Analytical Time Optimal Control Solution for Free Flying Objects with Drift Terms

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Abstract

In order to control gymnastic and jumping robots, we will derive the complete analytical solution to the posture control problem of a two-link free flying object with initial angular momentum. We will show that the solution involves singular control and derive formulae to calculate the optimal switching condition, optimal terminal time and optimal trajectories. As an application, a high diving motion is simulated.

1 Introduction

Motions of animals and gymnasts in the air as well as free flying space robots without thruster are subject to nonholonomic constraints generated by the conservation law of angular momentum. In particular, non-zero angular momentum is deliberately added before jumping for the somersault performance of gymnasts and running motion of some animals to rotate their bodies.

The purpose of this paper is to derive analytical posture control laws for free flying objects like above having non-zero angular momentum. We plan to use them to build a parallel bar gymnastic robot and running robot in the next stage of our research.

Such problems are little bit different from others [1] since the motion of the free flying objects having non-zero angular momentum is described by a nonholonomic system with a drift term having no equilibrium state. Kamon et. al [2] derived the minimum energy trajectory by numerical optimization method to simulate a 3D somersault motion. Godhavn et al. [3] proposed a potential numerical computation algorithm to show a somersault motion of a planar diver. Berkemeier et. al. [4] dealt with a two link hopping robot. However, since these are numerical solutions, we cannot obtain the closed form control formulae which are needed for experiments. In addition, we cannot know the nature of the optimal solutions, e.g., we cannot tell when a particular solution becomes singular.

By the way, the fundamental degree of freedom of the posture control problem for free flying objects is three when the initial angular momentum is zero [5], while is two when the initial angular momentum is not zero, provided some of their freedoms are fixed.

In this paper, we will deal with a two link robot with non-zero initial angular momentum and show an analytical solution of the time optimal control problem. The obtained solutions include simple closed form formulae of the control law. They also show that the problem leads to a singular optimal control problem depending upon the initial posture; the switching time is once when the singular solution does not occur while is twice when the singular solution is used. As an application, the somersault motion of a diver approximated by the two link system is simulated.

2 Canonical form and accessible region

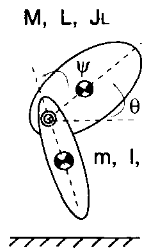

Fig.1 free flying robot

We'll deal with the posture control problem of a planar free flying robot as shown in Fig.1, where the robot is composed of the body and leg; θ is the absolute angle of the body measured counter-clock-wise relative to the frame of inertia; ψ is the relative angle between the body and leg measured counter-clock-wise; L and l are the distances between the joint and the centers of mass (CM) of the body and the leg, respectively; M and J_L are the weight of the body and the moment of inertia of the body around its CM; m and J_l are those for the leg.

Suppose that the robot has a nonzero constant angular momentum $P_0 \neq 0$ which is provided from the ground before takeoff as an initial angular momentum. Then the conservation law of angular momentum around CM of the whole robot leads to

$$P_0 = (M_1 + A_1\cos\psi)\dot{\theta} + (M_2 + A_2\cos\psi)\dot{\psi} \qquad (1)$$

where

$$M_1 = J_L + J_l + \frac{mM(L^2+l^2)}{m+M}, \quad A_1 = 2A_2$$
$$M_2 = J_l + \frac{mMl^2}{m+M}, \quad A_2 = \frac{mMlL}{m+M} \qquad (2)$$

which cannot be integrated when $P \neq 0$, $M_1 > A_1$ and $A_2 \neq 0$.

Defining $\dot{\psi}$ as a control, (1) can be described by

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{P_0}{M_1 + A_1\cos\psi} \end{bmatrix} + \begin{bmatrix} 1 \\ -\dfrac{M_2 + A_2\cos\psi}{M_1 + A_1\cos\psi} \end{bmatrix} u \qquad (3)$$

which is a nonlinear system with a drift term and has no equilibrium. Obviously, this system is locally accessible.

Since the rotational motion of the robot can not be stopped when $P_0 \neq 0$, the control problem is to make q passing through a given reference state $q_r = (\psi_r, \theta_r)^T$ at a given time T. Only ψ can be settled in ψ_r by putting $u = 0$ after arrival at the reference state.

