New Stability Results for Time-varying Systems Based on A Modified Detectability Condition

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Abstract

The uniformly asymptotical stability is investigated from the output-to-state viewpoint for general nonlinear time-varying systems. Several criterions are proposed using some integral inequalities involving the output function and a new detectability condition.

Furthermore, the existing results using the Lyapunov direct method such as the Krasovskii-LaSalle invariance principle, a theorem of Aeyels and a theorem of Khalil for time-varying systems are shown to be deduced for the proposed scheme. From these applications, it can be seen that as the invariance principle of LaSalle is used in studying the stability of time-invariant systems, our results can be also used in studying the stability of time-varying systems.

Keywords: Nonlinear time-varying systems, stability, detectability, Lyapunov stability

1. Introduction

Since the 1960's, Lyapunov function based approaches have been developed for the analysis of system stability (see [1-5,7-10,13,14-15]). Among these, a very useful criterion, called the "LaSalle invariance principle," was proposed in [8] and has been applied and extended to the study of many diverse areas in the recent literature. For instance, Byrnes and Martin [5] proposed an integral invariance principle to study the stability of nonlinear time-invariant autonomous systems. However, neither the LaSalle invariance principle nor the integral invariance principle can be applied to time-varying systems directly. This is due to the fact that the ω-limit set is not an invariant set in general time-varying systems (see, e.g., [7], p.193). Since the invariance principles have been proven to be important and useful in the analysis of system dynamics, the extension of these principles to general time-varying system has attracted much attention (e.g., [1,2,7,13]). In [13], results for some classes of time-varying systems such as almost periodic systems and asymptotically autonomous systems were obtained using the concept of pseudo-invariant set. However, no simple method was given for the determination of the pseudo-invariance set. Instead of using the concept of the invariance principles, two interesting results of employing the concept of "limit equation" [2] and the direct Lyapunov approach [1] were obtained for time-varying systems. Although the stability criteria proposed in previous literature can be used in some time-varying systems, their approaches are, in general, hard to check.

In this paper, several stability criterions will be proposed from the output-to-state view-point for general nonlinear time-varying systems. A new detectability condition, called the weak zero-state detectability condition, will be given in order to avoid the obstacle that the ω-limit set is not invariant in general time-varying systems. Furthermore, it will be shown that the asymptotically stability can be achieved for Lyapunov stable systems provided that the output function satisfies an integral inequality and the weak zero-state detectability condition.

Moreover, the existing results using the Lyapunov direct method such as the Krasovskii-LaSalle invariance principle [14], a theorem of Aeyels [1] and a theorem of Khalil [7] for time-varying systems are shown to be deduced for the proposed scheme. From these applications, it can be seen that as the invariance principle of LaSalle is used in studying the stability of time-invariant systems, our results can be also used in studying the stability of time-varying systems. ***Notations:***

1. $\Re_+$ is the set of nonnegative real number.
2. $|v| = \sqrt{v_1^2 + v_2^2 + \cdots + v_p^2}$, $\forall v = (v_1, v_2, \cdots, v_p) \in \Re^p$.
3. A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is said to belong to class K if it is strictly increasing and $\alpha(0) = 0$.
4. A function $V : \Re_+ \times \Re^p \to \Re_+$ is said to be a locally positive definite function (lpdf) if it is continuous, $V(t,0) = 0$ for $t \ge 0$, and there exist a constant $\gamma > 0$ and a function α of class K such that $\alpha(|x|) \le V(t,x), \forall t \ge 0, \forall |x| < \gamma$.
5. V is decrescent if there exists a function β of class K such that $V(t,x) \le \beta(|x|), \forall t \ge 0, \forall |x| < \gamma$.

2. Uniformly asymptotical stability for weakly zero-state detectabe time-varying systems

2.1 From time-invariant systems to time-varying systems: detectability and obstacles

In this paper, we consider nonlinear time-varying systems of the form

$$\dot{x} = f(t,x) \tag{1}$$

$$y = h(t,x) \tag{2}$$

where x is a state variable contained in an open subset X of $\Re^p$, $y \in \Re^q$ is an output variable and $f(t,jx)$ and $h(t,x)$ are both measurable functions defined on $\Re_+ \times X$. We assume that $0 \in X$ and $f(t,x)$ satisfies the Caratheodory condition [6] so that the existence theorem and extension theorem of solutions of (1) are satisfied.

