Internets in the Sky: Capacity of 3-D Wireless Networks[1]

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Abstract

Consider n nodes located in a sphere of volume V m^3, each capable of transmitting at a data rate of W bits/sec. Under a protocol based model for successful receptions, the entire network can carry only $\Theta\left(WV^{\frac{1}{3}}n^{\frac{2}{3}}\right)$ bit-meters/sec, where 1 bit carried a distance of 1 meter is counted as 1 bit-meter. This is the best possible even assuming the node locations, traffic patterns, and the range/power/timing of each transmission, are all optimally chosen. If the node locations and their destinations are randomly chosen, and all transmissions employ the same power/range, then each node only obtains a throughput of $\Theta\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ bits/sec, if the network is optimally operated. Similar results hold under an alternate physical model where a minimum signal-to-interference ratio is specified for successful receptions.

The proofs of these results require determination of the VC-dimensions of certain geometric sets, which may be of independent interest.

1 Introduction

In [2], the capacity of multi-hop wireless networks was analyzed when nodes are located in a disk on the plane. In this paper we obtain the traffic-carrying capacity of three dimensional (3-D) wireless networks. Such wireless networks arise when the network consists of both terrestrial and satellite-based or aircraft-based communication links, or in building networks where nodes are located on different floors.

Consider n nodes located in a sphere of volume V m^3, with each node capable of transmitting at W bits/sec. We show that in the random case where the n nodes are randomly located in the sphere and each node's destination is randomly chosen, the throughput obtained by each node is $\Theta(\frac{W}{(n\log^2 n)^{\frac{1}{3}}})$ bits/sec under a Protocol Model of non-interference, if the network is optimally operated. In the best case where the node locations, OD-pair assignments and traffic patterns are optimally chosen, and the network is optimally operated, i.e., the transmission ranges, routes, and schedules of all transmissions are optimal, the entire network can transport $\Theta(WV^{\frac{1}{3}}n^{\frac{2}{3}})$ bit-meters/sec. Thus, even under optimal conditions, the throughput still decreases as $\Theta(\frac{W}{n^{\frac{1}{3}}})$ bits/sec for each node for a destination nonvanishingly far away.

As in [2], we also consider an alternate Physical Model of non-interference where a required signal-to-interference ratio is specified for successful receptions. Under this model, the lower bounds on the capacity are the same as those above, while the upper bounds on throughput are $\Theta\left(\frac{W}{n^{\frac{1}{3}}}\right)$ for the random case, and $\Theta\left(\frac{W}{n^{\frac{1}{\alpha}}}\right)$ for the best case, where α is the signal power path loss exponent.

In both the random and best cases, the capacity of a wireless network is higher when the nodes are located in a sphere, than when they are located in a disk (or on the surface of a sphere). Nevertheless, the throughput obtained by each node still diminishes to zero as the number of nodes in the network is increased. Thus, the implications discussed in [2] continue to hold for 3-D wireless networks. In particular, wireless networks connecting fewer number of users, or allowing connections mostly with nearby neighbors, may be more likely to find acceptance.

While proving the above results, we also determine the VC-dimensions of the following geometric sets: The set of all spheres in $\Re^k$, the set of all discs on the surface of a sphere in $\Re^k$, and the collection of the sets of lines intersecting spheres in $\Re^k$. These results may be of independent interest.

The rest of the paper is organized as follows. In Section 2 we describe the model for Arbitrary 3-D Wireless Networks. In Section 3 we obtain upper bounds on the transport capacity of such networks, which are of the form $cWn^{\frac{2}{3}}$ bit-meters/sec and $c'Wn^{\frac{\alpha-1}{\alpha}}$ bit-meters/sec, under the Protocol and Physical Models,

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respectively. In Section 4 we show that a transport capacity of $c''Wn^{\frac{2}{3}}$ bit-meters/sec is also feasible for Arbitrary 3-D Networks. In Section 5 we discuss the model for Random 3-D Wireless Networks. In Section 6 we show that $\Theta\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ bits/sec and $\Theta\left(\frac{W}{n^{\frac{1}{3}}}\right)$ bits/sec are upper bounds on the throughput obtainable by each node in Random 3-D Networks, under the Protocol and Physical Models, respectively. In Section 7 we construct a scheme which provides a throughput of $\Theta\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ bits/sec with high probability for Random 3-D Networks.

