Output Feedback Realization of State Feedback Systems

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Abstract – This paper presents a general formulation for the realization of a linear state feedback system using output feedback. The proposed formulation can be applied to any linear state feedback design, provided that an observable output signal is available. We will show that scalar output is sufficient to achieve the realization, even in the case of MIMO state feedback systems. In addition, the proposed output feedback formulation achieves the same closed-loop response as a state feedback system, even when a non-relaxed initial state is involved. We present both discrete-time and continuous-time formulations for the proposed method.

1. Introduction

The state space model of a dynamic system is known to provide a complete description of the system structure (Patel and Munro, 1982). A state space control design which incorporates information provided by this model regarding the interactions among system states and between states and the inputs, and which uses state variables as feedback signal, is the design which is able to achieve maximum flexibility in reshaping system dynamics. The best-known example of state space control design is the linear quadratic (LQ) optimization of Kalman (Kalman, 1960) which has been the most widely followed approach to control system synthesis ever since its introduction decades ago.

On the other hand, state feedback control laws are difficult to realize because, in most applications, full state information is not available. In order to allow state feedback control laws to be executed in the absence of such information, the state estimator (Kalman and Bucy, 1961) was developed. However, inclusion of a state estimator in the feedback loop causes the closed-loop system to contain both the regulator poles and the poles of the estimator (Stengel, 1986; Callier and Desoer, 1991), and to thereby depart from a true state feedback design.

Alternative methods of producing a state feedback control law using output feedback have been proposed. For SISO systems, we have shown that an output feedback controller can realize any state feedback design (Chan, 1996a). For MIMO systems, a general output feedback controller to realize the state feedback design has yet to be developed. The one method which we previously proposed (Chan, 1996b) relies on redundant output signals to achieve the design and does not apply to general MIMO formulations. Moreover, there is no proof that any of these designs can achieve the same command response as a state feedback system when the initial condition is not relaxed.

In this paper, a general output feedback controller capable of realizing any state feedback design is presented. We will show that scalar output is sufficient to allow completion of the realization, even when it involves a MIMO system. In addition, the proposed output feedback formulation is able to achieve a closed-loop response which is identical to that of a state feedback system, even when the system begins from a non-zero initial state.

We will present both a discrete-time formulation and a continuous-time formulation for the proposed controller. We begin with the derivation of the discrete-time formulation, and extend the application to a continuous-time formulation in Section 4.

2. Formulation of the problem

2.1 Feedback control system in state space. In this work, we consider discrete-time linear dynamic systems which have the following state space form:

$$\mathbf{y}(k) = C\,\mathbf{x}(k), \quad \mathbf{x}(k+1) = A\mathbf{x}(k) + B\,\mathbf{u}(k) \qquad (1)$$

Note that the $m \times 1$ vector **u**, the $\overline{m} \times 1$ vector **y**, and the $n \times 1$ vector **x** represent, respectively, the input, the output, and the state of the system. In addition, A, B, and C are constant matrices of appropriate dimensions. For this discussion, a state feedback control law is defined as follows:

$$\mathbf{u}(k) = \mathbf{r}(k) - K\,\mathbf{x}(k) \qquad (2)$$

where K is an $m \times n$ gain matrix and **r** is an $m \times 1$ command vector.

2.2 Previous methods for implementation of a state feedback design. In general, direct implementation of a design based on (2) is difficult, due to the absence of full state information. As a result, estimated values for $\mathbf{x}(k)$, obtained from an estimator, is used in place of true $\mathbf{x}(k)$. However, an estimator-based design will not produce the same closed-loop response as the original state feedback design (Appendix A). Other methods to realize a state feedback control law have been proposed (Chan, 1996a; Chan, 1996b). However, these methods can be applied only to restricted classes of state feedback systems, and there is no proof that these methods can achieve the same closed-loop response as a state feedback design when the initial state is not relaxed.

In the following, we present a general method for the realization of state feedback systems with output feedback. The proposed realization preserves the closed-loop response, regardless of initial state.