Let $\phi(t,t_0,x_0)$ denote a trajectory of (1) starting at x_0 at time $t = t_0$ throughout this paper. The stability properties of trajectories (1) in terms of the detectability were first studied by Byrnes et al. [5] for time-invariant systems. In particular, an integral invariance principle was proposed by them and used it to prove a series of stability results. Using the zero-state detectability property, an interesting result, which can be derived from [5], is stated in the following to motivate our approach.

Proposition 1 (Byrnes and Martin [5]): Consider a time-varying system of the form (1)-(2) where $f(t,x)$ ($=f_0(x)$) and $h(t,x)$ ($=h_0(x)$) are both continuous and independent on t. Suppose the system is locally zero-state detectable and the following assumption

$$\textbf{(A1)} \quad \int_0^\infty \left|h_0(\phi(t,0,x_0))\right|^2 dt < \infty \tag{3}$$

holds for all $\phi(t,0,x_0)$ lying within a compact subset of X. Then, the origin is asymptotically stable if it is Lyapunov stable.

Remark 1: The original paper in [5], the asymptotic stability is guaranteed under a stronger condition- the zero-state observability, see Theorem 2.1 in the literature. However, it is possible to show that Proposition 1 also holds using Corollary 1.2 in [5] and the property of Lyapunov stability.

Proposition 1 has many interesting applications as shown in that paper. However, it can not be extended directly to general time-varying systems because the following obstacles will appear:

(1) It is well-known that the w-limit set is not an invariant set in general time-varying systems, and
(2) The origin is not attractive under a similar zero-state detectability (and zero-state observability) condition for some time-varying systems.

An illustrative example can be found in [10]. Thus, a modification will be necessary if one wants to extend Proposition 1 to time-varying systems. A possible way is to modify the definition of zero-state detectability. In this paper, the following modified detectability condition will be used to achieve a similar result.

Definition 1: A system of the form (1)-(2) is weakly zero-state detectable if for any given sequences $\{s_n\}$ and $\{t_n\}$ of real number, with $s_n \to \infty$ and $t_n \to \infty$, and any sequence $\{\phi_n(t,t_n,x_n)\}$ of solutions which is lying with a compact subset κ of X and satisfying the following equation

$$\lim_{n\to\infty} h(t+t_n, \phi_n(t+t_n,t_n,x_n)) = 0 \tag{4}$$

for almost all $t \in [0,\infty)$, we have

$$\inf\{\|\phi_n(t,t_n,x_n)\| \,|\, \forall t \in [t_n, t_n+s_n], \forall n \in \aleph\} = 0. \tag{5}$$

In the following, a technique lemma from [6] will be given. It will be used to show that the weak zero-state detectability is implied by the zero-state detectability for time-invariant systems.

Lemma 1: Let $f_1(t,x), f_2(t,x),\dots$ be a sequence of continuous functions defined on $\Re_+ \times X$ such that $\{f_n(t,x)\}$ converges uniformly to a function $f(t,x)$ on every compact subset of $\Re_+ \times X$. Let κ be a compact subset of X and $x_n(t)$, lying with κ, be a solution of $\dot{x} = f_n(t,x)$ starting at t=0. Then, there exist a solution $x(t)$ of $\dot{x} = f(t,x)$ starting at t=0, and a subsequence $\{n_k\}$ of $\{n\}$ such that $\{x_{n_k}(t)\}$ converges uniformly to $x(t)$ on every compact subset of $\Re_+$.

Lemma 2: Consider a time-invariant system of the form (1)-(2) where $f(t,x) = f_0(x)$ and $h(t,x) = h_0(x)$ are both independent on t and continuous. Then, the locally zero-state detectability condition implies that the weak zero-state detectability condition holds locally.