2 Arbitrary 3-D Networks

In Arbitrary 3-D Networks, n nodes are arbitrarily located in a sphere S of volume V m^3. Each node can have traffic to send to an arbitrary destination. Each node can transmit over any subset of M independent channels with capacities $W_1, W_2, \ldots, W_M$ bits/sec, where $\sum_{m=1}^{M} W_m = W$. Each node can use an arbitrary transmission range for each such transmission.

Let X_i, $1 \le i \le n$, denote the location of node i. Suppose $\{(X_k, X_{R(k)}) : k \in \mathcal{T}\}$ is the set of all active transmitter-receiver pairs at some instant over a certain channel. Then we consider the following two models for successful reception of a transmission over one hop.

The Protocol Model: The transmission from node X_i, $i \in \mathcal{T}$, is successfully received by its intended receiver $X_{R(i)}$ if for every $k \in \mathcal{T} \setminus i$

$$|X_k - X_{R(i)}| \geq (1+\Delta)|X_i - X_{R(i)}|. \quad (1)$$

The quantity $\Delta > 0$ models situations where a guard zone is specified by the protocol to prevent a neighboring node from transmitting on the same channel at the same time. It also allows for imprecision in the achieved range of transmissions.

The second model that is more related to physical layer considerations is the following:

The Physical Model: Let P_k, $k \in \mathcal{T}$, be the power level at which node X_k transmits. Then the transmission from node X_i, $i \in \mathcal{T}$, is successfully received by its intended receiver $X_{R(i)}$ if

$$\frac{\frac{P_i}{|X_i - X_{R(i)}|^{\alpha}}}{N + \sum_{\substack{k\in\mathcal{T}\\ k\neq i}} \frac{P_k}{|X_k - X_{R(i)}|^{\alpha}}} \geq \beta. \quad (2)$$

This models a situation where a minimum signal to interference ratio (SIR) of β is necessary for successful receptions, the ambient noise power level is N, and signal power decays with distance r as $\frac{1}{r^{\alpha}}$. For 3-D wireless networks we will suppose that $\alpha > 3$. The reason is that if $\alpha \le 3$, and nodes are uniform in space, then the interference level everywhere is unbounded as the number of nodes in the network increases.

3 Arbitrary 3-D Networks: An Upper Bound on Transport Capacity

Given a set of successful transmissions, we say that the network transports one *bit-meter* when one bit has been transported over a distance of one meter towards its destination. The *transport capacity* of a network is defined as the supremum of the bit-distance product that can be transported by the entire network per second.

Suppose an Arbitrary Network transports a total of λnT bits in T seconds. Suppose the average distance between the source and the destination of a bit is $\bar{L}$ meters. In other words, the network achieves a transport capacity of $\lambda n\bar{L}$ bit-meters/sec. Then, the following holds:

Theorem 3.1 *i) In the Protocol Model, the transport capacity $\lambda n\bar{L}$ of any Arbitrary 3-D Wireless Network is bounded by*

$$\lambda n\bar{L} \leq 2\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} \frac{1}{\Delta} W n^{\frac{2}{3}} \text{ bits-meters/sec.}$$

(ii) In the Physical Model,

$$\lambda n\bar{L} \leq \left(\frac{2\beta+2}{\beta}\right)^{\frac{1}{\alpha}} \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} W n^{\frac{\alpha-1}{\alpha}} \text{ bit-meters/sec.}$$

(iii) If the ratio $\frac{P_{\max}}{P_{\min}}$ between the maximum and minimum powers that transmitters can employ is strictly bounded above by β, then

$$\lambda n\bar{L} \leq 2\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} \frac{1}{(\frac{\beta P_{\min}}{P_{\max}})^{\frac{1}{\alpha}} - 1} W n^{\frac{2}{3}} \text{ bit-meters/sec.}$$

Proof: Let $h(b)$ be the number of hops taken by bit b for $1 \le b \le \lambda nT$, and r_b^h be the distance traversed by b in hop h. Then

$$\sum_{b=1}^{\lambda nT} \sum_{h=1}^{h(b)} r_b^h \geq \lambda nT\bar{L}. \quad (3)$$