3. General output feedback realization of state feedback systems

3.1 Z-domain analysis of a state feedback design. Let $\bar{\mathbf{u}}(z)$, $\bar{\mathbf{y}}(z)$, $\bar{\mathbf{x}}(z)$, and $\bar{\mathbf{r}}(z)$ represent the z-transforms of the sequences $\{\mathbf{u}(k)\}$, $\{\mathbf{y}(k)\}$, $\{\mathbf{x}(k)\}$, and $\{\mathbf{r}(k)\}$, respectively. As a result, the state feedback control law becomes

$$\begin{aligned} &\bar{\mathbf{u}}(z) = \bar{\mathbf{r}}(z) - K\,\bar{\mathbf{x}}(z) = \bar{\mathbf{r}}(z) - \mathbf{G}(z)\,\bar{\mathbf{u}}(z), \\ &\mathbf{G}(z) = K(zI - A)^{-1}B \end{aligned} \tag{3}$$

For left-invertible systems, Eq. (3) can be rewritten as $\bar{\mathbf{u}}(z) = \mathbf{H}(z)^{-1}\bar{\mathbf{y}}(z)$ where $\mathbf{H}(z)^{-1}$ is an left-inverse of $\mathbf{H}(z)$ and $\mathbf{H}(z) = C(zI - A)^{-1}B$. In this case, a simple output feedback form of (3) can be obtained, as follows:

$$\bar{\mathbf{u}}(z) = \bar{\mathbf{r}}(z) - \mathbf{M}(z)\bar{\mathbf{y}}(z), \quad \mathbf{M}(z) = \mathbf{G}(z)\mathbf{H}(z)^{-1}. \tag{4}$$

However, even this restricted design will suffer from internal instability when $\mathbf{H}(z)$ is inverse unstable, a condition that is commonly true (Astrom et al., 1984).

On the other hand, the truth of the matter is neither the invertibility of $\mathbf{H}(z)$ nor its zero locations determine whether or not a internally stable output feedback realization of (3) is possible. In fact, one such design for a general $\mathbf{H}(z)$ is always possible, and is presented as follows.

3.2 A general output feedback design. We can achieve an internally stable output feedback realization of (3) with the following design:

$$\begin{aligned} &\bar{\mathbf{u}}(z) = \bar{\mathbf{r}}(z) - \frac{\mathbf{P}(z)}{z^{\ell}}\bar{\mathbf{u}}(z) - \frac{\mathbf{Q}(z)}{z^{\ell}}\hat{\mathbf{y}}(z), \\ &\hat{\mathbf{y}}(z) = T\,\bar{\mathbf{y}}(z), \ T: m_r \times \overline{m}, \ m_r \le \overline{m} \end{aligned} \tag{5}$$

where $\mathbf{P}(z)$ is an $m \times m$ matrix of $(\ell - 1)^{\text{th}}$ order polynomials, $\mathbf{Q}(z)$ is an $m \times m_r$ matrix of ℓ^{th} order polynomials, $\hat{\mathbf{y}}(z)$ is the selected output for feedback, and T is a constant matrix. This design achieves internal stability by avoiding cancellation of unstable factors with the denominators in the equation. In addition, the design goal, $\bar{\mathbf{u}}(z) = \bar{\mathbf{r}}(z) - \mathbf{G}(z)\bar{\mathbf{u}}(z)$, is achieved for an appropriate value of ℓ and if the following equation is satisfied:

$$\begin{aligned} &\frac{\mathbf{P}(z)}{z^{\ell}} + \frac{\mathbf{Q}(z)}{z^{\ell}}\hat{\mathbf{H}}(z) = \mathbf{G}(z), \\ &\hat{\mathbf{H}}(z) = \hat{C}(zI - A)^{-1}B, \quad \hat{C} = TC \end{aligned} \tag{6}$$

We will show that a $\mathbf{P}(z)$ and $\mathbf{Q}(z)$ pair which satisfies (6) exists for all observable systems. Moreover, the solution will be unique if one condition is met. Proof for these statements requires introduction of the following proposition.

Proposition 1: The following equation is true for any constant matrix F of appropriate dimensions:

$$\begin{aligned} &z^{l}F(zI - A)^{-1}B = FA^{l}(zI - A)^{-1}B + \sum_{i=0}^{l-1}\Omega_{l,i}^{F}z^{i}, \\ &\Omega_{l,i}^{F} = FA^{l-1-i}B, \quad \forall l, \end{aligned} \tag{7}$$

Proof: Define $\bar{\mathbf{f}}(z) = F(zI - A)^{-1}B\,\bar{\mathbf{u}}(z)$ and $\bar{\mathbf{f}}_l^{\circ}(z) = z^{l}\bar{\mathbf{f}}(z)$. Also, let $\{\mathbf{f}(k)\}$ and $\{\mathbf{f}_l^{\circ}(k)\}$ denote the time sequences of $\bar{\mathbf{f}}(z)$ and $\bar{\mathbf{f}}_l^{\circ}(z)$, respectively. Then,

$$\begin{aligned} \mathbf{f}_l^{\circ}(k) &= \mathbf{f}(k + l) = F\,\mathbf{x}(k + l) \\ &= FA^{l}\,\mathbf{x}(k) + \sum_{i=0}^{l-1}\Omega_{l,i}^{F}\mathbf{u}(k + i) \end{aligned} \tag{8}$$

Proposition 1 can then be proven by taking the z-transform of (8). Q.E.D.