Since the translated motion cannot be controlled at all, we will not mention about it. Animals and gymnasts control their posture within the falling time.

In order to analyze the properties of the system, we will first introduce the following transformation which makes the second entry of the input vector zero:

$$\begin{aligned} x_2 &= \theta + w(\psi) - x_{2r} \ : x_{2r} = \theta_r + w(\psi_r) \\ x_1 &= \psi - \psi_r \end{aligned} \tag{4}$$

where ψ_r and θ_r are the reference states and

$$\begin{aligned} w(\psi) &= \int_0^{\psi} \frac{M_2 + A_2 \cos p}{M_1 + A_1 \cos p} dp = \frac{A_2}{A_1}\psi \\ &+ \frac{2(M_2 A_1 - M_1 A_2)}{A_1\sqrt{M_1^2 - A_1^2}} tan^{-1}\left(\sqrt{\frac{M_1 - A_1}{M_1 + A_1}}\tan\frac{\psi}{2}\right) \end{aligned} \tag{5}$$

Then a simple calculation yields a canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{P_0}{M_1 + A_1 \cos(x_1 + \psi_r)} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \tag{6}$$

and the control problem is transformed to drive $x(0)$ to the origin 0 at the time T.

We assume that $P_0 > 0$ hereafter. Then since $\dot{x}_2 > 0$ in (6), we may have

$$x_2(0) < 0 \tag{7}$$

as the accessible region to $x(T) = 0$. Actually, this will be shown to be sufficient later. However, in practice, we cannot distinguish x_2 from $x_2 \pm 2k\pi$. When considering in this sense, we can remove the restriction (7) at expense of spending longer access time.

3 Time optimal control problem

3.1 Problem formulation and solutions

We'll solve the time optimal control problem for (6). First of all, we define

$$p(x_1) := \frac{P_0}{M_1 + A_1 \cos(x_1 + \psi_r)} \tag{8}$$

in (6). Then, the control problem is to minimize

$$J = \int_0^T dt = T \tag{9}$$

while bringing $x(0)$ to $x(T) = 0$ under the constraint $|u| \le u_m$.

The Hamiltonian of this problem is

$$H = 1 + \lambda_1 u + \lambda_2 p(x_1) \tag{10}$$

and the principle of optimality with little algebra yield the following necessary conditions in the regular optimal control case:

$$\dot{x}_1 = -\mathrm{sgn}(\lambda_1) u_m \tag{11}$$

$$\dot{x}_2 = p(x_1) \tag{12}$$

$$\dot{\lambda}_1 = \alpha \frac{\partial p(x_1)}{\partial x_1} \tag{13}$$

$$H = 1 + \lambda_1 u - \alpha p(x_1) = 0 \quad (\forall t) \tag{14}$$

This problem can be solved analytically without seeking $\lambda_1(0)$ and α explicitly if we pay attention to the two invariant manifolds made by $u = \pm u_m$.

Examine the case $u = u_m$ ($\lambda_1 < 0$) first. In this case, it follows from $\dot{x}_1 = u_m$ that x_1 increases monotonously. Besides, using $\dot{x}_2/\dot{x}_1 = dx_2/dx_1 = p(x_1)/u_m$, x_2 satisfies the following manifold

$$x_2 = g(x_1)/u_m + C_1, \tag{15}$$

where C_1 is the integral constant and

$$g(x_1) = \int_0^{x_1} p(y) dy \tag{16}$$

which is a mono-valued function passing through the origin.

Similarly, when $u = -u_m(\lambda_1 > 0)$, we have another manifold

$$x_2 = -g(x_1)/u_m + C_2 \tag{17}$$

The integral constants C_1 and C_2 will be adjusted for (15) and (17) to pass through designated states.

As shown in Fig.2, (15) and (17) draw a bunch of curves rising in the right hand side and left hand side directions, respectively, when the integral constants C_1 and C_2 vary. We'll refer to the two manifolds, denoted by I and II in Fig.2, passing through the origin as the *switching line* hereafter.

