Extended Argument Principle and Integral Design Constraints Part I: A Unified Formula for Classical Results

Jie Chen* Gang Chen† Zhang Ren† Li Qiu‡

Abstract

In this paper we study Bode and Poisson type integral relations. We call for attention to a link between the well-known argument principle and Bode and Poisson integrals, which seems to have been unnoticed previously. We show how various integral constraints may be unified under an extended version of the argument principle. This enables us to derive the classical Bode and Poisson integral relations in a simple manner, and further to discover new integral formulas of significance for analysis of control design limitation and tradeoff.

1 Introduction

Bode and Poisson type integrals are known to play a fundamental role in the characterization and understanding of limitation and tradeoff in the design of feedback control systems [6, 12]. Specifically, when applied to sensitivity and complementary sensitivity functions, they quantify analytically how the magnitudes of these functions are inherently constrained pointwisely in frequency, which in turn mandates a tradeoff of performance in different frequency ranges. It is well-understood that this tradeoff is a necessary consequence of various, and often conflicting design objectives, and that it can be further exacerbated by undesirable system characteristics such as nonminimum phase zeros and unstable poles. It is also accepted that understanding of this tradeoff can aid significantly in feedback design, lending important insight and guidelines transcendent over design methods. While a classical, well-studied subject, for its profound as well as ever-lasting impact, the study on Bode and Poisson integral relations, and more generally, investigation of performance limitation and design tradeoff issues, has in recent years become a revitalized area. Numerous results have been reported, pushing its boundary to a far broader domain, to, e.g., multivariable systems [1, 2, 3, 4, 14], sampled-data systems [7], nonlinear systems [13], and filtering and estimation problems [8, 11].

This paper continues to examine Bode and Poisson type integral relations, focusing specifically on linear time-invariant, single-input single-output systems. We show that there is an intimate connection between Bode and Poisson integrals and the well-known argument principle, or a slightly extended version of the latter; in fact, it will be seen that the Bode and Poisson type integral relations can all be obtained readily from the extended argument principle. This discovery thus brings to light a somewhat hidden connection unnoticed previously, which enables us to unify the argument principle and the integral relations in a single formula. This unification bears some significance. From a conceptual standpoint, as the argument principle is the very foundation for the Nyquist stability criterion, it links performance constraints to stability in a direct fashion, and consequently provides a unified treatment for these concepts; indeed, it becomes rather clear that stability may be interpreted as a special instance of performance objectives of interest. Similarly, since the Nyquist criterion is a mandatory topic in control curriculum, this connection makes it possible to introduce more advanced Bode integrals without further labor. The educational benefit is thus substantial.

More importantly, the generality of the extended argument principle motivates, and indeed paves way for search of additional integral constraints beyond those already known, which should then complement the classical results and enable a deeper and more refined analysis of performance limitation and design tradeoff issues. In Part II of this series, we will present several novel results in this spirit. The present Part I focuses instead on the known, classical results. Furthermore, we shall also develop mathematical results to be used throughout the series.

The rest of this paper is organized as follows. In Section 2, we provide a brief review of the argument principle and its extension. Furthermore, we elaborate on a number of its modifications and forms for application to the control context. Based on this analysis, we show in Section 3 that the Bode and Poisson integral relations can all be derived in a straightforward manner, as a special instance of the extended argument principle. Section 4 concludes our discussion.

* Department of Electrical Engineering, University of California, Riverside, CA 92521-0425. Tel: (909)787-3688. Fax: (909)787-2425. Email: jchen@ee.ucr.edu. Author to whom all correspondences should be addressed.

† Department of Electrical Engineering, University of California, Riverside, CA 92521-0425.

‡ Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.

2 Argument Principles

The standard argument principle is a mathematical result routinely found in typical control textbooks. We quote this result from [9].

Lemma 2.1 (The Argument Principle) *Let $f(s)$ be a meromorphic function in a simply connected domain D contained in a Jordan contour ∂D. Suppose that $f(s)$ has neither zero nor pole on ∂D, but has N_z zeros and N_p poles in D, all counting the multiplicities. Then,*

$$\frac{1}{2\pi j}\int_{\partial D}\frac{f'(s)}{f(s)}ds = N_z - N_p. \quad (1)$$

More generally, an extended version of the argument principle is found in, e.g., [9].