A second proposition is presented in order to prove that the proposed controller exists whenever $(A, \hat{C})$ forms an observable pair.

Proposition 2: Let $\mathbf{P}(z)$ and $\mathbf{Q}(z)$ be expressed into the following forms:

$$\mathbf{P}(z) = \sum_{i=1}^{\ell}P_i z^{\ell-i}, \quad \mathbf{Q}(z) = \sum_{i=0}^{\ell}Q_i z^{\ell-i} \tag{9}$$

where P_i and Q_i represent constant matrices having dimensions of $m \times m$ and $m \times m_r$ respectively. If $\ell \ge n - 1$ and $(A, \hat{C})$ is observable, then the following set of Q_i and P_j will form the pair, $\mathbf{P}(z)$ and $\mathbf{Q}(z)$, needed to satisfy (6):

$$\begin{aligned} &[Q_{\ell} \ \cdots \ Q_0] = KA^{\ell}\begin{bmatrix} \hat{C} \\ \vdots \\ \hat{C}A^{\ell} \end{bmatrix}^{-R}, \quad P_{\ell-j} = \Omega_{\ell,j}^{K} - \Lambda_j, \\ &\Lambda_j = \sum_{i=0}^{\ell-j-1}Q_i\Omega_{\ell-i,j}^{\hat{C}}, \quad \Omega_{l,i}^{K} = KA^{l-1-i}B. \end{aligned} \tag{10}$$

Note that $[\ ]^{-R}$ denotes the right inverse of the $n \times (m_r(\ell + 1))$ matrix.

Proof: The following expression of (6) can be inferred from (9):

$$\begin{aligned} &\sum_{i=1}^{l}P_i z^{\ell-i} + \sum_{i=0}^{l}Q_i z^{\ell-i}\hat{C}(zI - A)^{-1}B \\ &= z^{\ell}K(zI - A)^{-1}B \end{aligned} \tag{11}$$

By application of Proposition 1, the above equation is transformed as follows:

$$\begin{aligned} &L M(zI-A)^{-1}B+\sum_{j=0}^{l-1}(P_{\ell-j}+\Lambda_j)z^j \\ &=K A^{\ell}(zI-A)^{-1}B+\sum_{j=0}^{l-1}\Omega_{\ell,j}^{K}z^j \end{aligned} \tag{12}$$

where

$$L=[Q_\ell \quad \cdots \quad Q_0], \quad M=\begin{bmatrix}\hat{C}\\ \vdots \\ \hat{C}A^{\ell}\end{bmatrix}, \quad \Omega_{l,i}^{\hat{C}}=\hat{C}A^{l-1-i}B$$

Then, since M is the observability matrix of the $(A,\hat{C})$ doublet, a right inverse of M will exist, and thus a set of Q_i and P_j which satisfies (12) can be formulated from (10), provided that the $(A,\hat{C})$ doublet is observable and $\ell \geq n-1$. Q.E.D.

It is noted that the right inverse of M becomes M^{-1}, and a unique solution of (10) results, when $m_r=1$ and $\ell=n-1$. Uniqueness of the solution will simplify the computation of (10).

Remark 1: Although **Proposition** 2 was proven for the case of $\ell \geq n-1$, it is possible that KA^{ℓ} may fall inside the row space of M, and that a solution may also exist for $\ell < n-1$. Consequently, a solution is ensured for $\ell \leq n-1$. Also, an increase in m_r will expand the row space of M at small values of ℓ, thereby enhancing the chance of a low order solution. However, having $m_r>1$ may also cause non-unique solutions to appear, thereby adding complexity to the computation of the controller (Appendix B).

3.3 Internal stability of the proposed design. We now examine the internal stability of the proposed realization by comparing its closed-loop poles with those of the state feedback system. Since the output and the control input of a linear feedback system share the same closed-loop poles, we can check stability by analyzing the closed-loop dynamics of the control input.