For simplicity in exposition, suppose that transmissions in the network are slotted into synchronized slots of length τ secs. Then, in any slot s, at most $n/2$ nodes can transmit over any channel m. Hence, we have

$$\sum_{b=1}^{\lambda nT} \sum_{h=1}^{h(b)} 1(h\text{th hop of bit } b \text{ is over channel } m \text{ in slot } s) \leq \frac{W_m \tau n}{2}.$$

Summing over the channels and the slots, and noting that there can be no more than $\frac{T}{\tau}$ slots in T secs, we get

$$H \quad := \quad \sum_{b=1}^{\lambda nT} h(b) \leq \frac{WTn}{2}. \tag{4}$$

Consider now the Protocol Model. Using triangle inequality as in Theorem 3.1 [2], the non-interference protocol requirement gives that in each slot s and over each channel m spheres centered at each active receiver and of radius Δ times half the distance from the corresponding transmitter are essentially disjoint. Taking edge effects into account and noting that a range greater than the diameter is unnecessary, we deduce that at least a quarter of such a sphere is within sphere S. Since at most $W_m\tau$ bits can be carried in slot s from a transmitter to a receiver over the channel m, we have

$$\sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)} 1(h\text{th hop of bit } b \text{ is over channel } m \text{ in slot } s)\frac{1}{4}\frac{4\pi}{3}\left(\frac{\Delta r_b^h}{2}\right)^3 \leq W_m \tau V. \tag{5}$$

Summing over the channels and the slots, we get

$$\sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)} \frac{\pi\Delta^3}{24}(r_b^h)^3 \quad \leq \quad WTV,$$

which can be rewritten as

$$\sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)} \frac{1}{H}(r_b^h)^3 \quad \leq \quad \frac{24WTV}{\pi\Delta^3 H}. \tag{6}$$

Now x^3 is a convex function over $x \geq 0$. Hence

$$(\sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)} \frac{1}{H}r_b^h)^3 \quad \leq \quad \sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)} \frac{1}{H}(r_b^h)^3. \tag{7}$$

Combining (6) and (7) yields,

$$\sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)} r_b^h \quad \leq \quad \left(\frac{24WTVH^2}{\pi\Delta^3}\right)^{\frac{1}{3}}. \tag{8}$$

Now substituting (3) in (8) gives,

$$\lambda nT\bar{L} \quad \leq \quad \left(\frac{24WTVH^2}{\pi\Delta^3}\right)^{\frac{1}{3}}. \tag{9}$$

Substituting (4) in (9) yields the result.

Proofs of (ii) and (iii) proceed along similar lines as in Theorem 2.1 [2], with the only difference arising due to the diameter $2\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$ of sphere S. ■

4 Arbitrary 3-D Networks: A Constructive Lower Bound on Transport Capacity

We will now show that the $O(n^{\frac{2}{3}})$ order of the upper bound on the transport capacity in the previous section is tight, by exhibiting a scenario where it is achieved.

Theorem 4.1 *Nodes can be placed in sphere S, and traffic patterns assigned, such that the network can achieve $\frac{WV^{\frac{1}{3}}}{1+2\Delta}n^{\frac{2}{3}}$ bit-meters/sec under the Protocol Model, and $\frac{WV^{\frac{1}{3}}}{\left(16\beta\left(21+2^{\frac{\alpha}{2}}+\frac{4\alpha-2}{\alpha-3}\right)\right)^{\frac{1}{\alpha}}}n^{\frac{2}{3}}$ bit-meters/sec under the Physical Model.*

Proof: First consider the Protocol Model. Let $r = \frac{V^{\frac{1}{3}}}{1+2\Delta}\frac{2}{n^{\frac{1}{3}}}$. With the center of sphere S taken as the origin, place transmitters at locations $(j(1+2\Delta)r \pm \Delta r, k(1+2\Delta)r, l(1+2\Delta)r)$, $(j(1+2\Delta)r, k(1+2\Delta)r \pm \Delta r, l(1+2\Delta)r)$ and $(j(1+2\Delta)r, k(1+2\Delta)r, l(1+2\Delta)r \pm \Delta r)$ where $|j+k+l|$ is odd. Also place receivers at $(j(1+2\Delta)r \pm \Delta r, k(1+2\Delta)r, l(1+2\Delta)r)$, $(j(1+2\Delta)r, k(1+2\Delta)r \pm \Delta r, l(1+2\Delta)r)$ and $(j(1+2\Delta)r, k(1+2\Delta)r, l(1+2\Delta)r \pm \Delta r)$, where $|j+k+l|$ is even. Each transmitter can transmit to its nearest receiver, which is at a distance r away, without interference from any other transmitter. Furthermore, the above allows for $\frac{n}{2}$ transmitter–receiver pairs to be placed within S. Under this placement of nodes, there are a total of $\frac{n}{2}$ concurrent transmissions, each of range r, and each at W bits/sec. This achieves the transport capacity indicated.