Lemma 2.2 (The Extended Argument Principle) *Let $f(s)$ be a meromorphic function in a simply connected domain D contained in a Jordan contour ∂D. Suppose that $f(s)$ has neither zero nor pole on ∂D, but has N_z zeros $z_i \in D$, $i = 1, \cdots, N_z$, and N_p poles $p_i \in D$, $i = 1, \cdots, N_p$, all counting the multiplicities. Then for any function $g(s)$ analytic on $\overline{D}$, where $\overline{D}$ is the closure of D,*

$$\frac{1}{2\pi j}\int_{\partial D}\frac{f'(s)}{f(s)}g(s)ds = \sum_{i=1}^{N_z} g(z_i) - \sum_{i=1}^{N_p} g(p_i). \quad (2)$$

Remark 2.1 In the context of control systems, the region D of interest is the right half of the complex plane or the exterior of the unit disc, which is the instability region for continuous-time or discrete-time systems. In the present paper we shall consider only continuous-time systems, and hence the right half plane. ■

In principle, it is possible to obtain all the existing integral relations directly from Lemma 2.2, by an elaborate selection of $f(s)$ and $g(s)$. In this regard, either $f(s)$ or $g(s)$ may be related to system transfer functions, depending on the integral relation under consideration; we shall demonstrate this point in the sequel. It will also be possible to restrict $f(s)$ alone to be a transfer function of interest, while taking $g(s)$ as a weighting function. The latter will prove to be more straightforward, but requires more technical deliberation. In particular, in view of the variety of the integral relations, it will be necessary to modify Lemma 2.2, so that more general functions $f(s)$ and $g(s)$ may be accommodated; for example, $g(s)$ may have singularities on ∂D and in D. For this need we develop below a number of versions of the generalized argument principle. The first result addresses the case where $g(s)$ has singularities on the imaginary axis but not in the open right half plane (ORHP). The second result examines $g(s)$ that has singularities in ORHP but not on the imaginary axis.

We denote the right half plane by $\mathbb{C}_+$ and its closure $\overline{\mathbb{C}}_+$. For $R > 0$, we shall also consider the semi-circle $C_R := \left\{z : \ z = Re^{j\theta}, \ \theta \in \left[-\frac{\pi}{2}, \ \frac{\pi}{2}\right]\right\}$. For a complex number s, its conjugate is denoted by $\overline{s}$ and its real and imaginary part by, $Im(s)$ and $Re(s)$ respectively. Furthermore, we make the following assumptions.

A 2.1 *$f(s)$ is meromorphic in $\mathbb{C}_+$, which does not have zero or pole on the imaginary axis.*

A 2.2 $\forall\theta \in [-\frac{\pi}{2}, \ \frac{\pi}{2}]$,
$$\lim_{R\to+\infty} R\left|g'(Re^{j\theta})\log f(Re^{j\theta})\right| = 0.$$

A 2.3 *At any singularity s_0 of $g(s)$ on $\overline{\mathbb{C}}_+$,*
$$\lim_{s\to s_0}(s - s_0)g'(s)\log f(s) \ \ \textit{exists.}$$

A 2.4 *$f(s)$ satisfies the conjugate symmetry property,* $\overline{f(s)} = f(\overline{s})$.

A 2.5 *$g(s)$ is odd on the imaginary axis,* $g(j\omega) = -g(-j\omega)$.

Here in making Assumption A 2.1, we intend to take $f(s)$ to be a certain system transfer function. The assumption can then be imposed with no loss of generality; indeed, it is a well-known fact that the imaginary zeros and poles have no effect on the known integral relations. The other assumptions, A 2.2-2.5, are also rather general and are widely applicable. In fact, A 2.3 is necessary for relevant integrals to be well-defined, and A 2.2, A 2.4 are standard in the context of control systems.

