Delay Effects on Static Output Feedback Stabilization

Silviu-Iulian Niculescu[1]
HEUDIASYC (UMR CNRS 6599),
Université de Technologie de Compiègne,
Centre de Recherche de Royallieu,
BP. 20529, 60205, Compiègne, France
E-mail: silviu@hds.utc.fr

Chaouki T. Abdallah[2]
EECE Department,
University of New Mexico,
Albuquerque, NM 87131, USA
E-mail: chaouki@eece.unm.edu

Abstract

This paper addresses conditions for characterizing static output feedback controllers including delays for some proper (finite-dimensional) transfer functions. The interest of such study is in controlling systems which can *not* be stabilized by the classical, nondelayed static output feedback, and its difficulty lies in computing delay intervals guaranteeing closed-loop stability, since stability switches/reversals may occur for the same (matrix) gain if the delay is seen as a 'free' (design) parameter. The derived conditions are expressed in terms of some appropriate matrix pencils or MIMO Nyquist tests. Illustrative examples are also presented.

Keywords: Stability, delay switches/reversals, matrix pencils, Nyquist.

1 Introduction and Problem Formulation

In this paper, we consider mainly the following:
Problem 1: *Given a strictly proper transfer function $H(s) \in \mathbf{C}^{\mathbf{p}\times\mathbf{m}}$ $(p, m \geq 1)$ with a state-space representation ($u \in \mathbf{R}^{\mathbf{m}}$, $y \in \mathbf{R}^{\mathbf{p}}$, $x \in \mathbf{R}^{\mathbf{n}}$):*

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases}, \qquad (1.1)$$

find all *pairs $(K, \tau) \in \mathbf{R}^{\mathbf{m}\times\mathbf{p}} \times \mathbf{R}^{+}$ such that the static delayed output feedback $u(t) = Ky(t-\tau)$ stabilizes the system (1.1).*

It is clear that for $\tau \equiv 0$, we have the 'classical' *static output feedback problem*, which has been thoroughly studied (See for example [31] and the references therein).We are interested in introducing a *delay* in the control law of the class of systems (1.1) for which the static output stabilization *fails*. It would seem that the class of the transfer functions which are closed-loop asymptotically stable may become larger if one uses *infinite-dimensional controllers*, for stabilizing finite-dimensional systems.

In practice, the delay effects on the system's phase may be sufficient in some cases to guarantee the stability of the control scheme in closed-loop (see, for example, in the stabilization of some *oscillatory systems*). Indeed, consider the following SISO system:

$$y(t) = H(s) \cdot u(t), \quad H(s) = \frac{1}{s^2 + \omega_0^2},$$

with $\omega_0 > 0$. It is clear that any feedback $u(t) = ky(t)$, $k \in \mathbf{R}$ does not stabilize the system. However the control $u(t) = ky(t-\tau)$, $k > 0$ may ensure the stability for sufficiently "small" delays [1, 24]. Even for such a simple system, the behavior with respect to the pair (k, τ) is very *complicated*: if τ is thought of as a parameter, then for a fixed 'gain' k we may have a sequence of stability and instability regions in the parameter space (k, τ).

Although the *destabilizing effect* of a delay in a system model is well known in the control literature, see, e.g. the reference list in [24], the 'switch' from instability to stability (called also reversal, see, e.g. [8]) has not been sufficiently addressed. Note however that, to the best authors' knowledge, it was first discussed by Minorsky [23] in the 40s for a second-order (delayed) friction equation. Further comments and remarks on delayed oscillatory systems can also be found in [3, 12].

A different but related problem to *problem 1* may be stated as follows:
Problem 2: *Assume that (1.1) can be stabilized by a static output feedback, then we want to know how* robust *is the closed-loop stability with respect to the delay.*

It seems natural that, for 'small gains', it is possible to ensure the closed-loop stability for any delay value, i.e. stability is a *delay-independent* property, and for 'large gains', stability may be guaranteed only in the first-delay interval $[0, \overline{\tau}(K))$, i.e. it is a *delay-dependent* property. The existence of *other delay intervals* guaranteeing stability in the closed-loop system is also analyzed in this paper, along with several robust-stability existence results.