For the Physical Model, a calculation of the SIR under the above placement shows that it is lower bounded at all receivers by $\frac{(1+2\Delta)^\alpha}{16\left(21+2^{\frac{\alpha}{2}}+\frac{4\alpha-2}{\alpha-3}\right)}$. Choosing Δ to make this lower bound equal to β yields the result. ■

5 Random 3-D Networks

Above we have determined the best case behavior where nodes can be optimally placed and traffic patterns optimally designed. We now address a scenario when the network itself is random.

In a random scenario, there are n nodes uniformly and independently distributed in a sphere S of volume 1 m^3. Each node sends data at $\lambda(n)$ bits/sec to a randomly chosen destination node. This destination node is picked as follows. A uniformly and independently distributed point in S is chosen, and the node nearest to this location is chosen as the destination node. Thus, the average separation between source-destination pairs is on the order of 1 meter.

In this random setting, we assume that all transmissions employ the same nominal range r or power level P. As in Arbitrary 3-D Networks, we consider two models for successful reception of a transmission.

The Protocol Model: All nodes employ a common *range* r for all their transmissions. Let $\{(X_k, X_{R(k)}) : k \in \mathcal{T}\}$ be the set of all active transmitter-receiver pairs at some time instant over a certain channel. Then transmission from $X_i, i \in \mathcal{T}$, is successfully received by $X_{R(i)}$ if:

$$|X_i - X_{R(i)}| \le r \quad (10)$$

and, for every $k \in \mathcal{T} \setminus i$,

$$|X_k - X_{R(i)}| \ge (1+\Delta)r. \quad (11)$$

The Physical Model: All nodes transmit at a common power level P. A transmission from a node X_i, $i \in \mathcal{T}$, is successfully received by node $X_{R(i)}$ if

$$\frac{\frac{P}{|X_i - X_{R(i)}|^\alpha}}{N + \sum_{\substack{k\in\mathcal{T} \\ k\neq i}} \frac{P}{|X_k - X_{R(i)}|^\alpha}} \ge \beta. \quad (12)$$

We say that the *throughput capacity* of Random 3-D Wireless Networks is of order $\Theta(f(n))$ bits/sec if there are constants $c > 0$ and $c' < +\infty$ such that

$$\lim_{n\to\infty} \text{Prob}(\lambda(n) = cf(n) \text{ is feasible}) = 1,$$
$$\lim_{n\to\infty} \text{Prob}(\lambda(n) = c'f(n) \text{ is feasible}) = 0.$$

We next obtain an $O\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ upper bound on the throughput capacity of random 3-D wireless networks. In Section 7 we will construct a scheme which achieves the same order of throughput capacity.

6 Random 3-D Networks: An Upper Bound on Throughput Capacity

An essential requirement for any positive throughput level to be feasible is that there exists a path between each node and its chosen destination. In particular, every node must have at least one node in its range with which it can communicate. We therefore first obtain a necessary condition on the transmission range $r(n)$ such that every node in the network has at least one node in its range with probability approaching one as the number of nodes $n \to +\infty$.

Given the n nodes in S, denote by $\mathcal{G}(n, r(n))$ the graph which results from connecting nodes separated by a distance less than $r(n)$ by an edge. $P^{(1)}(n, r(n))$ denote the probability that a graph $\mathcal{G}(n, r(n))$ has at least one isolated node. Then the following holds (see [1] for a proof).