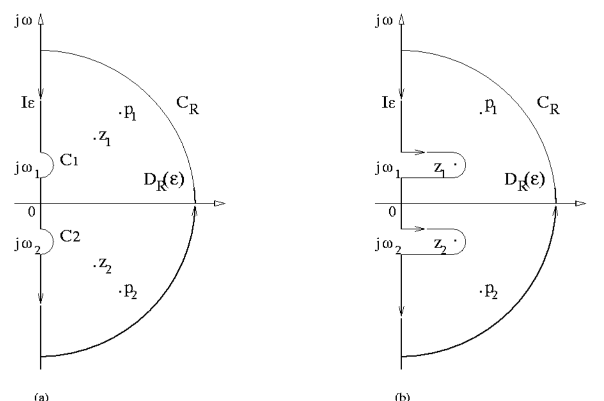

Figure 1: Integral contour $\partial D_{R(\epsilon)}$

Theorem 2.1 *Let $f(s)$ be a meromorphic function in $\mathbb{C}_+$. Suppose that $f(s)$ has N_z zeros $z_i \in \mathbb{C}_+$, $i = 1, \cdots, N_z$, and N_p poles $p_i \in \mathbb{C}_+$, $i = 1, \cdots, N_p$, all counting the multiplicities. Suppose also that $g(s)$ has N_s singularities at $j\omega_i$, $i = 1, 2, \cdots, N_s$, but is analytic in $\mathbb{C}_+$. Then under assumptions A 2.1-2.5,*

$$\int_{-\infty}^{+\infty} g'(j\omega)\log f(j\omega)d\omega$$

$$= 2\pi\left(\sum_{i=1}^{N_z} g(z_i) - \sum_{i=1}^{N_p} g(p_i)\right) - \pi\sum_{i=1}^{N_s}\gamma_i, \quad (3)$$

where $\gamma_i := \lim_{s\to j\omega_i}(s - j\omega_i)g'(s)\log f(s)$.

Proof. Under A 2.4 and A 2.5, the zeros and poles of $f(s)$ appear in conjugate pairs, so do the singularities of $g(s)$. It then suffices to prove the case that $f(s)$ has two simple poles p_1, $p_2 = \bar{p}_1$, and two simple zeros z_1, $z_2 = \bar{z}_1$ in ORHP, and $g(s)$ has two singularities $j\omega_1$, $j\omega_2 = -j\omega_1$. The proof will be divided into two steps. The first step will be to consider the situation where the imaginary parts of z_i and p_i do not coincide with the singularities of $g(s)$. Step 2 addresses the case when they do coincide, thus completing in full the proof.

Step 1: Consider the contour in Figure 1(a), where semicircles C_1 and C_2 avoid the singularities of $g(s)$. Let the radius R of C_R be sufficiently large and the radius ϵ of C_i be sufficiently small. Let I_ϵ consist of the remaining parts of the imaginary axis, with direction from jR to $-jR$. The whole indented region is denoted by $D_R(\epsilon)$, whose boundary $\partial D_R(\epsilon)$ is positively oriented.

We may apply directly Lemma 2.2 to $f(s)$ and $g(s)$,

$$\frac{1}{2\pi j}\int_{\partial D_R(\epsilon)} \frac{f'(s)}{f(s)} g(s) ds = \sum_{i=1}^{2} \left(g(z_i) - g(p_i)\right).$$

For the logarithmic function $\log f(s)$, z_i and p_i, $i = 1,\ 2$, are branch points. Let us make the branch cuts from these points to the left horizontally. Consider any $s \in \mathbb{C}_+$ such that $\log f(s)$ and $g(s)$ are analytic. Since

$$\frac{d[g(s)\log f(s)]}{ds} = \frac{f'(s)}{f(s)} g(s) + g'(s)\log f(s),$$

we have

$$\begin{aligned}
&\int_{\partial D_R(\epsilon)} \frac{f'(s)}{f(s)} g(s) ds \\
=\ & \int_{\partial D_R(\epsilon)} d[g(s)\log f(s)] - \int_{\partial D_R(\epsilon)} g'(s)\log f(s) ds.
\end{aligned}$$

Thus the integral of $g'(s)\log f(s)$ on I_ϵ gives

$$\begin{aligned}
& -\int_{I_\epsilon} g'(s)\log f(s) ds \\
=\ & 2\pi j \sum_{i=1}^{2} \left(g(z_i) - g(p_i)\right) - \int_{\partial D_R(\epsilon)} d[g(s)\log f(s)] \\
+\ & \sum_{i=1}^{2} \int_{C_i} g'(s)\log f(s) ds + \int_{C_R} g'(s)\log f(s) ds.
\end{aligned}$$