Proof: Let $U \subseteq X$ be an open set containing the origin such that the zero-state detectability condition holds in U. Let $\{\phi_n(t,t_n,x_n)\}$, lying within a compact subset of U, be any sequence of solutions such that the equation (4) is satisfied and let $x_n(t) = \phi_n(t+t_n,t_n,x_n)$ for all $n \in \aleph$. Then, $x_n(t)$ is also a solution of (1) starting at t=0 for each $n \in \aleph$. Moreover, we can assume that $x_n(t)$ is lying within a fixed compact subset of X in view of Definition 1. From Lemma 1, there exist a subsequence $\{n_k\}$ of $\{n\}$ and a solution $x(t)$ of (1) such that $\{x_{n_k}(t)\}$ converges uniformly to $x(t)$ on every compact subset of $\Re_+$. Then, $x(t) \in U, \forall t \ge 0$, and $h_0(x(t)) = \lim_{k\to\infty} h_0(x_{n_k}(t)) = 0$ for almost all t in $\Re_+$ in view of the equation (4). Since $h_0(x(t))$ is continuous, we have $h_0(x(t)) = 0$ for all t in $\Re_+$. Thus, $x(t) \to 0$ by the zero-state detectability condition. Note that,

$$\inf\{\|\phi_n(t,t_n,x_n)\| \,|\, \forall t \in [t_n,t_n+s_n], \forall n \in \aleph\}$$
$$\le \lim_{k\to\infty} \left|\phi_{n_k}(t_{n_k}+t, t_{n_k}, x_{n_k})\right| = x(t).$$

for any $t \ge 0$ and any sequence $\{s_n\}$, with $s_n \to \infty$. This implies that the equality (5) holds and the lemma is completed. □

2.2 Main results

In this subsection, our results will be proposed.

Motivated by the result of Proposition 1, the following assumption is given.

(A2) Suppose that for any constant l>0 and any compact subset κ of X, there exists a constants $M(l,\kappa)$ with $M > 2l$, such that every solution $\phi(t,t_0,x_0)$ of (1), lying within the compact set κ, will satisfy the following inequality

$$\int_0^l |h(\tau+t_0+T,\phi(\tau+t_0+T,t_0,x_0)|^2 d\tau < 1/l\,, \qquad (6)$$

for some constant T, with $l \le T \le M - l$. Now, we are in a position to give a generalization of Proposition 1 for time-varying systems in the following theorem. Its proof can be found in Appendix A.

Theorem 1: Consider a nonlinear system of the form (1)-(2). Suppose it is weakly zero-state detectable and assumption (A2) holds. Then, the origin is uniformly asymptotically stable if the origin is uniformly Lyapunov stable. Moreover, the origin is globally uniformly asymptotically stable if solutions of (1) are uniformly bounded.

To show that Theorem 1 is a generalization of Proposition 1, let us give the following lemma.

Lemma 3: Consider a time-invariant system of the form (1)-(2) ($f \equiv f_0(x), g \equiv g_0(x)$) and assume that the functions in the system are continuous. Then, the assumption (A1) implies the assumption (A2).

Proof: If assumption (A2) is false, there is a compact subset κ_0 of X and a $l_0 > 0$ such that for each $n \in \aleph$, with $n > 2l_0$, a solution $x_n(t)$ of (1) exists, which is lying within the compact set κ_0 and satisfies the following inequality

$$\int_0^{l_0} |h_0(x_n(\tau+t))|^2 d\tau \ge \frac{1}{l_0} \qquad (7)$$

for all $l_0 \le t \le n - l_0$. By Lemma 1, there exist a solution $x(t)$ of (1) and a subsequence $\{n_k\}$ of $\{n\}$ such that $\{x_{n_k}(t)\}$ converges uniformly to $x(t)$ on every compact subset of $\Re_+$. In view of the inequality (7) and taking the limit, we have

$$\int_0^{l_0} |h_0(x(\tau+t))|^2 d\tau \ge \frac{1}{l_0}, \qquad \forall t \ge l_0.$$

This implies

$$\int_0^{\infty} |h_0(x(\tau))|^2 d\tau \ge \sum_{m=1}^{\infty} \int_0^{l_0} |h_0(x(\tau+ml_0))|^2 d\tau = \infty.$$

Thus, we reach a contradiction in view of assumption (A1). The lemma is completed. □

Now, Proposition 1 follows from Theorem 1 in view of Lemmas 2 and 3. On practical applications, assumption (A2) seems complex. It is possible to give a more easily checked sufficient condition as given in the following.

(A3) Suppose that for any compact subset κ of X, there exists a positive constant $\overline{M}(\kappa)$ such that for every solution $\phi(t,t_0,x_0)$ of (1), lying within κ, we have the following integral inequality

$$\int_{t_0}^{\infty} |h(\tau,\phi(\tau,t_0,x_0))|^2 d\tau \le \overline{M}\,. \qquad (8)$$

Theorem 2: The same results hold as in Theorem 1 if assumption (A2) is replaced by assumption (A3).

