Receding Horizon Output Feedback Control for Linear Systems with Input Saturation

Y. I. Lee[1] and B. Kouvaritakis[2]

[1] Gyeongsang National Univ., Dept. of Control and Instr.
Research Center for Aircraft Parts Tech., Jinju Gyeong-Nam 660-701, Korea;
e-mail:ciman@nongae.gsnu.ac.kr
[2] Oxford University, Dept. of Engineering Science,
Parks Road, OX1 3PJ, UK; e-mail:basil.kouvaritakis@eng.ox.ac.uk

Abstract

In this paper, a constrained receding horizon output feedback control method which is based on a state observer is suggested. The proposed method adopts the receding horizon dual-mode paradigm which consists of a 'feasible invariant set' and 'free control moves'. Polyhedral feasible invariant sets of estimated state are derived along with guaranteed bounds on state estimation errors. The guaranteed bounds on the state estimation error are developed by considering invariant sets of state estimation errors which include possible initial estimation errors. Predictions of future states are made based on estimated current state and bounds on current estimation error. The free control moves are determined so that the predicted future state belongs to the polyhedral feasible invariant set, despite input constraints and measurement noise.

Keywords: Output feedback, Input saturation, Feasible and invariants sets, Observer

1 Introduction

The dual-mode paradigm provides an efficient way to guarantee the feasibility and stability of the receding horizon control of systems with input constraints [1][2]. The basic idea is to use a finite number, N, of feasible control moves to steer the state into an invariant and feasible target set, so that under the state feedback law $\mathbf{u} = F\mathbf{x}$, all future predicted states $\mathbf{x}$ remain within this set and so that the feedback law itself is feasible. This strategy forms an extension of [3] for which the target set was taken to be the origin; use of larger target sets not only increase the applicability of the receding horizon strategy, but also allows more degrees of freedom which can be deployed for the optimization of dynamic performance.

In the presence of uncertainties such as model mismatch and disturbances, the application of the dual-mode paradigm is likely to result in impracticable computational complexity because the propagation of states over the horizon should be checked for all possible values of the uncertainty. The method proposed in [4] avoids the computational complexity by not exploiting the benefits of N free control moves but instead considering feasible invariant sets for varying state feedback gains. The N free control moves were re-introduced in [5] [6] (in the presence of polytopic model uncertainty) and [7] (in the presence of bounded disturbances). In [5] and [6]-[7], computational complexity was avoided by employing an autonomous augmented system representation and a recursive state bounding, respectively.

In all the works mentioned above, it is assumed that state is measurable. When the state is not available, a state observer can be used to provide state estimates. State estimation errors can lead to infeasibility even though the nominal predictions are feasible. The effect of state estimation error is different from that of disturbances because its magnitude varies according to the observer dynamics while the bounds on the disturbances are assumed in advance. Output feedback receding horizon control used in conjunction with observer has not received a lot of research attention; one example of such work is [8] which deals with open-loop stable systems only without disturbances and noise. Shamma and Tu[10] developed an observer based constrained control method using 'set-valued' observers. These are infinite dimensional observers that construct a set of state estimates that are consistent with observer measurements. The real-time computational burden of set-valued observer limits severely its applicability.

In this paper, an observer based constrained output feedback control method deploying the dual-mode paradigm is developed. Our approach is novel in the sense that the estimation error bounds can be computed effectively by defining invariant sets of estimation errors in the presence of measurement noise and state disturbances. Utilizing invariant sets of estimation errors, we can extend the dual-mode paradigm to the case of constrained output feedback control systems. This is achieved through the combined definition of feasible invariant sets of estimated states and invariant sets of estimation errors. The proposed algorithm is formulated so that it can be solved by well known QP methods.

2 Problem Formulation

Consider the system with input constraints, state disturbances and measurement noise:

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k) + D\mathbf{v}(k), \quad (1) \\ & \quad -\overline{\mathbf{u}} \leq \mathbf{u} \leq \overline{\mathbf{u}},\ -\overline{\mathbf{v}} \leq \mathbf{v} \leq \overline{\mathbf{v}} \\ \mathbf{y}(k) &= C\mathbf{x}(k) + E\omega(k),\ -\overline{\omega} \leq \omega \leq \overline{\omega} \quad (2) \end{aligned}$$

where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$, $\mathbf{y} \in R^l$, $\mathbf{v} \in R^r$ $\omega \in R^q$. It is assumed that the pairs (A, B) and (A, C) are stabilizable and detectable, respectively. We shall use the notation $\mathbf{x} = \{x_i\}$ for vectors and $A = \{a_{ij}\}$ for matrices. Inequalities between vectors or matrices in this paper apply on an element-by-element basis.