The proposed approach is based on generalized eigenvalues distribution with respect to the unit circle of some appropriate matrix pencils associated to the system. The use of matrix pencils for characterizing the existence of static output feedback controllers for delay-free systems was already considered in the literature [18, 9]. Some constructive procedures were resented in [9]. Furthermore, matrix pencils techniques

[1]The research of S. I. Niculescu is partially supported by NSF-CNRS "Delay effects" Grant, 1999-2002
[2]The research of C.T. Abdallah is partially supported by NSF-CNRS "Delay effects" Grant, NSF INT-9818312

in the stability of time-delay systems is intimately related to the development of $2D$ stability analysis [6] in the commensurate delay case [22, 16] (variables on the imaginary axis, and on the unit circle in the complex plane), and to the *linearization* of some matrix polynomials [19, 20] after 'reducing' one of the variables [10, 24, 25, 26] (the variable on the imaginary axis). Note that such variable-reducing ideas were already encountered in the 1960s in [2] in a different framework.

Due to the difficulty of the general design problem, we shall analyze in this paper the analysis problem by considering delay-intervals ensuring closed-loop stability for a given, known K. Some remarks on the case $K = kI_m$ $(m = p)$ with k free are also included using the above approach, but also the MIMO Nyquist theorem. The notations are standard, except when otherwise noted.

2 Preliminary Results and Definitions

For a given $K \in \mathbf{R}^{\mathbf{m}\times\mathbf{m}}$, system (1.1), and the corresponding controller $u(t) = Ky(t-\tau)$, define the following matrix pencils $\Sigma_1 \in \mathbf{C}^{\mathbf{2p}\times\mathbf{2p}}$, $\Sigma_2 \in \mathbf{C}^{\mathbf{n}\times\mathbf{n}}$:

$$\begin{aligned}\Sigma_1(z,K) &= z\begin{bmatrix} I_p & 0 \\ 0 & \phi_\otimes(BKC, I_n)\end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & -I_p \\ \phi_\otimes(I_n,(BKC)^T) & \phi_\oplus(A,A^T)\end{bmatrix} \quad (2.2)\\ \Sigma_2(z,K) &= zBKC + A. \quad (2.3)\end{aligned}$$

where $\phi_\otimes, \phi_\oplus : \mathbf{R}^{\mathbf{n}\times\mathbf{n}} \times \mathbf{R}^{\mathbf{n}\times\mathbf{n}} \mapsto \mathbf{R}^{\mathbf{p}\times\mathbf{p}}$ are defined as follows: For all $P, Q \in \mathbf{R}^{\mathbf{n}\times\mathbf{n}}$,

$$p = n^2 : \quad \begin{cases} \phi_\otimes(P,Q) = P\otimes Q, \\ \phi_\oplus(P,Q) = P\oplus Q. \end{cases} \quad (2.4)$$

$$p = \frac{n(n-1)}{2} : \quad \begin{cases} \phi_\otimes(P,Q) = P\tilde{\otimes}Q, \\ \phi_\oplus(P,Q) = P\tilde{\oplus}Q. \end{cases} \quad (2.5)$$

The symbols $\otimes$ and $\oplus$ are the product and the sum of Kronecker respectively, and the symbols $\tilde{\otimes}$ and $\tilde{\oplus}$ are defined as follows [29]:

$$P\tilde{\otimes}Q = [c_{ij}] \in \mathbf{R}^{\mathbf{p}\times\mathbf{p}}, \quad \text{where}$$

$$c_{ij} = \frac{1}{2}\left(p_{i_1j_1}q_{i_2j_2} + p_{i_2j_2}q_{i_1j_1} - p_{i_2j_1}q_{i_1j_2} - p_{i_1j_2}q_{i_2j_1}\right),$$

with (i_1, i_2) the ith pair of the sequence

$$(1,2),\ (1,3),\ \ldots(1,n),\ (2,3),\ \ldots(2,n),\ \ldots(n,n)$$

and (j_1, j_2) is generated by duality. For $P\tilde{\oplus}Q$, we use the classical definition of the Kronecker sum:

$$P\tilde{\oplus}Q = P\tilde{\otimes}I_n + I_n\tilde{\otimes}Q.$$

Using the same arguments as in [24, 25], we may prove the following result:

Lemma 1 *The following are true:*
1) The complex number $z \in \mathbf{C}^*$, $|z| \neq 1$ is a generalized eigenvalue of the matrix pencil Σ_1 if and only if z^{-1} is an eigenvalue of Σ_1.
2) All the generalized eigenvalues on the unit circle of the matrix pencil Σ_2 are also eigenvalues on the unit circle for Σ_1.

