P_{k+1}] {M_{fk}}^i + \sum_{i=1}^{r} \mathrm{tr}[{T_{hk}}^i Q_k] {M_{fk}}^i \tag{9}$$

$$\Psi_k \doteq {B_k}^T P_{k+1} B_k + {D_k}^T Q_k D_k + {D_k}^T S_k + {S_k}^T D_k + R_k + \sum_{i=1}^{r} \mathrm{tr}[{T_{fk}}^i P_{k+1}] \, {W_{fk}}^i + \sum_{i=1}^{r} \mathrm{tr}[{T_{hk}}^i Q_k] \, {W_{hk}}^i \tag{10}$$

$${u_k}^o = -{\Psi_k}^\dagger \Gamma_k x_k \tag{11}$$

$$P_k = \Phi_k - {\Gamma_k}^T {\Psi_k}^\dagger \Gamma_k \tag{12}$$

with $P_N = I$. The following summarizes the main results of this paper:

Theorem 1. Consider the cost criterion (7), subject to the system constraint (1) - (3) and the performance output (4) - (6). Let $\Psi_k \geq 0$ hold for Ψ_k defined by (10). Assuming that the full state is available for feedback, the control that minimizes the cost criterion is given by (11) where P_k is calculated by (12) and the minimal cost is given by $J_N{}^o = E\{{x_0}^T P_0 x_0\}$.

Let us consider the above result for $Q_k = I$, $S_k = 0$, and $R_k = 0$, with $D^T D > 0$. This yields the previous H_2 problem solution that was obtained in [1].

Another relationship with earlier results can be established by taking $Q_k = 0$, $S_k = 0$, $R_k = 0$ in the time-invariant case as $N \to \infty$. In this case, the closed loop system is m.s. asymptotically stable or

$$\lim_{N\to\infty} E\{\| x_N \|^2\} = 0$$

if and only if $P_k \to 0$. Since this is the time-invariant case, the iteration direction can be reversed by taking $P_0 = I$ and looking to see if $P_k \to 0$ as $k \to \infty$, which exactly produces the semi-algorithm proposed in [4] for testing the state feedback m.s. stabilizability of this class of systems. This is another proof that our present results are generalizations of the past ones.

In the present context, we can in fact generalize the above comment to encompass a variety of "m.s. abilities" of the stochastic system not just m.s. stabilizability. A test procedure similar to that given in [4] can be given e.g. for testing whether we can find a controller that guarantees a bound on the m.s. l_2 to l_2 gain of the system with $Q_k = I$, $S_k = 0$ and $R_k = -\mu I$, $\mu > 0$ by considering the iteration (12) starting with $P_0 = I$ and looking to see if $P_k \to 0$ as $k \to \infty$.

Several dissipativity results are also possible using this formulation. For example, taking $x_0 = 0$, $Q_k = 0$, $S_k = -0.5I$, and $R_k = 0$ will give the stochastic (m.s.) passivity result

$$\sum_{k=0}^{N-1} E\{{z_k}^T w_k\} \geq E\{{x_N}^T Y_N x_N\} \geq 0 \qquad (13)$$

Other dissipativity results are possible. Setting $Q_k = 0$, $S_k = -0.5I$, and $R_k = \mu I$, $\mu > 0$, yields m.s. input strict passivity. If we set $Q_k = \nu I$, $\nu > 0$, $S_k = -0.5I$, and $R_k = 0$, then m.s. output strict passivity results. Also, setting

$Q_k = \nu I$, $\nu > 0$, $S_k = -0.5I$, $R_k = \mu I$, $\mu > 0$, gives strict m.s. passivity both for the input and the output (very strict passivity). Therefore, one can see that this formulation includes some of the most important performance criteria in a common framework.

3. References

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