The state $\mathbf{x}$ is not accessible and a full order observer of the plant (1-2):

$$\begin{aligned} \hat{\mathbf{x}}(k+1) &= A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) \quad (3) \\ &\quad + L(\mathbf{y}(k) - C\hat{\mathbf{x}}(k)) \end{aligned}$$

is used to provide estimated states instead. From (1-2) and (3), the dynamics of the state estimation error $\mathbf{e}(k) = \mathbf{x}(k) - \hat{\mathbf{x}}(k)$ can be derived as:

$$\mathbf{e}(k+1) = (A - LC)\mathbf{e}(k) + \mu(k), \quad (4)$$

where $\mu(k) = D\mathbf{v}(k) - LE\omega(k)$. Based on the bounds on $\mathbf{v}$ and ω, it is possible to derive bounds on μ:

$$-\overline{\mu} \leq \mu \leq \overline{\mu}, \quad \overline{\mu} = |D|\overline{\mathbf{v}} + |LE|\overline{\omega}.$$

Given a matrix M, the notation $|M|$ is defined by $|M| = \{|m_{ij}|\}$. It is assumed that the bounds on the initial estimation error is given as:

$$|\mathbf{e}(0)| \leq \overline{\mathbf{e}}_0, \quad (5)$$

where $|\mathbf{e}| = \{|e_i|\}$.

We would like to extend the 'dual-mode' paradigm to the case of output feedback control which is based on the state estimator (3). As a first step, a set of estimated state which is feasible and invariant with respect to the feedback control $\mathbf{u} = F\hat{\mathbf{x}}$ will be derived. Substituting $\mathbf{u}$ by $F\hat{\mathbf{x}}$ in (3) yields:

$$\hat{\mathbf{x}}(k+1) = (A + BF)\hat{\mathbf{x}}(k) + L(C\mathbf{e}(k) + E\omega(k)).$$

Provided we have bounds on $|L(C\mathbf{e}(k) + E\omega(k))|$, we can define a feasible invariant set of $\hat{\mathbf{x}}$ using the technique used for state measurable systems with bounded disturbances [7]. Considering the feasible invariant set of $\hat{\mathbf{x}}$ along with bounds on $\mathbf{e}$, we can compute a set of real states which remain bounded by $\mathbf{u} = F\hat{\mathbf{x}}$.

From (3) and (4), it is possible to represent $\hat{\mathbf{x}}(N)$ in terms of $\hat{\mathbf{x}}(0)$, $\mathbf{e}(0)$, $\mathbf{u}(i)$, $\mu(i)$ and $\omega(i)$ $(i = 0, 1, \cdots, N-1)$. For any initial state estimation $\hat{\mathbf{x}}(0)$, control inputs during $[0, N-1]$ and bounds on $|\mathbf{e}(0)|$, $|\mu(\cdot)|$ and $|\omega(\cdot)|$, it is possible to define upper and lower bounds on $\hat{\mathbf{x}}(N)$:

$$\overline{\hat{\mathbf{x}}}(N) = \max_{\substack{|\mathbf{e}(0)| \leq \overline{\mathbf{e}}_0 \\ |\mu| \leq \overline{\mu}}} \hat{\mathbf{x}}(N) \quad (6)$$

$$\underline{\hat{\mathbf{x}}}(N) = \min_{\substack{|\mathbf{e}(0)| \leq \overline{\mathbf{e}}_0 \\ |\mu| \leq \overline{\mu}}} \hat{\mathbf{x}}(N), \quad (7)$$

where the maximum and minimum of $\hat{\mathbf{x}}$ is applied on an element-by element basis. Using the upper and lower bounds of $\hat{\mathbf{x}}(N)$, future control moves can be determined so that $\hat{\mathbf{x}}(N)$ belongs to the feasible and invariant set of $\hat{\mathbf{x}}$.