In a state feedback design, the closed-loop dynamics of the control input is given by $\bar{\mathbf{u}}(z)=[\mathbf{I}+\mathbf{G}(z)]^{-1}\mathbf{r}(z)$. In the proposed design, the closed-loop dynamics of the control input can be obtained from (5) and (6), as follows:

$$\begin{aligned} \bar{\mathbf{u}}(z)&=[z^{\ell}\mathbf{I}+\mathbf{P}(z)+\mathbf{Q}(z)\hat{\mathbf{H}}(z)]^{-1}z^{\ell}\mathbf{r}(z) \\ &=[z^{\ell}\mathbf{I}+z^{\ell}\mathbf{G}(z)]^{-1}z^{\ell}\mathbf{r}(z)=[\mathbf{I}+\mathbf{G}(z)]^{-1}\mathbf{r}(z) \end{aligned} \tag{13}$$

It is seen that an ℓ^{th}-order output feedback design introduces ℓ extra poles at $z=0$ into the closed-loop system; these extra poles are then canceled by the z^{ℓ} factor in the numerator. Consequently, an internally stable design is ensured.

3.4 Closed-loop response of the proposed design. A second important criterion for the success of the proposed design is whether or not its closed-loop system produces the same response to an arbitrary initial state as a state feedback controller produces. For the state feedback system with an arbitrary initial state $\mathbf{x}(0)=\mathbf{x}_0$, the closed-loop command response of the control input is

$$\begin{aligned} \mathbf{u}(z)=[I+K(zI-A)^{-1}B]^{-1} \\ [\mathbf{r}(z)-K(zI-A)^{-1}\mathbf{x}_0] \end{aligned} \tag{14}$$

For the proposed output feedback design, the closed-loop response of the control input will be

$$\begin{aligned} \mathbf{u}(z)=[I+K(zI-A)^{-1}B]^{-1} \\ [\mathbf{r}(z)-(\mathbf{Q}(z)/z^{\ell})\hat{C}(zI-A)^{-1}\mathbf{x}_0] \end{aligned} \tag{15}$$

We will show that (15) implies (14). The derivation will require introduction of the following proposition.

Proposition 3: The following equation is true for any constant matrix F:

$$z^{l}F(zI-A)^{-1}\mathbf{x}_0=FA^{l}(zI-A)^{-1}\mathbf{x}_0, \quad \forall l. \tag{16}$$

Proof: Define $\bar{\mathbf{q}}(z)=F(zI-A)^{-1}\mathbf{x}_0$ and $\bar{\mathbf{q}}_l^{\circ}(z)=z^{l}\bar{\mathbf{f}}(z)$. Also, let $\{\mathbf{q}(k)\}$ and $\{\mathbf{q}_l^{\circ}(k)\}$ denote the time sequences of $\bar{\mathbf{q}}(z)$ and $\bar{\mathbf{q}}_l^{\circ}(z)$, respectively. Then,

$$\mathbf{q}(k)=FA^{k}\mathbf{x}_0$$

and

$$\mathbf{q}_l^{\circ}(k)=\mathbf{q}(k+l)=FA^{k+l}\mathbf{x}_0=(FA^{l})A^{k}\mathbf{x}_0; \tag{17}$$

therefore, the proposition is proven by taking the z-transform of (17). Q.E.D.

Because of Proposition 3, we obtain

$$\begin{aligned} \mathbf{Q}(z)\hat{C}(zI-A)^{-1}\mathbf{x}_0&=\sum_{i=0}^{\ell}Q_i z^{\ell-i}\hat{C}(zI-A)^{-1}\mathbf{x}_0 \\ &=\sum_{i=0}^{\ell}Q_i\hat{C}A^{\ell-i}(zI-A)^{-1}\mathbf{x}_0 \end{aligned} \tag{18}$$

From (12), we also infer that $\sum_{i=0}^{\ell}Q_i\hat{C}A^{\ell-i}=KA^{\ell}$. As a result, the following equation is obtained by referring to Proposition 3 once more:

$$\begin{aligned} &(\mathbf{Q}(z)/z^{\ell})\hat{C}(zI-A)^{-1}\mathbf{x}_0 \\ &=z^{-\ell}KA^{\ell}(zI-A)^{-1}\mathbf{x}_0 \\ &=z^{-\ell}Kz^{\ell}(zI-A)^{-1}\mathbf{x}_0=K(zI-A)^{-1}\mathbf{x}_0 \end{aligned} \tag{19}$$

Hence, (15) implies (14). In arriving at (19), we have assumed that data of $\hat{\mathbf{y}}(k),\cdots,\hat{\mathbf{y}}(k-\ell)$ and $\mathbf{u}(k)$, $\cdots,\mathbf{u}(k-\ell)$ are available at all k, including $k=0$.