3 Existence Results

Based on the continuity property with respect to the delay of the roots of the closed-loop characteristic equation [14], we may prove several *existence* stability results.
Denote $\sigma_i(K) = \sigma(\Sigma_i(z,K))$ $(i = \overline{1,2})$ the set of generalized eigenvalue of the matrix pencil $\Sigma_i(z,K)$, and let $\sigma(K) = \sigma_1(K) - \sigma_2(K)$ denote the set ofpoints in $\sigma_1(K)$ but not in $\sigma_2(K)$. Assume also that Σ_i are regular matrix pencils (see [28] for the exact meaning of this assumption).

Then we have the following results (see [28] for the complete proofs):

Proposition 1 (delay-independent) *Consider a gain matrix K such that Σ_1 is regular, and such that $\sigma = \emptyset$ on the unit circle of the complex plane. Then, the following assertions are equivalent:*
i) The static output feedback $u(t) = Ky(t)$ is a stabilizing law for (1.1);
ii) The static output feedback $u(t) = Ky(t-\tau)$ is a stabilizing law for (1.1) for any delay value τ.

Remark 1 (Generalized eigenvalue distribution) *Note that if the matrix pencil Σ_1 has no eigenvalues on the unit circle, then it is* dichotomically separable *with respect to the unit circle [25, 24]. Such condition guarantees $\sigma = \emptyset$, but it is* not sufficient *to guarantee the delay-independent type property, as remarked in [25]. In fact, Σ_1 may have generalized eigenvalues on the unit circle, but these eigenvalues should be identical to those of Σ_2.*

Remark 2 (Strong/weak delay-independent) *In terms of system's parameters, the* common *generalized eigenvalues of Σ_1 and Σ_2 ($\sigma(K)$ should be non-empty) generate* hypersurfaces *in the parameter-space, which have to be included in the corresponding* stability regions, *since they are related, in some sense, to the case $\tau \to +\infty$. Such aspect was pointed out in [7], and exploited in [24]. One may differentiate* strong *and* weak *delay-independent stability notions, by including or not the corresponding hypersurfaces in the stability regions [24] (and the references therein). In this framework, the results proposed in [22] are* strong, *and the results in [16] are* weak delay-independent, *etc.*
Note that the strong *stability notion allows a* complete decoupling *of the complex variables between those on the imaginary axis, and on the unit circle, respectively. The* weak *stability notion allows that* 0 *is an* accumulation point, *in some sense, for generalized eigenvalues (continuously depending on the delay values in the Datko's sense [14]), if the delay $\tau \to +\infty$. Such problems are better explained in a* hyperbolicity *framework (see also [25]).*

Remark 3 *In conclusion, if $\sigma(K) = \emptyset$ on the unit circle of the complex plane, the static delayed output feedback does not improve the closed-loop stability with respect to the delay-free case. The problem is* reduced *to a* static output feedback problem*[31].*

Proposition 2 (first-delay interval) *Consider a gain matrix K such that Σ_1 is regular, and such that $\sigma \neq \emptyset$ on the unit circle of the complex plane. Then, the following assertions are equivalent:*
i) The static output feedback $u(t) = Ky(t)$ is a stabilizing law for (1.1);
ii) The static output feedback $u(t) = Ky(t-\tau)$ is a stabilizing law for (1.1) for any delay value, $\tau \in [0, \overline{\tau}(K))$, where:

$$\begin{aligned} \overline{\tau}(K) &= \min_{1\le i\le n} \min_{1\le l\le p} \frac{\alpha_j}{\omega_{il}}, \\ & e^{-\alpha_j} \in \sigma_1(K), \quad j\omega_{il} \in \sigma(A + e^{j\alpha_{il}}(BKC)) \end{aligned}$$

Consider the closed-loop system of the strictly proper transfer function $H(s) \in \mathbf{C}^{\mathbf{p}\times\mathbf{m}}$ with the feedback $u(t) = Ky(t-r)$ for some gain $K \in \mathbf{R}^{\mathbf{m}\times\mathbf{p}}$, and some positive $r > 0$. Introduce now the sets:

$$\begin{aligned} \Lambda_{r,+,K} &= \Big\{(\tau_{k_i}, \alpha_k) : \tau_{k_i} = \frac{\alpha_k}{\omega_{ki}} > r : \\ & e^{-j\alpha_k} \in \sigma(K), \\ & j\omega_{ki} \in \sigma\left(A + e^{-jh\alpha_k} BKC\right) - \{0\}, \\ & 1 \le k \le 2p,\ 1 \le i \le n\Big\}, \qquad (3.6) \\ \Lambda_{r,-,K} &= \Big\{(\tau_{k_i}, \alpha_k) : \tau_{k_i} = \frac{\alpha_k}{\omega_{ki}} < r : \\ & e^{-j\alpha_k} \in \sigma(K), \\ & j\omega_{ki} \in \sigma\left(A + e^{-jh\alpha_k} BKC\right) - \{0\}, \\ & 1 \le k \le 2p,\ 1 \le i \le n\Big\}. \qquad (3.7) \end{aligned}$$