From Fig.2, we can see that we can find a controlled trajectory only when $x_2(0)$ exists in the region under the switching line I and II (the region is marked by slashes). When $x_2(0)$ is in this region, the optimal control strategy may be as follows. As is in Case A, when $x_1(0) > 0$, we first choose $u = u_m$ to ride the state on the manifold rising in the right hand side direction, then switch the control to $u = -u_m$ when the state reaches the switching line I. Similarly, as is in Case B, when $x_1(0) < 0$, we first choose $u = -u_m$ to ride the state on the manifold rising in the left hand side direction, followed by $u = u_m$ when the state reaches the switching line II. When $x_1 = 0$, we can choose either controls in Case A or Case B, $x_1 = 0$ gives the boundary between Case A and Case B.

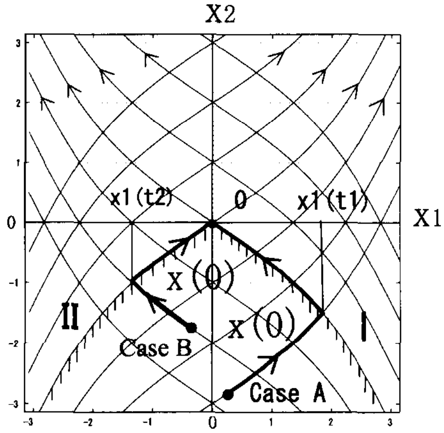

Fig.2 Switching line I and II and optimal trajectories

actually, we can prove the following [7]

1. When $\psi_r = 0$ and $-\pi < x_1(t) < \pi$, the mentioned control strategy is optimal.

2. When $\psi_r = 0$ and $x_1(t) = \pm\pi$ happens, the control problem becomes singular where λ_1 becomes identically zero. In this case, as is shown in Fig.3, the optimal control is given by $u = 0$ before riding the state on the switching lines I or II. Therefore, the control strategy explained using Fig. 2 is optimal only when $x(0)$ is located inside the area enclosed by the bold face curves depicted in Fig.3. We call this area a *basic region.*

3. When $\psi_r \neq 0$, the boundary of Case A and Case B is given by $x_1 = -2\psi_r$ as shown in Fig.4.

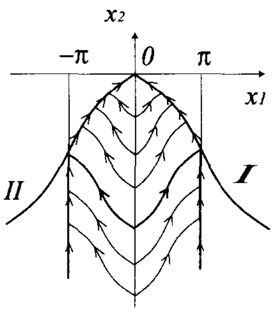

Fig.3 Optimal trajectories when $\psi_r = 0$

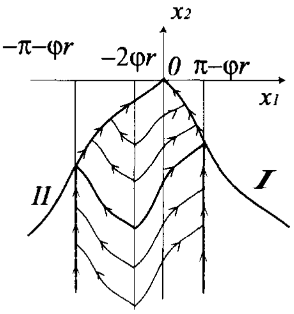

Fig.4 Optimal trajectories when $\psi_r \neq 0$

Let's examine the role of u_m. When u_m is chosen bigger, it follows from (15) and (17) that the lines I and II becomes flat and close to x_1 line. Therefore, the region of $x_2(0)$ which can be brought to the origin approaches $x_2 < 0$, which gives the sufficiency for (7).

3.2 Optimal switching conditions

In [7], we have derived the switching condition as well as the optimal switching time using Fig. 2 for $x(0)$ in the basic region, by paying attention to the fact that the change of x_1 divided by u_m yields the transition time. We can use Fig.2 even it corresponds to the case $\psi_r = 0$ because the derivation is independent of ψ_r.

The results are the following. The optimal switching time t_1, the state $x(t_1)$ and terminal time T are

$$\begin{aligned} t_1 &= \frac{x_1(t_1) - x_1(0)}{u_m} \\ g(x_1(t_1)) &= -\frac{u_m}{2}x_2(0) + \frac{g(x_1(0))}{2} + \frac{g(0)}{2} \\ x_2(t_1) &= \frac{x_2(0)}{2} - \frac{g(x_1(0))}{2u_m} + \frac{g(0)}{2u_m} \\ T_A &= \frac{2x_1(t_1) - x_1(0)}{u_m} \end{aligned} \tag{18}$$

when $x_1(0) > 0$, while they are given by

$$\begin{aligned} t_2 &= \frac{-x_1(t_2) + x_1(0)}{u_m} \\ g(x_1(t_2)) &= \frac{u_m}{2}x_2(0) + \frac{g(x_1(0))}{2} + \frac{g(0)}{2} \\ x_2(t_2) &= \frac{x_2(0)}{2} + \frac{g(x_1(0))}{2u_m} - \frac{g(0)}{2u_m} \\ T_B &= \frac{-2x_1(t_2) + x_1(0)}{u_m} \end{aligned} \tag{19}$$

when $x_1(0) < 0$. The $x_1(t_1)$ and $x_1(t_2)$ can be obtaied using inversion of the mono-valued function $g(x_1)$.