3.6 A sample design. The proposed method was tested on a future project aircraft (McLean, 1990), with modified lateral dynamics given as follows:

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.163 & 0 & -1 \\ 16.6 & -1.08 & -0.13 \\ 15.7 & -0.02 & -0.25 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -0.0027 & 0.05 \\ 21.15 & 6.88 \\ 0.54 & -11.7 \end{bmatrix} \mathbf{u},$$

$$\mathbf{y} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}.$$

With a sampling time of 0.1 second, the discrete time plant dynamics becomes

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.9079 & 0.0001 & -0.0954 \\ 1.5096 & 0.8977 & -0.0898 \\ 1.4965 & -0.0018 & 0.8996 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -0.0028 & -0.0618 \\ 2.0029 & 0.6942 \\ 0.0498 & -1.1225 \end{bmatrix} \mathbf{u}(k)$$

A discrete time LQ design produced the following state feedback gain:

$$K = \begin{bmatrix} 0.8630 & 0.3550 & 0.1171 \\ -0.3483 & 0.0513 & -0.5384 \end{bmatrix}.$$

For this state feedback design, three different choices of T were used for the computation of $\mathbf{P}(z)$ and $\mathbf{Q}(z)$, with the following results:

(a) $T = [1 \quad 0]$ - A solution was obtained at $\ell = 2$ as follows:

$$\mathbf{Q}(z) = \begin{bmatrix} 0.467z^2 + 0.406z\text{-}0.455 \\ 2.692z^2\text{-}5.653z + 2.948 \end{bmatrix},$$

$$\mathbf{P}(z) = \begin{bmatrix} -0.2216z\text{-}1.0168 & -0.156z\text{-}0.3406 \\ -5.314z + 6.581 & -1.250z + 2.204 \end{bmatrix}.$$

(b) $T = [0 \quad 1]$ - A 2nd-order solution was also obtained:

$$\mathbf{Q}(z) = \begin{bmatrix} 1851z^2\text{-}3341z + 1772 \\ 263.4z^2\text{-}476.7z + 252.9 \end{bmatrix},$$

$$\mathbf{P}(z) = \begin{bmatrix} -91.38z + 98.59 & 2078z - 2050 \\ -13.03z + 14.07 & 296.3z - 292.6 \end{bmatrix}.$$

(c) $T = I$ - A solution was obtained at $\ell = 1$ as follows:

$$\mathbf{Q}(z) = \begin{bmatrix} 0.0731z + 0.2546 & 0.9250z\text{-}0.8344 \\ -0.0133z + 0.0577 & -0.6845z + 0.1589 \end{bmatrix},$$

$$\mathbf{P}(z) = \begin{bmatrix} 0.5219 & 1.1558 \\ 0.1377 & -0.1407 \end{bmatrix}.$$

Closed-loop simulations under non-relaxed initial conditions also confirm that each of the proceeding output feedback solutions achieves the same closed-loop response as the state feedback system. The high gain design with $T = [0 \quad 1]$ reflects the fact that the second-output of the system is barely observable. The solution with $T = I$ is not unique, because a value of $m_r = 2$ was used.

4. Extension to a continuous-time formulation

4.1 Nomenclature. We will denote constants and variables in the continuous-time formulation by the same symbols as their counterparts in the discrete-time. The two formulations are distinguished only by the arguments used: real time t and the Laplace operator s for continuous-time versus k and z, respectively, for discrete-time.

4.2 Continuous-time state feedback system. A continuous-time state space plant equation can be expressed as

$$\mathbf{y}(t) = C\mathbf{x}(t) \quad \text{and} \quad \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t). \quad (20)$$

The corresponding state feedback control law is

$$\mathbf{u}(t) = \mathbf{r}(t) - K\mathbf{x}(t).$$

4.3 Realization of the state feedback design. Basically, the arguments made in Section 3.1 can be applied to the continuous-time. However, the implementation of $\mathbf{M}(s) = \mathbf{G}(s)\mathbf{H}(s)^{-1}$ is even less likely to be possible than it is in the case of discrete-time, for the following reason. Since K may be any matrix, we can expect that $\mathbf{G}(s) = K(sI - A)^{-1}B$ will have a relative order of one. As a result, $\mathbf{M}(s)$ will be proper, and thus implementable, only if $\mathbf{H}(s) = C(sI - A)^{-1}B$ also has a relative order of one, a condition that is seldom true (Astrom et al., 1984).