Under assumptions 2.2-2.3, it is straightforward to show that

$$\lim_{R\to+\infty} \int_{C_R} g'(s)\log f(s) ds = 0,$$
$$\lim_{\epsilon\to 0} \int_{C_i} g'(s)\log f(s) ds = -\pi j \gamma_i,$$

where $\gamma_i := \lim_{s\to j\omega_i} (s - j\omega_i) g'(s)\log f(s)$. Under assumptions A 2.4 and 2.5, as $R \to +\infty$,

$$\begin{aligned}
& \int_{\partial D_R(\epsilon)} d[g(s)\log f(s)] \\
=\ & \sum_{i=1}^{2} \int_{\partial D_R(\epsilon)} d[g(s)\log(s - z_i) - g(s)\log(s - p_i)] \\
=\ & 2\pi j \sum_{i=1}^{2} [g(jIm(z_i)) - g(jIm(p_i))] \\
=\ & 0.
\end{aligned}$$

We may then obtain (3) by taking the limit with $R \to +\infty$ and $\epsilon \to 0$.

Step 2: For simplicity, we will next assume the imaginary parts of z_1 and z_2 coincide with $j\omega_1$ and $j\omega_2$ respectively. The same derivation holds in general. We construct the close contour $\partial D_R(\epsilon)$ as in Figure 1(b). Note that the path C_i now differs. Following the same procedure as in Step 1, we get

$$\begin{aligned}
& \int_{-\infty}^{+\infty} g'(j\omega)\log f(j\omega) d\omega \\
& = -2\pi \sum_{i=1}^{2} g(p_i) + \lim_{\epsilon\to 0} \frac{1}{j} \sum_{i=1}^{2} \int_{C_i} g'(s)\log f(s) ds.
\end{aligned}$$

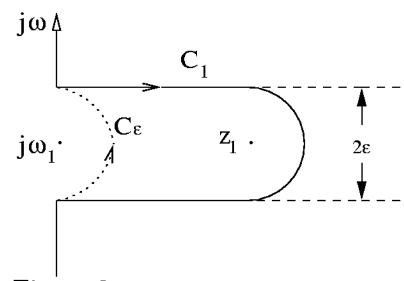

Figure 2: Indentation at $j\omega_1$ and z_1

Consider the integral on C_1, shown in detail in Figure 2. Let C_ϵ be the small positively oriented semicircle depicted by dotted curve. As in Step 1,

$$\begin{aligned}
& \int_{C_1} g'(s)\log f(s) ds \\
=\ & [g(s)\log f(s)]|_{j\omega_1+j\epsilon}^{j\omega_1-j\epsilon} - \int_{C_1} \frac{f'(s)}{f(s)} g(s) ds \\
=\ & [g(s)\log f(s)]|_{j\omega_1+j\epsilon}^{j\omega_1-j\epsilon} + 2\pi j g(z_1) + \int_{C_\epsilon} \frac{f'(s)}{f(s)} g(s) ds.
\end{aligned}$$

The integral of $\frac{f'(s)}{f(s)} g(s)$ on C_ϵ yields

$$\begin{aligned}
& \int_{C_\epsilon} \frac{f'(s)}{f(s)} g(s) ds \\
=\ & [g(s)\log f(s)]|_{j\omega_1-j\epsilon}^{j\omega_1+j\epsilon} - \int_{C_\epsilon} g'(s)\log f(s) ds.
\end{aligned}$$

Consequently,

$$\int_{C_1} g'(s)\log f(s)ds = 2\pi j g(z_1) - \pi j \gamma_1.$$

The proof is thus completed by applying the same derivation to the integral on C_2. ■

One special case of Theorem 2.1 arises when $f(s)$ is analytic on $\overline{\mathbb{C}}_+$. This corresponds to applications where $f(s)$ is a stable transfer function.

Corollary 2.1 *Let $f(s)$ be a meromorphic function in $\mathbb{C}_+$. Suppose that $f(s)$ has N_z zeros $z_i \in \mathbb{C}_+$, $i = 1, \cdots, N_z$, all counting the multiplicities, but has no poles in $\mathbb{C}_+$. Suppose also that $g(s)$ has N_s singularities at $j\omega_i$, $i = 1, 2, \cdots, N_s$. Then under assumptions A 2.1-2.5,*

$$\int_{-\infty}^{+\infty} g'(j\omega)\log f(j\omega)d\omega = 2\pi\sum_{i=1}^{N_z} g(z_i) - \pi\sum_{i=1}^{N_s}\gamma_i, \tag{4}$$

where $\gamma_i := \lim_{s\to j\omega_i}(s - j\omega_i)g'(s)\log f(s)$.