Example We have simulated the motion of a planar diver approximated by two links. The result will be shown in Fig.5 and Fig.6, where the parameters are given by

$$\begin{aligned} &M = 30(Kg),\ L = 0.75(m),\ J_L = 5.5(Kg\cdot m^2) \\ &m = 25(Kg),\ l = 0.8(m),\ J_l = 4.5(Kg\cdot m^2) \\ &P_0 = 170(Kg\cdot m^2/s),\ u_m = 5(1/s) \end{aligned}$$

The optimal control provides the minimum terminal time $T = 1.187(s)$ when $q_0 = (0, 3\pi/2)^T$ and $q_r = (0, 9\pi/2)^T$ Since $x_1(0) = 0$ and $x_2(0) = -3\pi$, the robot performs one and half somersault. The translated motions have been simulated under some appropriate conditions. In this problem setting, the minimum time control tells how long does it take at least to complete a specified motion within a given input magnitude u_m.

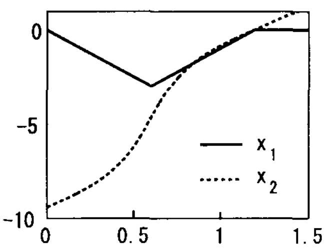

Fig.5 Time response

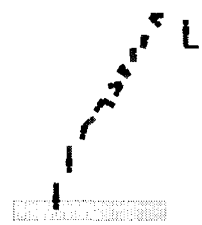

Fig.6 High diving motion

4 Proof of the facts

Assuming $\psi_r = 0$, we will prove Fig. 3, that is, we will prove that only one time switching is needed when the singular state does not occur while two time switching is needed otherwise.

To this end, we will analyze the response of λ_1 and the Hamiltonian (14).

Let's consider the case $\lambda_1 \neq 0$ first. If $\lambda_1 < 0$, (11) leads to $u = u_m$ and it follows from $\dot{x}_1 = u_m \rightarrow dt = dx_1/u_m$ that x_1 increases with the time and we can integrate (13) as

$$\begin{aligned} \lambda_1 &= \alpha \int^t \frac{dp(x_1)}{dx_1} dt = \frac{\alpha}{u_m} \int^{x_1(t)} dp(x_1) \\ &= \frac{\alpha}{u_m} p(x_1(t)) + K_1 \end{aligned} \tag{20}$$

Substituting this into (14) yields

$$1 + u_m \left[\frac{\alpha}{u_m} p(x_1) + K_1 \right] - \alpha p(x_1) = 0 \tag{21}$$

from which $K_1 = -1/u_m$ is obtained. Therefore, $\lambda_1(t)$ is expressed by

$$\lambda_1(t) = -\frac{1}{u_m}[1 - \alpha p(x_1(t))] \tag{22}$$

as a function of $x_1(t)$

Similarly, when $\lambda_1 > 0$, $u = -u_m$ leads to that x_1 decreases with the time and we can conclude

$$\lambda_1(t) = \frac{1}{u_m}[1 - \alpha p(x_1(t))] \tag{23}$$

It follows from the continuity of $\lambda_1(t)$ that (22) must intersect (23). Since the two λ_1's are the same except for their signs, they intersect only on the x_1 axis as shown in Fig.7.

Therefore, the input is switched at these intersection points. Let's denote one of the intersection point by $x_1 = x_{1s}$ (x_{1s} corresponds to $x_1(t_1)$ or $x_1(t_2)$ in the previous chapter). Equating two λ_1's at these points, we have

$$p(x_{1s}) = \frac{1}{\alpha} \tag{24}$$

where α may be determined by the initial condition.