Lemma 6.1 *If* $\frac{4\pi}{3}r^3(n) = \frac{\log n + \kappa_n}{n}$, *where* $\lim_{n\to\infty} \kappa_n = \kappa < +\infty$, *then*

$$\liminf_{n\to\infty} P^{(1)}(n, r(n)) \ge e^{-\kappa}\left(1 - e^{-\kappa}\right).$$

Next, as in Arbitrary Networks, the non-interference protocol requirement gives that spheres of radius $\frac{\Delta}{2}r(n)$ around concurrent receivers on a channel are disjoint. Hence, there can be at most $R(n) = \frac{1}{(1/4)(4/3\pi(\Delta r(n)/2)^3)}$ concurrent transmissions on any channel. Thus, the total transmission rate in the network at any time can be at most $WR(n)$. On the other hand, if L denote the mean length of the line from a source to its destination, then the total traffic in the network is at least $n\lambda(n)\frac{L-o(1)}{r(n)}$. For stability, we therefore need

$$\frac{(L - o(1))n\lambda(n)}{r(n)} \le \frac{24W}{\pi\Delta^3 r^3(n)}.$$

Thus,

$$\lambda(n) \le \frac{24W}{\pi\Delta^3(L - o(1))nr^2(n)}.$$

Now from Lemma 6.1 we have that $r(n) > (\frac{3\log n}{4\pi n})^{\frac{1}{3}}$ is necessary to ensure every node has at least one node in its range with probability approaching one as $n \to \infty$. Hence we get the following:

Theorem 6.1 *For Random 3-D Wireless Networks, there is a deterministic constant $c' < +\infty$, not depending on n, Δ or W, such that*

$$\lim_{n\to\infty} \textit{Prob}(\lambda(n) = \frac{c'W}{\Delta^3(n\log^2 n)^{\frac{1}{3}}} \textit{ is feasible}) = 0.$$

7 Random 3-D Networks: Constructive Lower Bound

In this section we describe a constructive scheduling and routing scheme similar to the one in [2], which shows that in a random 3-D wireless network each node can obtain a throughput of $\Theta(W/(n\log^2 n)^{\frac{1}{3}})$ with high probability. For this, we first determine the *VC-dimensions* [3] of certain geometric sets used in our constructive scheme (see [1] for proofs).

Theorem 7.1 *Let $\mathcal{S}(k)$ be the set of all spheres in $\Re^k$, then VC-dimension$(\mathcal{S}(k)) = k + 1$.*

Theorem 7.2 *The VC-dimension of the set of all disks on the surface of k-dimensional sphere, S^{k-1}, is $k + 1$.*

Let $\mathcal{L}(S)$ denote the set of all lines in $\Re^k$ which intersect a sphere $S \in \mathcal{S}(k)$. Then the following holds.

Theorem 7.3 *VC-dim*($\{\mathcal{L}(S) : S \in \mathcal{S}(k)\}$) $\leq (k^2 + 3k + 4)/2$.

Using the above, we next give a constructive scheme similar to [2] that achieves the same order of throughput capacity as upper bounded by Theorem 6.1.

Theorem 7.4 *(i) For Random 3-D Wireless Networks under the Protocol Model, there is a deterministic constant $c > 0$ not depending on n, Δ or W, such that*

$$\lambda(n) = \frac{c}{(1+\Delta)^3} \frac{W}{(n \log^2 n)^{\frac{1}{3}}} \quad \textit{bits/sec}$$

is feasible with probability approaching 1 as $n \to \infty$.

(ii) For Random 3-D Wireless Networks under the Physical Model, there is a deterministic constant $c' > 0$ not depending on n, α, β or W, such that

$$\lambda(n) = \frac{c'}{\left(2\left(18\beta\left(3+\frac{1}{\alpha-1}+\frac{1}{\alpha-2}+\frac{1}{\alpha-3}\right)\right)^{\frac{1}{\alpha}}-1\right)^3} \cdot \frac{W}{(n \log^2 n)^{\frac{1}{3}}} \quad \textit{bits/sec}$$

is feasible with probability approaching 1 as $n \to \infty$.

Proof: The proof proceeds along lines similar to the one given in [2] for the 2-D case. In the following we give the main steps involved in the proof without repeating the detailed arguments of [2], except where they do not exactly carry over to the 3-D case.

First consider the Protocol Model.