More generally, we shall also encounter the scenario where $g(s)$ has singularities in $\mathbb{C}_+$. The following corollary can be established by mimicking the proof of Theorem 2.1, provided that $g(s)$ does not have singularities on the imaginary axis. For simplicity we state the result only for $f(s)$ analytic on $\overline{\mathbb{C}}_+$. The ORHP poles of $f(s)$, nevertheless, can evidently be accounted for. Note here that since $g(s)$ is well defined on the imaginary axis, assumptions A 2.4 and A 2.5 can be relaxed.

Corollary 2.2 *Let $f(s)$ be analytic in $\mathbb{C}_+$. Suppose that $f(s)$ has N_z zeros $z_i \in \mathbb{C}_+$, $i = 1, \cdots, N_z$, all counting the multiplicities. Suppose also that $g(s)$ has singularities at $s_i \in \mathbb{C}_+$, $i = 1, 2, \cdots, N_s$, which are different from the zeros z_i of $f(s)$. Under assumptions A 2.1-2.3, the following statements are true.*

1. If the imaginary parts of $z_i, i = 1, \cdots, N_z$ do not coincide with those of s_i, $i = 1, \cdots, N_s$, then

$$\begin{aligned} &\int_{-\infty}^{+\infty} g'(j\omega)\log f(j\omega)d\omega \\ =\;& 2\pi\sum_{i=1}^{N_z} g(z_i) - 2\pi\sum_{i=1}^{N_z} g\left(jIm(z_i)\right) - 2\pi\sum_{i=1}^{N_s}\gamma_i; \end{aligned} \tag{5}$$

2. Suppose that z_i have the same imaginary parts as those of s_i, $i = 1, \cdots, m$, for some $m \le N_z$. Then,

$$\begin{aligned} &\int_{-\infty}^{+\infty} g'(j\omega)\log f(j\omega)d\omega \\ =\;& 2\pi\sum_{i=1}^{N_z} g(z_i) - 2\pi\sum_{i=m+1}^{N_z} g\left(jIm(z_i)\right) - 2\pi\sum_{i=1}^{N_s}\gamma_i, \end{aligned} \tag{6}$$

where $\gamma_i := \lim_{s\to s_i}(s - s_i)g'(s)\log f(s)$.

The following theorem is also immediate from Lemma 2.2 under a related, albeit different, assumption:

A 2.6 $\forall\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \lim_{R\to+\infty} R\left|\frac{f'(Re^{j\theta})}{f(Re^{j\theta})}g(Re^{j\theta})\right| = 0.$

Theorem 2.2 *Let $f(s)$ be a meromorphic function in $\mathbb{C}_+$. Suppose that $f(s)$ has N_z zeros $z_i \in \overline{\mathbb{C}}_+$, $i = 1, \cdots, N_{I_z}, \cdots N_z$, and N_p poles $p_i \in \overline{\mathbb{C}}_+$, $i = 1, \cdots, N_{I_p}, \cdots, N_p$, all counting the multiplicities, in which z_i, $i = 1, \cdots, N_{I_p}$, and p_i, $i = 1, \cdots, N_{I_z}$, are on the imaginary axis. Then under assumption A 2.6, whenever $g(s)$ is analytic in $\overline{\mathbb{C}}_+$,*

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{f'(j\omega)}{f(j\omega)} g(j\omega)d\omega &= -\pi\left(\sum_{i=1}^{N_{I_z}} g(z_i) - \sum_{i=1}^{N_{I_p}} g(p_i)\right) \\ &\quad -2\pi\left(\sum_{i=N_{I_z}+1}^{N_z} g(z_i) - \sum_{i=N_{I_p}+1}^{N_p} g(p_i)\right). \end{aligned} \tag{7}$$

It is then clear that if $f(s)$ and $g(s)$ satisfy the additional assumptions A 2.4-2.5, then the integral on the left hand side of (7) will not be affected by the imaginary zeros and poles of $f(s)$. This is the case with all known Bode and Poisson integrals.

