Optimal Approximation Rates for Lipschitz Controls

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Abstract

For nonlinear nonconvex control systems we investigate the approximation properties of Lipschitz controls. In case that the control range is connected, any trajectory produced by a measurable control can be approximated by trajectories produced by Lipschitz controls. The approximation is of order $O(M^{-1/2})$, where M is the Lipschitz constant. An example shows that this approximation order is optimal.

1 Introduction

We consider nonlinear nonconvex control systems

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x, \quad u(t) \in \Omega \qquad (1)$$

in the Euclidean space R^n. As usual, a solution $t \mapsto x(t) = x(t; x, u(\cdot))$ is an absolutely continuous function with $x(0) = x$ such that the differential equation of (1) is satisfied for (Lebesgue) almost all times $t \in [0, \infty)$. We define two families of controls: $\mathcal{U}$, the set of all measurable functions $u(\cdot) : [0, \infty) \to \Omega$ and, for a Lipschitz constant $M > 0$, $\mathcal{U}_M$, the set of all Lipschitz continuous functions $w(\cdot) : [0, \infty) \to \Omega$ with Lipschitz constant $M > 0$. It is well known that, in a weak sense, $\bigcup_{M>0} \mathcal{U}_M$ is dense in $\mathcal{U}$ and that any trajectory produced by a measurable control $u(\cdot) \in \mathcal{U}$ can be approximated by trajectories produced by Lipschitz controls $u_M(\cdot) \in \bigcup_{M>0} \mathcal{U}_M$, provided that the control range Ω is connected. We are interested in optimal approximation rates. It turns out that, for given initial value $x \in R^n$ and time interval $[0, H]$, the expression

$$\sup_{u(\cdot)\in\mathcal{U}} \left(\inf_{w(\cdot)\in\mathcal{U}_M} \left(\max_{t\in[0,H]} \|x(t, x, u(\cdot)) - x(t, x, w(\cdot))\| \right) \right)$$

is of order $O(M^{-1/2})$, as $M \to \infty$. This estimate cannot be improved, as shown in an specific example. The approximation of measurable controls by more regular controls is not only of theoretical but also of practical interest. Basically two classes of regular controls usually are considered. Firstly, the class of piecewise continuous controls and within this group the piecewise constant controls. Secondly, the class of Lipschitz controls. Whereas the approximation order for the former class has been investigated in a variety of articles, see [1, 5, 6, 7, 9, 10], the latter class seems to lack a comparable investigation. The question of approximating with Lipschitz controls is of some importance for practical problems, since for most mechanical systems measurable controls, or even piecewise constant controls, cannot be realized, due to the inertness of the controlling mechanism, see [4]. Furthermore, Lipschitz controls play a prominent part in game theory, where they are used to characterize the existence of a value. However, in this context one usually works with compactness arguments in order to achieve a convergence of the trajectories, see [3].

2 Optimal Approximation Rates

The setting is as follows.

- The control range Ω is a compact metric space.
- The vector fields $x \mapsto f(x, \omega)$ are uniformly Lipschitz continuous on R^n with Lipschitz constant $L \geq 0$.
- The map $(x, \omega) \mapsto f(x, \omega)$ is continuous on $R^n \times \Omega$.
- For any time $H > 0$ and any initial value $x \in R^n$ there is a constant $P \geq 0$ such that $\|f(y, \omega)\| \leq P$ for all $y \in Y := \{x(t; x, u(\cdot)) : \quad t \in [0, H], u(\cdot) \in \mathcal{U}\}$ and all $\omega \in \Omega$.

The last point is redundand, since for any initial state $x \in R^n$ and any interval $[0, H]$ the set of reachable states Y is bounded in R^n (by the continuity properties of the vector fields together with the compactness of the control range). For $\omega \in \Omega$ we set

$$\mathcal{U}_M^\omega := \{u(\cdot) \in \mathcal{U}_M : \quad u(0) = \omega\}.$$

The following assumption is essential.

Assumption. There is a time $T \geq 0$ such that for all $\omega_1, \omega_2 \in \Omega$ the inclusion

$$\mathcal{U}_1^{\omega_2} \subset \{u(T + \cdot) \in \mathcal{U}_1 : \quad u(\cdot) \in \mathcal{U}_1^{\omega_1}\}$$

is valid.

Obviously, this implies connectedness of the control range Ω, in the sense that any two points $\omega_1, \omega_2 \in \Omega$ can be connected by a continuous path within Ω.