Nevertheless, the output/input feedback controller proposed in Section 3.2 does have a continuous-time counterpart, which is defined as follows:

$$\bar{\mathbf{u}}(s) = \bar{\mathbf{r}}(s) - \frac{\mathbf{P}(s)}{\mathbf{g}(s)}\bar{\mathbf{u}}(s) - \frac{\mathbf{Q}(s)}{\mathbf{g}(s)}\hat{\mathbf{y}}(s), \qquad (21)$$
$$\hat{\mathbf{y}}(s) = T\,\bar{\mathbf{y}}(s), \quad \mathbf{g}(s) = \sum_{i=0}^{\ell} g_i s^{\ell-i}$$

Note that $\mathbf{g}(s)$ is a free polynomial. However, $\mathbf{g}(s) = s^{\ell}$ is not allowed: because that $\mathbf{g}(s)$ will enter the closed-loop system in hidden modes, it must be Hurwitz.

In order to achieve the design goal, we need a $\mathbf{P}(s)$ and $\mathbf{Q}(s)$ pair which satisfies the following equations:

$$\mathbf{P}(s)\hat{\mathbf{H}}(s) + \mathbf{Q}(s) = \mathbf{g}(s)\mathbf{G}(s), \qquad (22)$$
$$\hat{\mathbf{H}}(s) = \hat{C}(sI - A)^{-1}B$$

The existence of such a pair can be proven by formulating continuous-time versions of Propositions 1 and 2, a procedure which involves four steps:

(1) Replace the operator z with the operator s

(2) Replace $\mathbf{f}(k)$ and $\mathbf{f}_l^\circ(k)$ in Proposition 1 with $\mathbf{f}(t)$ and $\mathbf{f}_l^\circ(t)$, respectively. Then, replace (8) with the following equation:

$$\begin{aligned} \mathbf{f}_l^\circ(t) &= \frac{d^l\mathbf{f}(t)}{dt^l} = F\frac{d^l\mathbf{x}(t)}{dt^l} \\ &= FA^l\,\mathbf{x}(t) + \sum_{i=0}^{l-1}\Omega_{l,i}^F\frac{d^i\mathbf{u}(t)}{dt^i} \end{aligned}, \tag{23}$$

(3) Replace (12) with the following equation:

$$\begin{aligned} &LM(sI-A)^{-1}B + \sum_{j=0}^{l-1}(P_{\ell-j}+\Lambda_j)s^j \\ &= \Psi(sI-A)^{-1}B + \sum_{j=0}^{l-1}\Phi_j s^j \end{aligned} \tag{24}$$

where

$$\Psi = K\sum_{i=0}^{\ell}A^{\ell-i}g_i \quad \text{and} \quad \Phi_j = \sum_{i=0}^{\ell-j-1}\Omega_{\ell-i,j}^K g_i .$$

Note that (24) contains the parameterized version of the output feedback design.

(4) Replace the solution formulas for P_i and Q_j, given in (10), as follows:

$$\begin{aligned} &[Q_\ell \;\cdots\; Q_0] = \Psi M^{-R}, \\ &P_{\ell-j} = \Phi_j - \Lambda_j,\; j = 0,\cdots,\ell-1 \end{aligned} \tag{25}$$

where M^{-R} is a right-inverse of M. This continuous-time solution exists whenever $(A,\hat{C})$ forms an observable pair and $\ell \geq n-1$.

Note that a unique solution of P_i and Q_j will also result, when M^{-R} becomes M^{-1} at $m_r = 1$ and $\ell = n-1$. In addition, the comments of ***Remark 1*** apply to the continuous-time solution, as well as to the discrete-time solution.

4.4 Closed-loop analysis of the continuous-time design. In a continuous-time output feedback realization, the following equation for the closed-loop system dynamics is inferred from (21) and (22):

$$\begin{aligned} \bar{\mathbf{u}}(s) &= [\mathbf{g}(s)\mathbf{I} + \mathbf{P}(s) + \mathbf{Q}(s)\hat{\mathbf{H}}(s)]^{-1}\mathbf{g}(s)\mathbf{r}(s) \\ &= [\mathbf{g}(s)\mathbf{I} + \mathbf{g}(s)\mathbf{G}(s)]^{-1}\mathbf{g}(s)\mathbf{r}(s) \\ &= [\mathbf{I} + \mathbf{G}(s)]^{-1}\mathbf{r}(s) \end{aligned} \tag{26}$$

It is seen that the continuous-time realization introduces an extra factor, $\mathbf{g}(s)$, into the denominator of the closed-loop system, which is canceled by another $\mathbf{g}(s)$ factor in the numerator. The resulting closed-loop system is therefore internally stable and contains the same poles as the state feedback design.