Proof. It suffices to first construct a closed contour as in Figure 1(a), with indentations at the imaginary poles of $\frac{f'(s)}{f(s)}$, or equivalently, at the imaginary zeros and poles of $f(s)$. The proof then proceeds similarly, by noting that

$$\frac{1}{2\pi j}\int_{\partial D_R(\epsilon)} \frac{f'(s)}{f(s)} g(s)ds = \sum_{i=N_{I_z}+1}^{N_z} g(z_i) - \sum_{i=N_{I_p}+1}^{N_p} g(p_i),$$

and that

$$\begin{aligned} \lim_{R\to+\infty}\int_{C_R} \frac{f'(s)}{f(s)} g(s)ds &= 0, \\ \lim_{\epsilon\to 0}\int_{C_{z_i}} \frac{f'(s)}{f(s)} g(s)ds &= -\pi j g(z_i), \\ \lim_{\epsilon\to 0}\int_{C_{p_i}} \frac{f'(s)}{f(s)} g(s)ds &= \pi j g(p_i), \end{aligned}$$

where C_{z_i} denotes the semicircle around the imaginary zero z_i, with radius ϵ, and C_{p_i} the semicircle with respect to the imaginary pole p_i. ■

In closing, we point out that each of Theorem 2.1, its corollaries, and Theorem 2.2 can be utilized to derive integral constraints on system transfer functions, and each may be more advantageous than the other depending on the instance.

3 Bode/Poisson Integrals

In this section we demonstrate how Bode and Poisson type integral relations may be readily derived using the extended argument principle. We will accomplish this goal using all Corollary 2.1, 2.2 and Theorem 2.2, showing how each may be used to advantage.

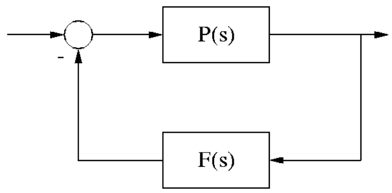

Figure 3: The feedback system

Consider the single-input single-output linear time-invariant feedback system depicted in Figure 3. Let the plant and compensator transfer functions be denoted by $P(s)$ and $F(s)$, respectively. We shall assume that $P(s)$ and $F(s)$ are both proper rational. Define the open loop transfer function by $L(s) := P(s)F(s)$, and the sensitivity and complementary sensitivity function by

$$S(s) := \frac{1}{1+L(s)}, \qquad T(s) := \frac{L(s)}{1+L(s)}.$$

With no loss of generality, we assume that there exists no unstable pole-zero cancellation in $L(s)$. Whenever this is the case, closed-loop stability of the system implies that both $S(s)$ and $T(s)$ are analytic in the closed right half plane. We assume throughout that the system is stable. Furthermore, we assume that $L(s)$ satisfies the conjugate symmetry property.

Suppose that the open loop transfer function $L(s)$ has right half plane poles $p_i \in \mathbb{C}_+$, $i = 1, \cdots, N_p$, counting the multiplicities. Suppose also that it has right half plane zeros $z_i \in \mathbb{C}_+$, $i = 1, \cdots, N_z$, counting the multiplicities. Then $L(s)$ can be factored as

$$L(s) = L_m(s) B_p^{-1}(s) B_z(s), \tag{8}$$

where $B_p(s)$ and $B_z(s)$ are the Blaschke products associated with the zeros and poles of $L(s)$, respectively, defined by

$$B_p(s) = \prod_{i=1}^{N_p} \frac{p_i - s}{\overline{p}_i + s}, \quad B_z(s) = \prod_{i=1}^{N_z} \frac{z_i - s}{\overline{z}_i + s}.$$

Hence, the sensitivity and complementary sensitivity functions admit the factorizations

$$S(s) = S_m(s) B_p(s), \quad T(s) = T_m(s) B_z(s). \tag{9}$$

We shall assume throughout this paper that $L(s)$ has neither zero nor pole on the imaginary axis. Under this assumption, $L_m(s)$ is stable and minimum-phased, so are $S_m(s)$ and $T_m(s)$. It is worth noting that imaginary zeros or poles of $L(s)$ have no effect on integral relations of all known kinds, whenever the integrals in question are appropriately defined, specifically in terms of the so-called Cauchy principal values [2, 5]. Indeed, since the function $g(s)$ to be selected satisfies A 2.5 in all the cases, this assumption can be imposed with no loss of generality.

We now develop Bode and Poisson integral relations using the argument formulas. We state these results in theorems, which can be found in e.g., [5, 6, 10, 12]. The statements are rather standard, but the proofs are entirely based on the extended argument principle.