In the next sections, details of obtaining 'feasible invariant' set of $\hat{\mathbf{x}}$ and guaranteeing the terminal membership of $\hat{\mathbf{x}}(k+N)$ will be described.

3 Invariant Sets

Application of the state feedback control law $\mathbf{u} = F\hat{\mathbf{x}}$ and use of the state transformation $\hat{\mathbf{x}} = W\hat{\mathbf{z}}$ yields the closed-loop estimated state equation as:

$$\hat{\mathbf{z}}(k+1) = \Phi^W \hat{\mathbf{z}}(k) + VLE\omega(k) + VLC\mathbf{e}(k), \quad (8)$$
$$\Phi^W = V(A+BF)W, \; V = W^{-1}.$$

Using the transformation $\hat{\mathbf{z}} = V\hat{\mathbf{x}}$, define a polyhedral set as:

$$\mathcal{R}_F^W(\alpha) = \{\hat{\mathbf{x}} \in R^n | \; |\hat{\mathbf{z}}| \leq \alpha\} \quad (9)$$

and suppose that $\hat{\mathbf{x}}(k) \in \mathcal{R}_F^W(\alpha)$. In order to ensure $\hat{\mathbf{x}}(k+1) \in \mathcal{R}_F^W$, we need $|\hat{\mathbf{z}}(k+1)| \leq \alpha$ *i.e.*:

$$|\Phi^W \hat{\mathbf{z}}(k) + VLE\omega(k) + VLC\mathbf{e}(k)| \leq \alpha. \quad (10)$$

Let us assume that $\mathbf{e}(k)$ is bounded as:

$$|\mathbf{e}(k)| \leq \overline{\mathbf{e}}, \; \forall \; k \geq 0, \quad (11)$$

Condition (10) is satisfied for any $\hat{\mathbf{x}}(k) \in \mathcal{R}_F^W(\alpha)$, $|\omega(k)| \leq \overline{\omega}$ and $|\mathbf{e}(k)| \leq \overline{\mathbf{e}}$ if and only if:

$$|\Phi^W|\alpha + |VLE|\overline{\omega} + |VLC|\overline{\mathbf{e}} \leq \alpha. \quad (12)$$

Inequality (12) defines the invariance condition for the set $\mathcal{R}_F^W(\alpha)$ with respect to F and L in the presence of bounded measurement noise and disturbances. In addition to this invariance condition, we also need feasibility which requires that the feedback law $\mathbf{u} = F\hat{\mathbf{x}}$ for any $\hat{\mathbf{x}} \in \mathcal{R}_F^W(\alpha)$ stays within bounds, namely

$$-\overline{\mathbf{u}} \leq F\hat{\mathbf{x}}(k) = FW\hat{\mathbf{z}}(k) \leq \overline{\mathbf{u}} \quad (13)$$

which leads to the feasibility condition

$$|FW|\alpha \leq \overline{\mathbf{u}}. \quad (14)$$

Note that unlike $\overline{\mathbf{e}}_0$ which is available (by assumption), the bound $\overline{\mathbf{e}}$ needs to be computed and this can be achieved through the use of an appropriate invariant set. Thus using another state transformation $\mathbf{e} = \mathcal{W}\mathbf{d}$, we define a polyhedral set as:

$$\mathcal{E}_L^{\mathcal{W}}(\beta) = \{\mathbf{e} \in R^n | \; |\mathbf{d}| \leq \beta\} \quad (15)$$

and suppose that $\mathbf{e}(k) \in \mathcal{E}_L^{\mathcal{W}}(\beta)$. In order to ensure that the error remains in the set at the next time, (4) implies that:

$$|\Psi^{\mathcal{W}}\mathbf{d}(k) + \mathcal{V}\mu(k)| \leq \beta, \quad (16)$$

where $\Psi^{\mathcal{W}} = \mathcal{V}(A - LC)\mathcal{W}$, $\mathcal{V} = \mathcal{W}^{-1}$. It is easy to see that (16) is satisfied for any $\mathbf{e}(k) \in \mathcal{E}_L^{\mathcal{W}}$ and $|\omega(k)| \leq \overline{\omega}$ if and only if:

$$|\Psi^{\mathcal{W}}|\beta + |\mathcal{V}|\overline{\mu} \leq \beta. \quad (17)$$