Theorem. Let $M > 0$ and $H \geq 0$. For any measurable control $u(\cdot) \in \mathcal{U}$ and any $\omega \in \Omega$ there is a Lipschitz control $w(\cdot) \in \mathcal{U}_M^\omega$ such that

$$\max_{t\in[0,H]} \|x(t,x,w(\cdot)) - x(t,x,u(\cdot))\|$$
$$\leq \frac{1}{\sqrt{M}} 4e^{LH}\sqrt{(PLH+P)HPTn}.$$

The proof of this statement relies on a generalized version of Caratheodory's theorem, see [2], applied to the set-valued averages

$$F_M(S,y,\omega) = \{\frac{1}{S}\int_0^S f(y,u(t))dt : \quad u(\cdot) \in \mathcal{U}_M^\omega\},$$

and can be found in [8].

We present an example that shows that the approximation order $O(M^{-1/2})$, as $M \to \infty$, is optimal.

Example. Consider the system in R^2

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} f_1(u_1(t)) \\ f_2(u_2(t)) \end{pmatrix},$$

where the control range $\Omega \subset R^2$ is a union $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ with $\Omega_1 := [-2,-1] \times [0,1]$, $\Omega_2 := [-1,1] \times \{1\}$ and $\Omega_3 := [1,2] \times [0,1]$. Hence, for a fixed control value $(\omega_1, \omega_2) \in \Omega$, the corresponding vector field is constant. We define

$$f_1(\omega_1) := \begin{cases} \omega_1 + 1 & \text{for} \quad \omega_1 \in [-2,-1] \\ 0 & \text{for} \quad \omega_1 \in [-1,1] \\ \omega_1 - 1 & \text{for} \quad \omega_1 \in [1,2] \end{cases},$$

$$f_2(\omega_2) := \omega_2.$$

For simplicity we choose the initial state $x = (x_1, x_2)^T = (0,0)^T \in R^2$ and the time horizon $[0,H] = [0,1]$. In the sequel we make use of a more restrictive version of the Landau symbol O. For $\lambda \in R$ we say that a function $M \mapsto h(M) \in R$ is of order $\tilde{O}(M^\lambda)$, as $M \to \infty$, if there are two constants $0 < c_1 \leq c_2$ such that for $M > 0$ large enough the estimates $c_1 M^\lambda \leq |h(M)|$ and $|h(M)| \leq c_2 M^\lambda$ are valid. In contrast, the standard Landau symbol O requires the second estimate only. Now we assume that there is a $\kappa > 1/2$ such that the expression

$$\sup_{u(\cdot)\in\mathcal{U}} \left(\inf_{w(\cdot)\in\mathcal{U}_M} \left(\max_{t\in[0,H]} \|x(t,x,u(\cdot)) - x(t,x,w(\cdot))\| \right)\right)$$

is of order $O(M^{-\kappa})$, as $M \to \infty$. For $M > 0$ and $\epsilon \in (0,\kappa)$ we define a periodic measurable control $t \mapsto u(t) = (u_1(t), u_2(t))^T$ (with period $4M^{-\kappa+\epsilon}$) by

$$u_1(t) := \begin{cases} 2 & \text{for} \quad t \in [M^{-\kappa+\epsilon}, 3M^{-\kappa+\epsilon}) \\ -2 & \text{for} \quad t \in [3M^{-\kappa+\epsilon}, 5M^{-\kappa+\epsilon}) \end{cases},$$

$$u_2(t) := 0 \quad \text{for} \quad t \in R.$$

This control produces a periodic trajectory $t \mapsto x(t;x,u(\cdot))$ with $x_2(t;x,u(\cdot)) \equiv 0$ and $t \mapsto x_1(t;x,u(\cdot))$ periodically oscillating between the values $-M^{-\kappa+\epsilon}$ and $+M^{-\kappa+\epsilon}$. Since $\epsilon > 0$, for any $c > 0$, we have $2M^{-\kappa+\epsilon} > cM^{-\kappa}$ for $M > 0$ large enough. Hence, for $M > 0$ large enough, the x_1-trajectory only can be approximated by Lipschitz controls $w(\cdot) = (w_1(\cdot), w_2(\cdot)) \in \mathcal{U}_M$ if $t \mapsto w_1(t)$ changes the sign. The number of changes is of order $\tilde{O}(M^{\kappa-\epsilon})$, as $M \to \infty$. Notice that any change of sign causes a vertical deviation of order $\tilde{O}(M^{-1})$, as $M \to \infty$, since the Lipschitz control has to pass through Ω_2. So, for any $\epsilon \in (0,\kappa)$, we collect an overall vertical deviation of order $\tilde{O}(M^{-1+\kappa-\epsilon})$, as $M \to \infty$, which is a contradiction.

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