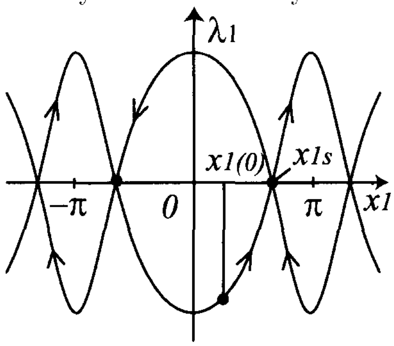

Fig. 7 $x_{1s} \neq \pm\pi$ (ordinary solution)

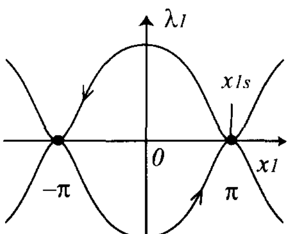

Fig. 8 $x_{1s} = \pm\pi$ (singular solution)

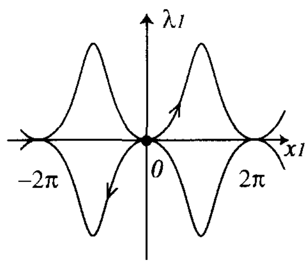

Fig. 9 $x_{1s} = 0$ (trivial solution)

It will be seen from Fig.7 that x_{1s} must satisfy $|x_{1s}| \leq \pi$ and only one switching instant is allowed before x_1 arrive at 0 from $x_1(0)$. The reason is as follows. If x_1 would move across x_{1s}, it should move to the next intersection point which increases the travelling time to $x_1 = 0$. Therefore,

$$|x_1(t)| \leq |x_{1s}| \leq \pi \tag{25}$$

must hold.

Next, suppose that λ_1 becomes identically zero for some finite time interval. In this case, since H is independent of λ_1, u cannot be determined by $u = -sgn(\lambda_1)$ and the problem becomes singular [6]. However, from Fig.7, if there exists such a λ_1, it should be at the intersection point such that $\lambda_1(t) = 0$ and $x_1(t) = x_{1s}$ hold for some period.

Besides, in the singular period, the derivatives of $\lambda_1(t)$ of any order must be zero. Its first and second derivatives are given by

$$\dot{\lambda}_1 = \alpha\frac{dp(x_1)}{dx_1}, \quad \ddot{\lambda}_1 = \alpha\frac{d^2p(x_1)}{dx_1^2}\dot{x}_1 \tag{26}$$

where

$$\frac{dp(x_1)}{dx_1} = \frac{P_0 A_1 \sin x_1}{(M_1 + A_1\cos x_1)^2} \tag{27}$$

Therefore, the following conditions

$$x_1 = x_{1s} = 0, \pm\pi \quad \text{and} \quad \dot{x}_1 = u = 0 \tag{28}$$

make them zero and the converse also holds. It is direct to check that $\lambda_1^{(i)}(t)$ $(\forall i \geq 3) = 0$ hold under (28). In addition, when $\lambda_1(t) = 0$, Hamiltonian (14) becomes

$$H = 1 - \alpha p(x_1) \tag{29}$$

which is ensured to be zero under the conditions (24) and (28). Therefore (24) and (28) are the necessary conditions for singular control.

Let's consider the case $x_1 = x_{1s} = \pm\pi$. This corresponds to the λ_1 depicted in Fig.8. In this case, the robot rotates by the fastest angular velocity with holding the leg upon the body to minimize the moment of inertia of the robot.

Then consider the case $x_{1s} = 0$ which corresponds to the λ_1 shown in Fig.9. This case means that $x_1(0) = x_{1s} = 0$ and $u = 0$ and it will be a singular solution only if $x_2(0) = 0$ as follows. When $x_2(0) < 0$, $u = 0$ leads to a trajectory keeping $x_1 = 0$ before the convergence which contradicts the optimal trajectory in the basic region shown in Fig.3. However, $x_1(0) = x_2(0) = 0$ and $u = 0$ is a trivial solution. We cannot count this as a singular solution since the optimal time interval is zero.

Then, the following conclusions are obtained. When $x(0) \neq 0$ and $|x_1| < \pi$, the singular solution does not occur and one time switching is optimal; when the state comes across $|x_1| = \pi$, $u = 0$ gives the singular control and the optimal trajectory is produced by two time switching as shown in Fig.3, since the continuity of the trajectory must hold.

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