Suppose that (17) is satisfied, then $\mathbf{e}(0) \in \mathcal{E}_L^{\mathcal{W}}$ guarantees $\mathbf{e}(k) \in \mathcal{E}_L^{\mathcal{W}}, k \geq 0$ because of the invariance of $\mathcal{E}_L^{\mathcal{W}}$. Conditions (12) can be rederived in terms of α, β and $\overline{\omega}$ if $\mathbf{e}(k)$ in (10) is replaced by $\mathcal{W}\mathbf{d}(k)$ in conjunction with assumption that $\hat{\mathbf{x}} \in \mathcal{R}_F^W(\alpha)$ and $\mathbf{e} \in \mathcal{E}_L^{\mathcal{W}}(\beta)$. On the other hand $\mathbf{e}(0) \in \mathcal{E}_L^{\mathcal{W}}$ for any $|\mathbf{e}(0)| \leq \overline{\mathbf{e}}_0$ if and only if:

$$\overline{\mathbf{d}}_0 = |\mathcal{V}|\overline{\mathbf{e}}_0 \leq \beta. \quad (18)$$

The arguments above can be summarized as per the following theorem:

THEOREM 1 *Consider the plant (1-2) and its state estimator (3) with initial estimation error* $\mathbf{e}(0)$ *satisfying (5). For the given state transformations* $\hat{\mathbf{x}} = W\hat{\mathbf{z}}$ *and* $\mathbf{e} = \mathcal{W}\mathbf{d}$*, the polyhedral set* $\mathcal{R}_F^W(\alpha)$ *is feasible and invariant with respect to* F *and* L *if there exist positive vectors* α *and* β *that satisfy:*

$$|\Phi^W|\alpha + |VLE|\overline{\omega} + |VLC\mathcal{W}|\beta \leq \alpha \quad (19)$$
$$|FW|\alpha \leq \overline{\mathbf{u}} \quad (20)$$
$$|\Psi^{\mathcal{W}}|\beta + |\mathcal{V}|\overline{\mu} \leq \beta \quad (21)$$
$$|\mathcal{V}|\overline{\mathbf{e}}_0 \leq \beta. \quad (22)$$

■

REMARK 1 *The transformation matrix* W *and* $\mathcal{W}$ *can be considered as design parameters to be chosen to yield non-empty* $\mathcal{R}_F^W(\alpha)$ *and* $\mathcal{E}_L^{\mathcal{W}}(\beta)$*. Choosing* W *and* $\mathcal{W}$ *to be the eigenvector matrices of* $A+BF$ *and* $A+LC$ *respectively, guarantees the Perron-Frobenius norms of* $|\Phi^W|$ *and* $|\Psi^{\mathcal{W}}|$ *be less than 1 [2].*

Since the feasibility of $\mathbf{u} = F\hat{\mathbf{x}}$ is guaranteed for any $\hat{\mathbf{x}} \in \mathcal{R}_F^W(\alpha)$ when (19- 22) are satisfied, the closed-loop stability can be established using the invariance property of sets $\mathcal{R}_F^W$ and $\mathcal{E}_L^W$ in the following corollary.

COROLLARY 1 *If (19-22) are satisfied,* $\mathbf{u} = F\hat{\mathbf{x}}$ *stabilize the system (1) for any initial estimation* $\hat{\mathbf{x}}(0) \in \mathcal{R}_F^W(\alpha)$ *with initial estimation error* $\mathbf{e}(0)$ *satisfying (5).*

In the next section, we will develop a receding horizon control strategy which deploys the feasible invariant set as a target set.