We can also follow steps similar to those presented in Section 3.4 to analyze the closed-loop command response of the continuous-time design. First, the closed-loop control input response of a non-relaxed continuous-time state feedback system is

$$\begin{aligned} \mathbf{u}(s) &= [I + K(sI-A)^{-1}B]^{-1} \\ &\quad [\mathbf{r}(s) - K(sI-A)^{-1}\mathbf{x}_0] \end{aligned} \tag{27}$$

For the continuous-time output feedback design, the closed-loop response of the control input can be written as follows:

$$\begin{aligned} \mathbf{u}(s) &= [I + K(sI-A)^{-1}B]^{-1} \\ &\quad [\mathbf{r}(s) - (\mathbf{Q}(s)/\mathbf{g}(s))\hat{C}(sI-A)^{-1}\mathbf{x}_0] \end{aligned} \tag{28}$$

We can show that (28) implies (27). For this derivation, a continuous-time version of Proposition 3 is introduced without proof as follows.

Proposition 4: The following equation is true for any constant matrix F:

$$s^lF(sI-A)^{-1}\mathbf{x}_0 = FA^l(sI-A)^{-1}\mathbf{x}_0, \quad \forall l . \tag{29}$$

Because of **Proposition** 4, the following equation is obtained:

$$\begin{aligned} \mathbf{Q}(s)\hat{C}(sI-A)^{-1}\mathbf{x}_0 &= \sum_{i=0}^{\ell}Q_is^{\ell-i}\hat{C}(sI-A)^{-1}\mathbf{x}_0 \\ &= \sum_{i=0}^{\ell}Q_i\hat{C}A^{\ell-i}(sI-A)^{-1}\mathbf{x}_0 \end{aligned} \tag{30}$$

From (25), we also infer that

$$\sum\nolimits_{i=0}^{\ell}Q_i\hat{C}A^{\ell-i} = \Psi = K\sum\nolimits_{i=0}^{\ell}A^{\ell-i}g_i .$$

Then, by applying Proposition 4 a second time, the following equation is obtained:

$$\begin{aligned} &\mathbf{Q}(s)\hat{C}(sI-A)^{-1}\mathbf{x}_0 = \sum_{i=0}^{\ell}Q_i\hat{C}A^{\ell-i}(sI-A)^{-1}\mathbf{x}_0 \\ &= K\sum_{i=0}^{\ell}g_iA^{\ell-i}(sI-A)^{-1}\mathbf{x}_0 \\ &= K\sum_{i=0}^{\ell}g_is^{\ell-i}(sI-A)^{-1}\mathbf{x}_0 = \mathbf{g}(s)K(sI-A)^{-1}\mathbf{x}_0 . \end{aligned} \tag{31}$$

Hence, (28) implies (27). Note that this result is obtained by assuming that the data of $\hat{\mathbf{y}}(t)$ and $\mathbf{u}(t)$ are available at all t, including $t = 0^-$.

4.5 A sample design. A continuous-time LQ regulator was computed for the same plant used in Section 3.5. The resulting state feedback gain is as follows:

$$K = \begin{bmatrix} 0.7608 & 0.9241 & 0.1801 \\ 0.3252 & 0.2274 & -0.9603 \end{bmatrix}.$$

Two second-order output feedback solutions of this state feedback design were computed for two different values of T and $g(s)=(s+3)^2$, with the following results:

(a) $T=[1 \quad 0]$:

$$\mathbf{Q}(s)=\begin{bmatrix} 1.264s^2+6.606s+9.064 \\ -0.093s^2-4.278s-3.577 \end{bmatrix},$$

$$\mathbf{P}(s)=\begin{bmatrix} -7.086s-14.59 & -4.405s-8.448 \\ 6.255s+109.27 & 13.46s+105.08 \end{bmatrix}.$$

(b) $T=[0 \quad 1]$:

$$\mathbf{Q}(s)=10^3\times\begin{bmatrix} 0.1959s^2+0.0902s+3.0644 \\ 0.0473s^2+0.0161s+0.7454 \end{bmatrix},$$

$$\mathbf{P}(s)=10^2\times\begin{bmatrix} -0.8616+1.65 & 22.9680s+3.863 \\ -0.2124s+0.406 & 5.6603s+0.9602 \end{bmatrix}$$

Note that the poor observability of the second output resulted in a high gain solution with $T=[0 \quad 1]$. In addition, closed-loop simulations under various non-relaxed initial states confirm that the proceeding output feedback solutions faithfully reproduce the same closed-loop response of the state feedback system.

5. Concluding remarks

We have shown that state feedback control laws for any number of input and output can be realized by feeding back dynamically compensated output signal. Specifically, a state feedback control law can be realized with feedback consisting of scalar output, even when a MIMO solution is considered. Moreover, the proposed output feedback realization achieves the same closed-loop response as the state feedback control law regardless of initial state. The proposed realization is applicable to any state feedback system in which at least one measurable output is observable.