The main (existence) result may be written as follows:

Proposition 3 (general delay-intervals) *The strictly proper transfer function $H(s)$ can be stabilized by delayed output feedback of the form $u(t) = Ky(t-\tau)$ on the delay interval $(\underline{\tau}, \overline{\tau})$ if and only if:*
i) it can be stabilized by the same law for some delay τ_0 in the same interval, and
ii) the following inequalities hold simultaneously:

$$\begin{aligned} inf\,\{\tau : (\tau, \alpha) \in \Lambda_{\tau_0,-,K}\} &\le \underline{\tau}, \qquad (3.8) \\ sup\,\{\tau : (\tau, \alpha) \in \Lambda_{\tau_0,+,K}\} &\ge \overline{\tau}. \qquad (3.9) \end{aligned}$$

Furthermore, based on [13], we may prove the following general result:

Proposition 4 (instability persistence) *Let K be a real matrix, such that:*
a) the set $\sigma(K)$ is not empty, and
b) the imaginary axis eigenvalues of the complex matrix $A + BKz(K)$ where $z(K) \in \sigma(K)$ are simple.
Then:
i) there exists at least one delay interval $(\underline{\tau}(K), \overline{\tau}(K))$ such that the control law $u(t) = Ky(t-\tau)$ is a stabilizing output feedback for the transfer function $H(s) \in \mathbf{C}^{\mathbf{m}\times\mathbf{m}}$ for any delay $\tau \in (\underline{\tau}(K), \overline{\tau}(K))$, and
ii) there exists a positive τ_{max}, such that for all $\tau \ge \tau_{max}$, the closed-loop system is unstable.
Furthermore, when the delay τ varies from 0 to τ_{max}, at most a finite number of stability switches may occur.

4 Constructing delayed output feedbacks

Consider now the case $m = p$, with the output feedback $u(t) = Ky(t-\tau)$, where $K = kI_m$ for some real k, that is only *one parameter* to compute, and the problem becomes simpler due to the fact that the parameter space (k, τ) can be graphically represented. We shall redefine $\Sigma_{1,2}$ with respect to k, that is:

$$\begin{aligned} \Sigma_1(z,k) &= z \begin{bmatrix} I_p & O \\ 0 & \phi_\otimes(BC, I_n) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & -I_p \\ k^2\phi_\otimes(I_n, (BC)^T) & \phi_\oplus(A, A^T) \end{bmatrix} \qquad (4.10) \\ \Sigma_2(z,k) &= zBC + A. \qquad (4.11) \end{aligned}$$

In this case, the variable z will be on the circle $|z| = k$, so we will have a family of circles in $\mathbf{C}$. All the existence problems, may be rewritten in the *new variable* k. The idea behind such transformations is to have an *invariant* matrix pencil, and to use it to define some "bands" in the complex plane (see comments in Remark 1).

An *algorithm* for the delayed output feedback may be stated as follows:

i) first, compute the generalized eigenvalues $\lambda_{i,2}$ of Σ_2, and next compute $k_{i,2} = |\lambda_{i,2}|$, for all $i = \overline{1,2}$. These $k_{i,2}$ values may define the complex plane circles for which one may have *delay-independent* type results in closed-loop.

ii) next, compute the corresponding generalized eigenvalues $\lambda_{j,1}$ of Σ_1 for k taking values in the set: $\{k_{i,2}\}$, and next compute $\sigma(k_{i,2})$ with respect to the complex circle of radius $k_{i,2}$.

Then, we may apply the *existence results* given above.