4 Constrained Receding Horizon Output Feedback Control

Suppose that (19-22) are satisfied for some α and β. Then for any initial estimate $\hat{\mathbf{x}}(0) \in \mathcal{R}_F^W(\alpha)$ with $\mathbf{e}(0) \in \mathcal{E}_L^{\mathcal{W}}(\beta)$, stability can be obtained by using state feedback control $\mathbf{u} = F\hat{\mathbf{x}}$ as shown in Corollary 1. In the case however when the initial state $\hat{\mathbf{x}}(0)$ is not included in $\mathcal{R}_F^W(\alpha)$ while $\mathbf{e}(0) \leq \overline{\mathbf{e}}_0$, the control $\mathbf{u} = F\hat{\mathbf{x}}$ does not guarantee closed loop stability due to the input constraints. One stabilizing control strategy for this case is to use N predicted feasible moves to steer the N-steps-ahead estimated predicted state $\hat{\mathbf{x}}(k+N|k)$ into $\mathcal{R}_F^W(\alpha)$ for some α and β satisfying (19-20) and (21-22). For a given bounds on the initial state estimation error $\mathbf{e}_0$, it is possible to update the estimation error bounds at each time step using the estimation error dynamics (4) and to derive current estimation error bound $\overline{\mathbf{e}}_k$. Since the membership of $\mathcal{R}_F^W(\alpha)$ is shifted N-steps into the future, we can exploit the contractive nature of (4) further and replace (22) by the more appropriate condition the $|\mathcal{V}|\overline{\mathbf{e}}_{k+N} \leq \beta$

Considering all α satisfying (19-20), we will denote the union of $\mathcal{R}_F^W(\alpha)$ as $\mathcal{R}_{F,\overline{\mathbf{d}}}^W$:

$$\mathcal{R}_{F,\overline{\mathbf{d}}_N}^W = \bigcup_{\substack{\alpha \in S_{\alpha(\beta)} \\ \beta \in \mathcal{S}_{\beta(\overline{\mathbf{d}}_N)}}} \mathcal{R}_F^W(\alpha), \qquad (23)$$

where $S_{\alpha(\beta)}$ denotes the set of α satisfying (19-20) for some β and $\mathcal{S}_{\beta(\overline{\mathbf{d}}_N)}$ denotes the set of β satisfying (21-22) with $\overline{\mathbf{d}}_0 = |\mathcal{V}|\overline{\mathbf{e}}_0$ replaced by $\overline{\mathbf{d}}_N = |\mathcal{V}|\overline{\mathbf{e}}_N$. Thus, rather than require a predicted terminal estimation $\hat{\mathbf{x}}(N|0) \in \mathcal{R}_F^W(\alpha)$, for a given α, we will use the much less restrictive constraint that $\hat{\mathbf{x}}(N|0) \in \mathcal{R}_{F,\overline{\mathbf{d}}_N}^W$.

In this section, we will develop a control algorithm which deploys this strategy in a receding horizon manner: (i) at time instant k, determine N future control moves $\mathbf{u}(k|k), \mathbf{u}(k+1|k), \cdots, \mathbf{u}(k+N-1|k)$, which are feasible, so that:

$$\hat{\mathbf{x}}(k+N|k) \in \mathcal{R}_{F,\overline{\mathbf{d}}_{k+N}}^W; \qquad (24)$$

(ii) apply $\mathbf{u}(k|k)$ to the plant; and (iii) repeat this procedure at the next time instant $k+1$.

Using (4), it is possible to represent $\mathbf{d}(k)$ in terms of $\mathbf{d}(0)$:

$$\mathbf{d}(k) = [\Psi^w w]^k \mathbf{d}(0) + \sum_{p=1}^{k} [\Psi^w w]^{k-p} \mathcal{V}\mu(p-1), \qquad (25)$$

where $[\Psi^w w]^k$ represents the k^{th} power of $\Psi^w w$. Over the allowable class of ν and ω and all $\mathbf{e}(\mathbf{0})$ satisfying (5), the least (attainable) upper bound on $\mathbf{d}(\mathbf{k})$ can be derived as:

$$|\mathbf{d}(k)| \leq |[\Psi^w w]^k|\overline{\mathbf{d}}_0 + \sum_{p=0}^{k-1} |[\Psi^w w]^p \mathcal{V}|\overline{\mu} = \overline{\mathbf{d}}_k \qquad (26)$$

From (26), we can see that the bounds on $|\mathbf{d}(k)|$, $\overline{\mathbf{d}}_k$, can be obtained in terms of $\overline{\mathbf{d}}_0$ using the following recursive formulation:

$$\overline{\mathbf{d}}_i = |[\Psi^w w]^i|\overline{\mathbf{d}}_0 + M_i \overline{\mu}, \qquad (27)$$

where M_i is obtained recursively as:

$$M_i = M_{i-1} + |[\Psi^w w]^i \mathcal{V}|, \quad M_0 = I \qquad (28)$$

It is obvious that $[\Psi^w w]^i$ can be computed recursively as well and $\overline{\mathbf{d}}_k$ can be computed online using (27-28).