- Construct a Voronoi tessellation $\mathcal{V}_n$ of sphere S (similar to the one in Section IV.A of [2], but now in 3-D) such that each cell V contains a ball of radius $\rho(n)$ and is contained in a ball of radius $2\rho(n)$, where $\rho(n)$ is such that $\frac{4\pi}{3}\rho^3(n) = \frac{100 \log n}{n}$.
- Choose the range of each node to be $r(n) = 8\rho(n)$.
- This range allows any two nodes in neighboring cells to directly communicate (Lemma 4.2 in [2]).
- As in Lemma 4.3 [2], every cell in $\mathcal{V}_n$ has at most a constant number $c_1 = O((1+\Delta)^3)$ of interfering neighbors.
- Using a result from graph theory on vertex coloring, the above allows us to construct a transmission schedule such that each cell gets to transmit at least once in every $(c_1 + 1)$ consecutive slots (Lemma 4.4(i) [2]).
- Let $\{Y_i\}_{i=1}^n$ be independently and uniformly distributed (i.i.d.) points in S chosen independently of $\{X_i\}_{i=1}^n$. The destination node $X_{\text{dest}(i)}$ for the traffic generated at source node X_i is chosen as the node closest to Y_i.
- Let L_i be the line joining X_i and Y_i. Then, $\{L_i\}_{i=1}^n$ are i.i.d.
- Packets originating at source node X_i are routed to their destination node $X_{\text{dest}(i)}$ to follow L_i. That is, the packets from X_i are relayed from one cell to another in the order in which the cells intersect L_i. On reaching the cell containing Y_i, the packets are sent on to their final destination $X_{\text{dest}(i)}$, which is within one hop of Y_i with high probability.
- From Theorem 7.1, the VC-dimension of the set of all spheres in $\Re^3$ is 4. Hence, as in Lemma 4.8 [2], every cell in the Voronoi tessellation $\mathcal{V}_n$ contains at least one node with probability approaching one as $n \to +\infty$.
- We next obtain a uniform bound on the amount of traffic that needs to be carried by each cell V of $\mathcal{V}_n$. For this, we first bound the expected number of lines intersecting cell V as follows

$$\begin{aligned}
&E[\text{Number of lines in } \{L_i\}_{i=1}^n \text{ intersecting cell } V] \\
&\quad = n \text{ Prob(Line } L_i \text{ intersects } V) \\
&\quad \leq n\Big(\text{Prob}(d(X_i,V) \leq c_0\rho(n)) \\
&\qquad + \text{Prob}(L_i \text{ intersects } V \;\Big|\; d(X_i,V) > c_0\rho(n))\Big) \\
&\quad \leq n\left(\frac{4\pi}{3}((c_0+2)\rho(n))^3 + \int_{(c_0+1)\rho(n)}^{2r_0} \text{Prob}(L_i \right. \\
&\qquad \left. \text{intersects } V \Big| |X_i - c(V)| = x)\ 4\pi x^2 dx\right),
\end{aligned} \tag{13}$$

where we have used the facts that V contains a ball of radius $\rho(n)$ and is contained in a ball of radius $2\rho(n)$, c_0 is a constant to be specified later, $c(V)$ is the center of the ball containing the cell V, and $r_0 = (3/(4\pi))^{\frac{1}{3}}$ is the radius of the unit-volume sphere S. For the conditional probability in the second term, we upper bound the volume of the cone at X_i and touching the ball of radius $2\rho(n)$ containing V, by that of a cylinder as shown in Figure 1.

$$\begin{aligned}
&\text{Prob}(L_i \text{ intersects } V \;\Big|\; |X_i - c(V)| = x) \\
&\quad \leq \pi y^2(d+x) \\
&\quad \leq \pi\Big((d+x)\tan\theta\Big)^2(d+x) \\
&\quad \leq \pi\left(2r_0 \cdot 2\frac{2\rho(n)}{x}\right)^2 2r_0,
\end{aligned} \tag{14}$$

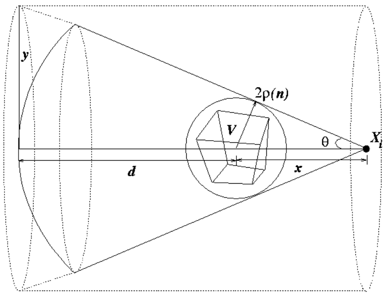

Figure 1: Computing the probability that a line L_i intersects cell V. Note that $d + x \leq 2r_0$.