References

[1] Patel, R.V., and Munro, N. (1982) *Multivariable System Theory and Design* Pergamon Press, Oxford.

[2] Kalman, R.E. (1960) *Contributions to the theory of optimal control* Bol. Soc. Mathematica Mexicana, pp.102-119.

[3] Kalman, R.E. and Bucy, R.S. (1961) *New Results in Linear Filtering and Prediction Theory*, Journal of Basic Engineering, March, pp.95-108.

[4] Stengel, R.F. (1986) *Stochastic Optimal Control - Theory and Application*, John Wiley & Sons, New York.

[5] Callier, F.M and Desoer, C.A. (1991) *Linear System Theory*, Springer-Verlag, New York.

[6] Chan, J.T. (1996a) *Optimal output feedback regulator - A numerical synthesis approach for input-output data*, ASME/JDSMC, **118**, pp.360-366.

[7] Chan, J.T. (1996b) *Data-Based Synthesis of Multivariable LQ Regulator*, Automatica **32-3**, pp.403-407.

[8] Astrom, K.A., P. Hagander and J. Sternby (1984) *Zeros of sampled systems*, Automatica **20-1**, pp.31-38.

[9] McLean, D. (1990) *Automatic Flight Control Systems*, Prentice Hall International, New York.

Appendix A: Closed-loop response of an estimator-based design

For an estimator-based design (Kalman and Bucy, 1961), we have

$$\begin{aligned} \mathbf{u}(k) &= \mathbf{r}(k)-K\hat{\mathbf{x}}(k), \\ \hat{\mathbf{x}}(k+1) &= A\hat{\mathbf{x}}(k)+B\mathbf{u}(k)+D[\mathbf{y}(k)-C\hat{\mathbf{x}}(k)] \end{aligned} \quad \text{(A1)}$$

where D is a constant estimator gain of appropriate dimension. With $\hat{\mathbf{x}}(0)=\hat{\mathbf{x}}_0$, let $\tilde{\mathbf{x}}_0=\mathbf{x}_0-\hat{\mathbf{x}}_0$; we thus obtain

$$\hat{\mathbf{x}}(z)=[zI-(A-DC)]^{-1}[B\,\mathbf{u}(z)+\hat{\mathbf{x}}_0+D\,\mathbf{y}(z)], \quad \text{(A2)}$$

and

$$\begin{aligned} \mathbf{u}(z) &= \mathbf{r}(z)-K[zI-(A-DC)]^{-1}[B\,\mathbf{u}(z)+\hat{\mathbf{x}}_0+D\,\mathbf{y}(z)] \\ &= \mathbf{r}(z)-K[zI-(A-DC)]^{-1} \\ &\qquad [B\,\mathbf{u}(z)+\hat{\mathbf{x}}_0+DC(zI-A)^{-1}(B\mathbf{u}(z)+\mathbf{x}_0)] \\ &= \mathbf{r}(z)-K(zI-A)^{-1}\{B\,\mathbf{u}(z)+\mathbf{x}_0\} \\ &\qquad -K[zI-(A-DC)]^{-1}DC(zI-A)^{-1}\tilde{\mathbf{x}}_0. \end{aligned}$$

Consequently, the closed-loop response of the control input becomes

$$\begin{aligned} \mathbf{u}(z) &= [I+K(zI-A)^{-1}B]^{-1}\{\mathbf{r}(z)-K(zI-A)^{-1}\mathbf{x}_0\} \\ &\quad -[I+K(zI-A)^{-1}B]^{-1} \\ &\qquad K[zI-(A-DC)]^{-1}DC(zI-A)^{-1}\tilde{\mathbf{x}}_0. \end{aligned}$$

Because the first term on the right hand side of the above equation is the closed-loop control input response of a true state feedback control law, an estimator-based design can never achieve true state feedback control, except when $\tilde{\mathbf{x}}_0=0$. However, $\tilde{\mathbf{x}}_0=0$ implies that $\mathbf{x}(k)$ is measurable, and that the estimator is an unnecessary construction.

Appendix B: Controller computation with $m_r \geq 2$

With $m_r \geq 2$, a solution for an output feedback realization can still be obtained by removing the linearly dependent row(s) of *M*, thereby nullifying the corresponding coefficient of $\mathbf{Q}(z)$. In this case, depending on which row(s) of *M* are deleted, different solutions will result from (10), or from (25) in the continuous-time formulation. A final design can be obtained from the linear combination of these solutions. The way these results are combined will depend on the nature of the design task.