Remark 4 *It is clear that a direct analysis of the generalized eigenvalues of Σ_1 with respect to some arbitrarily fixed k leads to a very difficult problem, since we such eigenvalues have no simple dependence on the parameter k. Furthermore, working directly with $|z| = 1$ gives no particular choice on the parameter k.*

A different analysis can be done using directly the MIMO Nyquist theorem. The corresponding result can be resumed as follows (see [28] for the proof):

Theorem 1 *Let $H(s)$ be a square transfer matrix with P_0 unstable Smith-Macmillan poles. Let $\lambda_i(s); i = 1, \cdots, m$ be the eigenvalues of $G(s)$. Then, the closed-loop system with feedback input $u = -Ky(t-\tau_0)$*

with $K = kI$ is stable if and only if the graphs of $e^{-s\tau_0}\lambda_i(s); i = 1, \cdots, n$ taken together encircle the -1 point, P_0 times in the counterclockwise direction.

5 Examples

We shall apply the previous results to three different examples.

5.1 Stabilizing oscillations using delays

Consider a simple second-order oscillatory system [1]:

$$\ddot{y}(t) + \omega_0^2 y(t) = u(t), \tag{5.12}$$

with $\omega_0, \in \mathbf{R}^*$. As specified, it is not possible to stabilize it by static delayed output feedback of the form:

$$u(t) = ky(t-\tau), \tag{5.13}$$

By taking

$$A = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}, \; bkc^T = \begin{bmatrix} 0 & 0 \\ +\frac{k}{\omega_0} & 0 \end{bmatrix}. \tag{5.14}$$

we may apply directly the proposed approach, and thus, we have the following stabilization result:

Proposition 5 *The system (5.12) can be stabilized by delayed output feedback $u(t) = ky(t-\tau)$ for all the pairs (k, τ) satisfying simultaneously:*
i) the gain $k \in (0, \omega_0^2)$, and
ii) the delay $\tau \in (\underline{\tau}_i(k), \overline{\tau}_i(k))$ where:

$$\begin{cases} \underline{\tau}_i(k) = \dfrac{2i\pi}{\sqrt{\omega_0^2 - k}} \\ \overline{\tau}_i(k) = \dfrac{(2i+1)\pi}{\sqrt{\omega_0^2 + k}} \end{cases},$$

for $i = 0, 1, \ldots$. Furthermore, if $\tau = \underline{\tau}_i(k)$ or $\tau = \overline{\tau}_i(k)$, the corresponding characteristic equation in closed-loop has at least one eigenvalue on the imaginary axis.
The regions of stabilizing k shrink as the delay τ gets larger, and furthermore for each k there exists a value $\tau^*(k)$, such that for any $\tau > \tau^*(k)$ the closed-loop system is unstable.

So, if we take $i = 0$, we see that the first delay interval guaranteeing the *closed-loop* stability is given by: $\tau \in \left(0, \frac{\pi}{\sqrt{\omega_0^2+k}}\right)$. Using a different argument to those already proposed in the paper, let us prove that for sufficiently small delays $\tau = \varepsilon > 0$ the closed-loop system is stable for any $k \in (0, \omega_0^2)$. Consider the characteristic equation associated to the system:

$$s^2 + \omega_0^2 - ke^{-s\tau} = 0. \tag{5.15}$$

If $\tau = 0$, the corresponding roots are on the imaginary axis $s = \pm j\omega_0$. Consider now a delay $\tau = \varepsilon > 0$. Simple computations give:

$$\frac{ds}{d\tau} = -\frac{s(s^2 + \omega_0^2)}{\varepsilon(s^2 + \omega_0^2) + 2s}. \tag{5.16}$$

Since the roots on the imaginary axis of (5.15) are *simple* $\omega = \sqrt{\omega_0^2 \pm k}$, it follows that the *crossing direction* of the roots (from left to right, or from right to left) is given by the $\mathbf{sgn}\left\{\mathbf{Re}\left(\frac{\mathbf{ds}}{\mathbf{d}\tau}\right)\right\}$ when $s = j\omega$: "+1" from stability to instability, and "-1" from instability to stability (Note that the condition on the simplicity of the roots is *necessary*, since if not the corresponding derivative will be 0, etc.).

Simple computations in (5.16) lead to the following:

$$Re\left(\frac{ds}{d\tau}\,|_{s=j\omega,\omega>0}\right) = -\frac{\omega^2(\omega_0^2 - \omega^2)}{4\omega^2 + \varepsilon^2(\omega_0^2 - \omega^2)^2},$$

which is *always negative*, for sufficiently small value $\varepsilon > 0$, and for any $k \in (0, \omega_0^2)$, etc.