Now, we will consider how to determine the feasible future control moves so that (24) is satisfied. Without loss of generality, it is convenient [9] to represent the N predicted control moves as perturbations on the state feedback law:

$$\begin{aligned} \mathbf{u}(k+l|k) &= F\hat{x}(k+l|k) + \mathbf{c}(k+l|k) \qquad (29) \\ &= FW\hat{\mathbf{z}}(k+l|k) + \mathbf{c}(k+l|k), \end{aligned}$$

for $l = 0, 1, \cdots, N-1$. Because of the input constraints, we have the following feasibility conditions on $\mathbf{c}(k+l|k)$:

$$|F^W \hat{\mathbf{z}}(k+k|k) + \mathbf{c}(k+l|k)| \leq \overline{\mathbf{u}}, \qquad (30)$$

where $F^W = FW$. Considering the dynamics of state estimator (3), let $\hat{\mathbf{x}}(k+N|k)|_{C(k)} = W\hat{\mathbf{z}}(k+N|k)|_{C(k)}$ be the prediction of $\hat{\mathbf{x}}(k+N)$ based on $\hat{\mathbf{x}}(k)$ and future perturbations $C(k) = [\mathbf{c}(k|k)' \; \mathbf{c}(k+1|k)' \cdots \mathbf{c}(k+N-1|k)']'$.

Replacing $\mathbf{u}$ in (3) by $F\hat{\mathbf{x}} + \mathbf{c}$, we have closed-loop estimated state equation in the transformed space:

$$\begin{aligned} \hat{\mathbf{z}}(k+1) &= \Phi^W \hat{\mathbf{z}}(k) + B\mathbf{c}(k) + VLE\omega(k) \qquad (31) \\ &\quad + VLC\mathcal{W}\mathbf{d}(k). \end{aligned}$$

From (31), the predictions can be written as:

$$\begin{aligned}\hat{\mathbf{z}}(k+i|k)|_{C(k)} &= [\Phi^W]^i\hat{\mathbf{z}}(k) \quad (32)\\ &+\textstyle\sum_{j=1}^i[\Phi^W]^{i-j}B\mathbf{c}(k+j-1|k)\\ &+\textstyle\sum_{j=1}^i[\Phi^W]^{i-j}VLE\omega(k+j-1)\\ &+\textstyle\sum_{j=1}^i[\Phi^W]^{i-j}VLC\mathcal{W}\mathbf{d}(k+j-1).\end{aligned}$$

Because of the uncertainties on $\mathbf{d}(\cdot)$, $\omega(\cdot)$ and $\mu(\cdot)$, the exact value of $\hat{\mathbf{z}}(k+i|k)|_{C(k)}$ cannot be determined. However, the bounds on $\hat{\mathbf{z}}(k+i|k)$ can be computed from (32) using the bounds on $\mathbf{d}(\cdot)$, $\omega(\cdot)$ and $\mu(\cdot)$ and the following obvious lemma.

LEMMA 1 *[6] Let $\underline{\mathbf{z}} \le \mathbf{z} \le \overline{\mathbf{z}}$, then the minimum and maximum value of $\Phi\mathbf{z}$ can be obtained as:*

$$\min_{\underline{\mathbf{z}} \le \mathbf{z} \le \overline{\mathbf{z}}} \Phi\mathbf{z} = \Phi^+\underline{\mathbf{z}} - \Phi^-\overline{\mathbf{z}} \quad (33)$$

$$\max_{\underline{\mathbf{z}} \le \mathbf{z} \le \overline{\mathbf{z}}} \Phi\mathbf{z} = \Phi^+\overline{\mathbf{z}} - \Phi^-\underline{\mathbf{z}}, \quad (34)$$

where $\Phi^+ = max(\Phi, 0)$ and $\Phi^- = max(-\Phi, 0)$. ■

From (32) we have:

$$\begin{aligned}&\max_{|d(k)| \le \overline{\mathbf{d}}_k, |\omega(\cdot)| \le \overline{\omega}, |\mu(\cdot)| \le \overline{\mu}} \hat{\mathbf{z}}(k+i|k)|_{C(k)}\\ &= \overline{\hat{\mathbf{z}}}(k+i|k) \quad (35)\\ &= [\Phi^W]^i\hat{\mathbf{z}}(k) + \sum_{j=1}^{i}[\Phi^W]^{i-j}B\mathbf{c}(k+j-1|k)\\ &+\sum_{j=1}^{i}|P_{i-j}|\overline{\omega} + \sum_{j=1}^{i}|Q_{i-j}|\overline{\mathbf{d}}_{k+j-1}\end{aligned}$$

$$\begin{aligned}&\min_{|d(k)| \le \overline{\mathbf{d}}_k, |\omega(\cdot)| \le \overline{\omega}, |\mu(\cdot)| \le \overline{\mu}} \hat{\mathbf{z}}(k+i|k)|_{C(k)}\\ &= \underline{\hat{\mathbf{z}}}(k+i|k) \quad (36)\\ &= [\Phi^W]^i\hat{\mathbf{z}}(k) + \sum_{j=1}^{i}[\Phi^W]^{i-j}B\mathbf{c}(k+j-1|k)\\ &-\sum_{j=1}^{i}|P_{i-j}|\overline{\omega} - \sum_{j=1}^{i}|Q_{i-j}|\overline{\mathbf{d}}_{k+j-1},\end{aligned}$$

where $P_i = [\Phi^W]^iVLE$ and $Q_i = [\Phi^W]^iVLC\mathcal{W}$. Using the bounds $\overline{\hat{\mathbf{z}}}(\cdot)$ and $\underline{\hat{\mathbf{z}}}(\cdot)$ in conjunction with Lemma 1, it is also possible compute bounds on control inputs $\mathbf{u} = F^W\mathbf{z} + \mathbf{c}$:

$$\begin{aligned}\overline{\mathbf{u}}(k+i|k) &= F^{W+}\overline{\hat{\mathbf{z}}}(k+i|k) - F^{W-}\underline{\hat{\mathbf{z}}}(k+i|k)\\ &\quad +\mathbf{c}(k+i|k) \quad (37)\\ &\ge F^W\mathbf{z}(k+i|k) + \mathbf{c}(k+i|k)\\ \underline{\mathbf{u}}(k+i|k) &= F^{W+}\underline{\hat{\mathbf{z}}}(k+i|k) - F^{W+}\overline{\hat{\mathbf{z}}}(k+i|k)\\ &\quad +\mathbf{c}(k+i|k) \quad (38)\\ &\le F^W\mathbf{z}(k+i|k) + \mathbf{c}(k+i|k),\end{aligned}$$

where $F^{W+} = max(F^W, 0)$ and $F^{W-} = max(-F^W, 0)$.
It is obvious that $-\alpha \le \underline{\hat{\mathbf{z}}}(k+N|k) \le \overline{\hat{\mathbf{z}}}(k+N|k) \le \alpha$ and $-\overline{\mathbf{u}} \le \underline{\mathbf{u}}(k+i|k) \le \overline{\mathbf{u}}(k+i|k) \le \overline{\mathbf{u}}(i = 0, 1, \cdots, N-1)$ guarantee $\hat{\mathbf{x}}(k+N) \in \mathcal{R}_F^W(\alpha)$. The relevant inequalities are listed in the theorem below; this theorem is a direct consequence of the arguments above and therefore will be given without proof.

THEOREM 2 *Consider the plant (1-2) and its state estimator (3). For a given state estimate $\hat{\mathbf{x}}(k)$ and estimation error bound $\overline{\mathbf{d}}_k (\ge |\mathcal{V}\mathbf{e}(k)|)$, $\hat{\mathbf{x}}(k+N|k) \in \mathcal{R}^W_{F,\overline{\mathbf{d}}_{k+N|k}}$ is implied by:*

$$[\Phi^W]^i\hat{\mathbf{z}}(k) + \sum_{j=1}^{i}[\Phi^W]^{i-j}B\mathbf{c}(k+j-1|k) + \sum_{j=1}^{i}|P_{i-j}|\overline{\omega} + \sum_{j=1}^{i}|Q_{i-j}|\overline{\mathbf{d}}_{k+j-1} \le \alpha \quad (39)$$