for sufficiently large c_0. Substituting (14) in (13), we get

$$E[\text{Number of lines in } \{L_i\}_{i=1}^n \text{ intersecting a cell } V] \leq n\left(\frac{4\pi}{3}((c_0+2)\rho(n))^3 + \pi(4\rho(n))^2 \cdot 4\pi(2r_0)^4\right) \leq nc_2\rho^2(n) \leq c_3(n\log^2 n)^{\frac{1}{3}}, \quad (15)$$

for some constants c_2 and c_3, and sufficiently large n.

Next, from Theorem 7.3, we get that the Vapnik-Chervonenkis Theorem [4] is applicable to the collection of sets of lines intersecting spheres in $\Re^3$, $\{\mathcal{L}(S) : S \in \mathcal{S}(3)\}$. Thus, for some $\delta(n) \to 0$,

$$\text{Prob}\left(\sup_{V\in\mathcal{V}_n} (\text{Traffic needing to be carried by cell } V) \leq c_4\lambda(n)(n\log^2 n)^{\frac{1}{3}}\right) \geq 1-\delta(n).$$

- Since each cell can transmit at $W/(c_1+1)$ bits/sec, while it needs to transmit at at most $c_4\lambda(n)(n\log^2 n)^{\frac{1}{3}}$ bits/sec with high probability, $\lambda(n)$ can be accommodated in the network with probability approaching one as $n \to +\infty$, if

$$\lambda(n) \leq \frac{W}{(1+c_1)c_4(n\log^2 n)^{\frac{1}{3}}}.$$

Next consider the Physical Model. An argument along lines similar to Lemma 4.4(ii) [2] shows that if the transmitters in the above constructive scheme use power level P, the SIR at each receiver is lower bounded by

$$\frac{\frac{P}{r^\alpha(n)}}{N + \sum_{k=1}^{+\infty}\left((k+2)^3-(k-1)^3\right)\frac{P}{k^\alpha(1+\frac{\Delta}{2})^\alpha r^\alpha(n)}} = \frac{1}{\frac{Nr^\alpha(n)}{P} + \frac{9}{(1+\frac{\Delta}{2})^\alpha}\sum_{k=1}^{+\infty}\frac{k^2+k+1}{k^\alpha}}.$$

For $\alpha > 3$, the sum in the denominator is smaller than $(3+\frac{1}{\alpha-1}+\frac{1}{\alpha-2}+\frac{1}{\alpha-3})$. Choosing $P = 2\beta r^\alpha(n)\cdot \max\{N,1\}$ and $\Delta = 2\Big(\big(18\beta(3+\frac{1}{\alpha-1}+\frac{1}{\alpha-2}+\frac{1}{\alpha-3})\big)^{\frac{1}{\alpha}} -1\Big)$, ensures that the lower bound on the SIR at each receiver is at least β. The result follows. ■

8 Conclusions

We have obtained the capacity of 3-D wireless networks. We have shown that under a Protocol Model of non-interference, in a random 3-D network of n nodes randomly located in a sphere, with each node capable of transmitting at W bits/sec and using a common range, the throughput that each node can obtain for a randomly chosen destination is $\Theta\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ bits/sec. Even under optimal choices for node locations, traffic patterns, and origin-destination pairs, and optimal operation by choosing transmission schedules, ranges and routes, each node cannot obtain a throughput of more than $\Theta\left(\frac{W}{n^{\frac{1}{3}}}\right)$ bits/sec for a destination on the order of 1 meter away. Under a Physical Model of non-interference, the lower bounds are the same as those above for the Protocol Model, while the upper bounds on throughput are $\Theta\left(\frac{W}{n^{\frac{1}{3}}}\right)$ for Random 3-D Networks and $\Theta\left(\frac{W}{n^{\frac{1}{\alpha}}}\right)$ for Arbitrary 3-D Networks.

In both the random and best case scenarios, 3-D wireless networks have higher capacity than 2-D networks. However, the throughput obtained by each node still diminishes to zero as the number of nodes in the network is increased. Hence, wireless networks connecting fewer number of users, or allowing connections mostly with nearby neighbors, may be more likely to find acceptance.

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