Remark 5 *The same results can be obtained using different approaches, as for example, the study of the corresponding characteristic equation [21], or using Nyquist criterion [1].*

Remark 6 *A similar analysis can be done if we assume that $k < 0$. Note that for such situation the system is* still unstable *for* sufficiently small *delay values $\tau = \varepsilon > 0$, and any $k < 0$, $|\, k \,| < \omega_0^2$. However, it will be stabilized on some delay intervals, etc.*

5.2 Delay measurements in active displacement

In active displacement control (flexible structures), a time delay τ always exists between measuring the deflection and applying the active displacement feedback. Since the corresponding delay-free, closed-loop model is, in general, stable, the problem is to study the *delay effects* on the closed-loop stability, with respect to the two parameters: a) the (point or lumped) delay; b) the gain of the active displacement feedback.

Based on the study proposed in [30], the stability (instability) problem can be reduced to the analysis of the following transcendental equation:

$$s^2 + \mu_n s + \lambda_n + ke^{-s\tau} = 0, \tag{5.17}$$

with μ_n, λ_n (associated eigenvalues corresponding to some orthogonal eigenvectors of some self-adjoint operators, etc.) and k (displacement control feedback parameter) positive.
Some algebraic manipulations combined with the existence results given above lead to the following results:

Proposition 6 (Delay-independent results) *The following statements are equivalent:*
1) The system (5.17) is delay-independent asymptotically stable.
2) The parameters (μ_n, λ_n, k) satisfy the following con-

straints:

$$\begin{cases} \mu_n \in \mathbf{R}^+ \\ \lambda_n \in \mathbf{R}^+ \\ k \in \begin{cases} [-\lambda_n, \lambda_n], \\ \text{ if } \lambda_n \in (0, 2\mu_n^2] \\ (-2\mu_n\sqrt{\lambda_n - \mu_n^2}, 2\mu_n\sqrt{\lambda_n - \mu_n^2}), \\ \text{ if } \lambda_n \in (2\mu_n^2, +\infty) \end{cases} \end{cases} . \quad (5.18)$$

If the conditions above are not satisfied, the only possibility to have is a *delay-dependent* type result. We shall consider two cases: only one switch (with no reversal), and several switches and reversal. The first case is already encountered in the scalar case, and the second one appears firstly with the second order systems. The proposed results can be summarized as follows (the proofs are in the full version of the paper [28]):

Proposition 7 (Only one switch) *The following statements are equivalent:*
1) The system (5.17) is delay-dependent stable, and there exists only one switch from stability to instability without any reversal.
2) The parameters (μ_n, λ_n, k) satisfy the following constraints:

$$\begin{cases} \mu_n & \in & \mathbf{R}^+ \\ \lambda_n & \in & (2\mu_n^2, +\infty) \\ k & \in & (-\infty, -\lambda_n] \cup [\lambda_n, +\infty) \end{cases} . \quad (5.19)$$

Proposition 8 (Delay bound) *Consider the second order system (5.17), and assume $k > 0$.*
If the parameters (k, μ_n, λ_n) satisfy the constraints (5.19), then the system (5.17) is asymptotically stable for all delays $\tau \in (0, \tau_{switch})$, and unstable for any $\tau > \tau_{switch}$, where: $\tau_{switch} =$

$$\frac{1}{\omega_+} arccos\left(\frac{\sqrt{k^2 + 4\mu_n^4 - 4\mu_n^2\lambda_n} - 2\mu_n^2}{k}\right). \quad (5.20)$$

with

$$\omega_+ = \sqrt{\lambda_n - 2\mu_n^2 + \sqrt{k^2 + 4\mu_n^4 - 4\mu_n^2\lambda_n}}.$$

At $\tau = \tau_{switch}$, the characteristic equation has two complex conjugate eigenvalues on the imaginary axis.

Proposition 9 (several switches and reversals)
The following statements are equivalent:
1) The system (5.17) is delay-dependent stable, and there exists several switches and reversals.
2) The parameters (μ_n, λ_n, k) satisfy the following constraints:

$$\begin{cases} \mu_n \in \mathbf{R}^+ \\ \lambda_n \in (2\mu_n^2, +\infty) \\ k \in (-\lambda_n, -2\mu_n\sqrt{\lambda_n - \mu_n^2}) \cup \\ \quad \cup (2\mu_n\sqrt{\lambda_n - \mu_n^2}, \lambda_n) \end{cases} . \quad (5.21)$$

The *first switch* from stability to instability, and the *first reversal* from instability to stability are given by:

Proposition 10 (Delay bounds) *Consider the second order system (5.17), and assume $k > 0$.*
If the parameters (k, μ_n, λ_n) satisfy the constraints (5.21), then the system (5.17) is asymptotically stable for all delays $\tau \in (0, \tau_{switch})$, and unstable for all delays $\tau \in (\tau_{switch}, \tau_{reversal})$, where the delay switch *τ_{switch} is given by:*

$$\frac{1}{\omega_+} arccos\left(\frac{\sqrt{k^2 + 4\mu_n^4 - 4\mu_n^2\lambda_n} - 2\mu_n^2}{k}\right), \quad (5.22)$$

and the delay reversal *$\tau_{reversal}$ is given by:*

$$\frac{1}{\omega_-} arccos\left(\frac{-\sqrt{k^2 + 4\mu_n^4 - 4\mu_n^2\lambda_n} - 2\mu_n^2}{k}\right) \quad (5.23)$$

with:

$$\omega_\pm = \sqrt{\lambda_n - 2\mu_n^2 \pm \sqrt{k^2 + 4\mu_n^4 - 4\mu_n^2\lambda_n}} \quad (5.24)$$

At $\tau \in \{\tau_{switch}, \tau_{reversal}\}$, the characteristic equation has two complex conjugate eigenvalues on the imaginary axis.

Remark 7 *If we take $\mu_n = 0$, we recover the bounds proposed in §4.1 (stabilizing oscillations).*

5.3 Integrodifferential models for commodity markets

In [4], the following distributed-discrete delay model $(R, Q > 0)$:

$$\dot{x}(t) + \frac{Q}{R}\int_{-\infty}^{0} e^{\frac{\theta}{R}} x(t+\theta)d\theta + x(t-\tau) = 0 \quad (5.25)$$

has been used for describing interactions between consumer memory and price fluctuations on commodity markets. Simple computations prove that (5.25) has the same characteristic equation as the differential equation with *discrete delays*:

$$\ddot{x}(t) + \frac{1}{R}\dot{x}(t) + \dot{x}(t-\tau) + \frac{Q}{R}x(t) + \frac{1}{R}x(t-\tau) = 0, \quad (5.26)$$

which is a second-order delay differential system. The analysis of the stability regions in the parameters space (Q, R, τ) may be easily transformed to an *output static delay feedback problem*:

Given the system $\frac{y(s)}{u(s)} = \frac{s + \frac{1}{R}}{s^2 + \frac{1}{R}s + \frac{Q}{R}}$, *find all the delay-intervals* $(\underline{\tau_i}(Q,R), \underline{\tau_i}(Q,R))$, $i = 0, \ldots,$ *such that the closed-loop is stable via the output feedback* $u(t) = -y(t-\tau)$.

Using the results presented above the complete characterization of the stability regions becomes an easy task.

6 Conclusions

This paper presented a compilation of the effects of delay on the static output feedback problem. We have generalized our earlier reseacrh on the presence of stability "switches" using a matrix pencil approach and have laid the foundation for future work on multivariable static controllers. We have also provided numerous examples to illustrate the applicability of our results.

References

[1] Abdallah, C., Dorato, P., Benitez-Read and Byrne, R.: Delayed positive feedback can stabilize oscillatory systems. *Proc. American Contr. Conf.* (1993) 3106-3107.

[2] Ansell, H. C.: On certain two-variable generalizations of circuit theory, with applications to network of transmission lines and lumped reactances. *IEEE Trans. Circuit Theory*, **11** (1964) 214-223.

[3] Ansoff, H. I. and Krumhansl, J. A.: A general stability criterion for linear oscillationg systems with constant time lag. *Quart. Appl. Math.* **6** (1948) 337-341.

[4] Bélair, J. and Mackey, M. C.: Consumer memory and price fluctuations in commodity markets: an integrodifferential model. *J. Dyn. Diff. Eqs.* **1** (1989) 299-325.

[5] Bellman, R. E. and Cooke, K. L.: *Differential-Difference Equations* (Academic Press, New York, 1963).

[6] Bose, N. K.: *Applied multidimensional systems theory* (Van Nostrand Reinhold, New York, 1982).

[7] Boese, F. G.: Stability conditions for the general linear difference-differential equation with constant coefficients and one constant delay. *J. Math. Anal. Appl.*, **140** (1989) pp. 136-176.

[8] Brauer, F.: Characteristic return times for harvested population models with time lag. *Math. Biosciences* **45** (1979) 295-311.