$$[\Phi^W]^i\hat{\mathbf{z}}(k) + \sum_{j=1}^{i}[\Phi^W]^{i-j}B\mathbf{c}(k+j-1|k) - \sum_{j=1}^{i}|P_{i-j}|\overline{\omega} - \sum_{j=1}^{i}|Q_{i-j}|\overline{\mathbf{d}}_{k+j-1} \ge -\alpha \quad (40)$$

$$\overline{\mathbf{d}}_{k+N} \le \beta \quad (41)$$

for some α and β satisfying (19-21), where $\overline{\mathbf{d}}_{k+N}$ is obtained using (27-28). Furthermore, the feasibility of control moves $\mathbf{u}(k+i|k)(i = 0, 1, \cdots, N-1)$ in (29) is guaranteed if:

$$F^{W+}\overline{\hat{\mathbf{z}}}(k+i|k) - F^{W-}\underline{\hat{\mathbf{z}}}(k+i|k) + \mathbf{c}(k+i|k) \le \overline{\mathbf{u}} \quad (42)$$

$$F^{W+}\underline{\hat{\mathbf{z}}}(k+i|k) - F^{W-}\overline{\hat{\mathbf{z}}}(k+i|k) + \mathbf{c}(k+i|k) \ge -\overline{\mathbf{u}}, \quad (43)$$

for $i = 0, 1, \cdots, N-1$

It is important to note that all the constraints above are linear in the variables α, β, $\mathbf{c}$, $\underline{\hat{\mathbf{z}}}(\cdot|k)$ and $\overline{\hat{\mathbf{z}}}(\cdot|k)$. We can effectively check whether $\hat{\mathbf{x}}(k+N|k)$ is included in $\mathcal{R}^W_{F,\overline{\mathbf{d}}_{k+N|k}}$ using QP or LP.

Under the assumption that F is chosen (off-line) so as to give optimal performance in some appropriate sense (*e.g.* nominal or worst case), an obvious choice for the available degrees of freedom is given by the minimization of the cost index:

$$J(C(k)) = \sum_{l=1}^{N} \mathbf{c}(k+l|k)'\mathbf{c}(k+l|k) \qquad (44)$$

with constraints (19-21) and (39-43) with α, β, $\mathbf{c}$, $\underline{\hat{\mathbf{z}}}(\cdot|k)$ and $\overline{\hat{\mathbf{z}}}(\cdot|k)$ as variables.

A receding horizon control algorithm based on this problem is summarized as:
Algorithm 1

Step 1 : Calculate $C(k)$ by solving QP (44), subject to (19-21) and (39-43).

Step 2 : Apply $\mathbf{u}(k) = F\hat{\mathbf{x}}(k) + \mathbf{c}(k|k)$ to the system.

Step 3 : At the next step, repeat the Step 1 and 2.

The feasibility and stability properties of Algorithm 1 can be established as per the following theorem.

THEOREM 3 *Consider the plant (1-2) and its state estimator (3) with transformation matrix W and $\mathcal{W}$. Algorithm 1 is feasible and guarantees boundedness of state for the closed-loop system if a feasible sequence $C(0)$ can be found for the given initial state estimate $\hat{\mathbf{x}}(0)$ and $|\mathcal{V}\mathbf{e}(0)| \leq \overline{\mathbf{d}}_0$.*

REMARK 2 *The use of $\hat{C}(1)$ at time $k = 1$ would result in a cost $J(\hat{C}(1)) \leq J(C^*(0))$, so that upon further optimization at $k = 1$, we have $J(C^*(1)) \leq J(C^*(0))$, where $*$ indicates optimality. This argument can be applied recursively to get:*

$$J(C^*(0)) \geq \sum_{j=0}^{L-1} \mathbf{c}^*(j|j)'\mathbf{c}^*(j|j) + J(C^*(L)). \quad (45)$$

From (45), we have boundedness of $\sum_{j=0}^{\infty} \mathbf{c}^*(j|j)'\mathbf{c}^*(j|j)$ since (45) is satisfied for all $L > 0$. It is obvious that $\mathbf{c}^*(L|L)$ converges to zero as L increases and the behaviour of $\mathbf{u}(L|L)$ can be considered as $F\hat{\mathbf{x}}(L) + o(L)$, which is feasible, where $o(L)$ approach zero as L grows to infinity. This implies the fact that the closed-loop performance will getting closer to that of $\mathbf{u} = F\hat{\mathbf{x}}$.

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