Theorem 3.1 (Bode Integral for S) *Assume that*

$$\lim_{R\to+\infty} R \sup_{\theta\in[-\pi/2,\ \pi/2]} \left| L\left(Re^{j\theta}\right)\right| = 0. \tag{10}$$

Then,

$$\int_0^{+\infty} \log|S(j\omega)| d\omega = \pi \sum_{i=1}^{N_p} p_i. \tag{11}$$

Proof. Let $f(s) = S(s)$, $g(s) = s$. Clearly, $g(s)$ is analytic in $\overline{\mathbb{C}}_+$. The condition (10) guarantees that assumption A 2.2 is satisfied. The result follows by directly applying Corollary 2.1. ■

Theorem 3.2 (Bode Integral for T) *Under the condition that* $T(0) \neq 0$,

$$\int_0^{+\infty} \frac{1}{\omega^2} \log\left|\frac{T(j\omega)}{T(0)}\right| d\omega = \frac{\pi}{2}\frac{T'(0)}{T(0)} + \pi \sum_{i=1}^{N_z} \frac{1}{z_i}. \tag{12}$$

Proof. Define $f(s) = \frac{T(s)}{T(0)}$, $g(s) = \frac{1}{s}$. Clearly, assumption 2.2 is satisfied. At the singularity $s = 0$ of $g(s)$, we have,

$$\lim_{s\to 0} s g'(s) \log f(s) = -\frac{T'(0)}{T(0)}.$$

Hence A 2.3 is satisfied. Again, invoking Corollary 2.1 yields the result directly. ■

Theorem 3.3 (Poisson Integral for S and T) *Let* $z_0 = x_0 + jy_0 \in \mathbb{C}_+$ *be a zero of* $L(s)$. *Then,*

$$\int_{-\infty}^{+\infty} \log|S(j\omega)| \frac{x_0}{x_0^2 + (y_0 - \omega)^2} d\omega = \pi \log|B_p^{-1}(z_0)|. \tag{13}$$

Similarly, let $p_0 = \sigma_0 + j\omega_0 \in \mathbb{C}_+$ *be a pole of* $L(s)$. *Then,*

$$\int_{-\infty}^{+\infty} \log|T(j\omega)| \frac{\sigma_0}{\sigma_0^2 + (\omega_0 - \omega)^2} d\omega = \pi \log|B_z^{-1}(p_0)|. \tag{14}$$

Proof. We will only prove (13), the proof for (14) is similar. Construct $f(s) = S(s)$, $g(s) = \log\frac{\overline{z}_0 + s}{z_0 - s}$. Note that the real part of $g(j\omega)$ is always 0. For $s \in \mathbb{C}_+$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$\begin{aligned} \lim_{R\to+\infty} R \left| \frac{2x_0}{(\overline{z}_0 + s)(z_0 - s)} \log S(s) \right| &= 0, \\ \lim_{s\to z} (s - z_0) \frac{2x_0}{(\overline{z}_0 + s)(z_0 - s)} \log S(s) &= 0. \end{aligned}$$

As such A 2.2 and 2.3 are satisfied. Now the proof is completed by applying Corollary 2.2 and taking the real part of both sides. ■

Bode's gain-phase relation is proved here using Theorem 2.2 to give an example of its usage.

Theorem 3.4 (Bode's Gain-phase Relationship) *Assume that $L(s)$ is stable and $L(j\omega) \neq 0$ for $\omega \in (-\infty, +\infty)$. Then at any frequency ω_c, the phase $\phi(j\omega_c)$ of L satisfies*

$$\phi(j\omega_c) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{d\log|L(\nu)|}{d\nu} \log\coth\frac{|\nu|}{2}d\nu + \sum_{i=1}^{N_z} \arg\frac{z_i - j\omega_c}{z_i + j\omega_c}, \qquad (15)$$

where $\nu := \log(\omega/\omega_c)$.

Proof. It suffices to take

$$f(s) = \frac{s + j\omega_c}{s - j\omega_c}, \qquad g(s) = \log\frac{L_m(s)}{|L(j\omega_c)|},$$

where $L_m(s)$ is the minimum phase part of $L(s)$. With $f(s)$ and $g(s)$ so given, it is easy to see that Assumption A 2.6 is satisfied. A direct application of Theorem 2.2 then gives rise to

$$\int_{-\infty}^{+\infty} \frac{-2j\omega_c}{(j\omega - j\omega_c)(j\omega + j\omega_c)} \log\frac{L_m(j\omega)}{|L(j\omega_c)|}d\omega = \pi\log\frac{L_m(j\omega_c)}{L_m(-j\omega_c)}.$$

The proof is completed by taking the imaginary part of both sides and changing to the variable $\nu := \log(\omega/\omega_c)$. ∎

4 Conclusion

Our main contribution in this paper lies in the recognition that there is a intimate connection between the argument principle and Bode and Poisson type integral relations. This recognition enables us to unify various integral results together with Nyquist stability criterion, under a single category captured by an extended argument principle. The following table summarizes the cases examined from this perspective.