[9] Chen, J.: Static output feedback stabilization for SISO systems and related problems: solutions via generalized eigenvalues. *Control - Theory and Advanced Tech.* **10** (1995) 2233-2244.

[10] Chen, J., Gu, C. and Nett, C. N.: A new method for computing delay margins for stability of linear delay systems, in *Proc. 33rd IEEE Conf. Dec. Contr.*, Lake Buena Vista, Florida, US, (1994) 433-437.

[11] Chung, L. L., Reinhor, A. M. and Soong, T. T.: "Experiments on active control of seismic structures," *ASCE J. Eng. Mech.* **114** (1988) 241-256.

[12] Cooke, K. L. and Grossman, Z.: Discrete delay, distributed delay and stability switches. *J. Math. Anal. Appl.* **86** (1982) 592-627.

[13] Cooke, K. L. and van den Driessche, P.: On zeroes of some transcendental equations. in *Funkcialaj Ekvacioj* **29** (1986) 77-90.

[14] Datko, R.: A procedure for determination of the exponential stability of certain differential-difference equations. *Quart. Appl. Math.* **36** (1978) 279-292.

[15] Hale, J. K.: Dynamics and delays. in *Delay differential equations and dynamical systems* (S. Busenberg, M. Martelli, Eds.), Springer-Verlag: Berlin, LNM (1991) 16-30.

[16] Hale, J. K., Infante, E. F. and Tsen, F. S. -P.: Stability in linear delay equations. in *J. Math. Anal. Appl.*, **105** (1985) 533-555.

[17] Hale, J. K. and Verduyn Lunel, S. M.: *Introduction to Functional Differential Equations* (Applied Math. Sciences, **99**, Springer-Verlag, New York, 1993).

[18] Helmke, U. and Anderson, B. D. O.: Hermitian pencils and output feedback stabilization of scalar systems, *Int. J. Contr.* **56** (1992) 857-876.

[19] Hertz, D., Jury, E. I. and Zeheb, E.: Stability independent and dependent of delay for delay differential systems. *J. Franklin Inst.*, **318** (1984) 143-150.

[20] Hertz, D., Jury, E. I. and Zeheb, E.: Root exclusion from complex polydomains and some of its applications. *Automatica*, **23** (1987) 399-404.

[21] Hsu, C. S. and Bhatt, S. J.: Stability charts for second-order dynamical systems with time lag. *J. Appl. Mech* (1966) 119-124.

[22] Kamen, E. W.: On the relationship between zero criteria for two-variable polynomials and asymptotic stability of delay differential equations. *IEEE Trans. Automat. Contr.*, **AC-25** (1980) 983-984.

[23] Minorsky, N.: Self-excited oscillations in dynamical systems possessing retarded actions. *J. Applied Mech.* **9** (1942) A65-A71.

[24] Niculescu, S. -I.: *Systèmes à retard. Aspects qualitatifs sur la stabilité et la stabilisation* (Diderot, Paris, Nouveaux Essais, 1997).

[25] Niculescu, S. -I.: Stability and hyperbolicity of linear systems with delayed state: A matrix pencil approach. *IMA J. Math. Contr. Information* **15** (1998) 331-347.

[26] Niculescu, S. -I.: Delay-interval stability and hyperbolicity of linear time-delay systems: A matrix pencil approach. in *Proc. 4th European Control Conference*, Brussels, Belgium, July 1997.

[27] Niculescu, S. -I., Verriest, E. I., Dugard L. and Dion, J. -M.: Stability and robust stability of time-delay systems: A guided tour. in *Stability and Control of Time-Delay Systems* (L. Dugard and E. I. Verriest, Eds.), LNCIS, Springer-Verlag, London, **228** (1997) 1-71.

[28] Niculescu, S. -I., and Abdallah, C.T.: Static Output Feedback in the Presence of Delay. *EECE Technical Report, EECE2000-002, In Preparation*, 2000.

[29] Qiu, L. and Davison, E. J.: The stability robustness determination of state space models with real unstructured perturbations. *Math. Contr. Signals, Syst.* **4** (1991) 247-267.

[30] Sloss, J. M., Sadek, I. S., Bruch Jr., J. C. and Adali, S.: "The effects of time delayed active displacement control on damped structures," *Control - Theory and Advanced Technology*, **10** (1995) 973-992.

[31] Syrmos, V., Abdallah, C.T., Dorato, P., and Grigoriadis, K.: Static Output feedback: A Survey, *Automatica*, vol. 33, No. 2, pp. 125–137, 1997.