	$f(s)$	$g(s)$
Bode Integral	$S(s)$	s
	$\frac{T(s)}{T(0)}$	$\frac{1}{s}$
Poisson Integral	$S(s)$	$\log\frac{\bar{z}+s}{z-s}$
	$T(s)$	$\log\frac{\bar{p}+s}{p-s}$
Gain-Phase Relation	$\log\frac{L(s)}{\lvert L(j\omega_c)\rvert}$	$\log\frac{s+j\omega_c}{s-j\omega_c}$

The mathematical results developed herein, which contain several versions of the extended argument principle applied to the right half plane, can be utilized to develop new integral results for the sensitivity and complementary sensitivity functions. The new integral relations along with their interpretations will be presented in Part II of this series.

Acknowledgement

This research was supported in part by the NSF and by the Hong Kong Research Grants Council.

References

[1] S. Boyd and C.A. Desoer, "Subharmonic functions and performance bounds in linear time-invariant feedback systems," *IMA J. Math. Contr. and Info.*, vol. 2, pp. 153-170, 1985.

[2] J. Chen, "Sensitivity integral relations and design tradeoffs in linear multivariable feedback systems," *IEEE Trans. Auto. Contr.*, vol. AC-40, no. 10, pp. 1700-1716, Oct. 1995.

[3] J. Chen, "Multivariable gain-phase and sensitivity integral relations and design tradeoffs," *IEEE Trans. Auto. Contr.*, vol. AC-43, no. 3, pp. 373-385, March 1998.

[4] J. Chen, "On logarithmic complementary sensitivity integrals for MIMO systems," *Proc. 1998 Amer. Contr. Conf.*, pp. 3529-30, Philadelphia, PA, June 1998.

[5] J.S. Freudenberg and D.P. Looze, "Right half plane zeros and poles and design tradeoffs in feedback systems," *IEEE Trans. Auto. Contr.*, vol. AC-30, no. 6, pp. 555-565, June 1985.

[6] J.S. Freudenberg and D.P. Looze, *Frequency Domain Properties of Scalar and Multivariable Feedback Systems*, Lecture Notes in Contr. and Info. Science, vol. 104, Berlin: Springer-Verlag, 1988.

[7] J.S. Freudenberg, R.H. Middleton, and J.H. Braslavsky, "Inherent design limitations for linear sampled-data feedback systems," *Int. J. Contr.*, vol. 61, no. 6, pp. 1387-1421, 1995.

[8] G.C. Goodwin, D. Mayne, and J. Shim, "Tradeoffs in linear filter design," *Automatica*, vol. 31, no. 10, pp. 1367-1376, Oct. 1995.

[9] N. Levinson and R.M. Redheffer, *Complex Variables*, Baltimore: Holden-Day, 1970.

[10] R.H. Middleton, "Trade-offs in linear control system design," *Automatica*, vol. 27, no. 2, pp. 281-292, Feb. 1991.

[11] L. Qiu and J. Chen, "Time domain performance limitations of feedback control", in *Mathematical Theory of Networks and Systems*, eds. A. Beghi, L. Finesso, and G. Picci, Il Poligrafo, 1998, pp. 369-372.

[12] M.M. Seron, J.H. Braslavsky, and G.C. Goodwin, *Fundamental Limitations in Filtering and Control,* London: Springer-Verlag, 1997.

[13] M.M. Seron, J.H. Braslavsky, P.V. Kokotovic, and D.Q. Mayne, "Feedback limitations in nonlinear systems: from Bode integrals to cheap control," *IEEE Trans. Auto. Contr.*, vol.AC-44, no.4, pp.829-33, April, 1999.

[14] H.K. Sung and S. Hara, "Properties of sensitivity and complementary sensitivity functions in single-input single-output digital control systems," *Int. J. Contr.*, vol. 48, no. 6, pp. 2429-2439, 1988.