Proof: Let us show that assumption (A2) is implied by assumption (A3). For any compact subset κ of X, let $\overline{M}$ be the positive constant such that the integral inequality (8) holds. For any l>0, let k in $\aleph$ such that $k \ge l\overline{M} + 2$. Let $M = kl$ (only depending on l and κ). Note that for every solution $\phi(t,t_0,x_0)$ lying within the compact set κ, we have

$$\begin{aligned}&\sum_{n=1}^{k-1} \int_0^l |h(t+t_0+nl,\phi(t+t_0+nl,t_0,x_0)|^2 dt \\ &= \int_l^{kl} |h(t+t_0,\phi(t+t_0,t_0,x_0)|^2 dt \le \overline{M}\end{aligned} \qquad (9)$$

by the integral inequality (8). Since $k - 1/l > \overline{M}$ by the choice of k, there exists a constant n_0 in $\aleph$, with $1 \le n_0 \le k-1$, such that

$$\int_0^l |h(t+t_0+n_0 l,\phi(t+t_0+n_0 l,t_0,x_0)|^2 dt < \frac{1}{l}$$

in view of the inequality (9). Let $T = n_0 l$. Then, $l \le T \le M - l$ and the inequality (6) holds. The theorem follows from Theorem 1. □

In the following, an example is given to illustrate the application of Theorem 2.

Example 1: Consider the following system

$$\dot{x} = -Q(t)x \qquad (10)$$

where $x \in \Re^p$ and $Q(t)$ is a piecewise continuous and bounded function. Assume that $Q(t)$ is a positive-semidefinite matrix and the following "persistence excitation" condition holds.

(PE) Positive constants t_0, T_0 and ε exists such that for all unit vectors w,

$$\int_t^{t+T_0} w^T Q(\tau) w d\tau \ge \varepsilon_0, \forall t \ge t_0. \qquad (11)$$

Let $V(x) = \frac{1}{2}\|x\|^2$. Then $\dot{V} \le -x^T Q(t)x$. Since V is positive definite and proper, the origin is uniformly Lyapunov stable and system is globally uniformly bounded. Define an output map $y = h(t,x) = Q^{1/2}(t)x$. Then, the assumption (A3) holds. In view of Theorem 2, we only need to check the weak detectability to guarantee the globally uniformly asymptotical stability. Let $x_n(t) = \phi_n(t+t_n,t_n,x_n^0)$ where $\{\phi_n(t,t_n,x_n^0)\}$, lying within a compact subset of $\Re^p$, is any sequence of solutions such that the equation (4) is satisfied. Then, integrating the two-side of (10), we have

$$\begin{aligned}x_n(t) - x_n(s) &= -\int_t^s Q(\tau+t_n)x_n(\tau)d\tau \\ &= -\int_t^s Q^{1/2}(\tau+t_n)h(\tau+t_n,x_n(\tau))d\tau, \forall t,s \ge 0.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} x_n(t) - x_n(s) = -\int_t^s \lim_{n\to\infty} Q^{1/2}(\tau+t_n)h(\tau+t_n,x_n(\tau))d\tau = 0$$

for all $t,s \ge 0$ by the equation (4) in the definition of weak zero-state detectability.

Since $\{\phi_n\}$ is lying within a compact subset, there is a subsequence $\{n_k\}$ of $\{n\}$ such that $\{x_{n_k}(0)\}$ converges to a vector $w_0 \in \Re^p$. Then $\lim_{k\to\infty} x_{n_k}(t) = \lim_{k\to\infty} x_{n_k}(0) = w_0$. Using the equation (4), the equality $\lim_{k\to\infty} Q^{1/2}(t+t_{n_k})w_0 = \lim_{k\to\infty} Q^{1/2}(t+t_{n_k})x_{n_k}(t) = 0$ holds for almost all t in $[0,\infty)$. By the assumption of (PE),

$$\int_0^{T_0} w^T Q(\tau + t_{n_k}) w d\tau \geq \varepsilon$$

for some $\varepsilon > 0$, each $k \in \aleph$ and each unit vector w. However,

$$\lim_{k\to\infty}\int_0^{T_0} w_0^T Q(\tau + t_{n_k}) w_0 d\tau = \lim_{k\to\infty}\int_0^{T_0} |Q^{1/2}(\tau + t_{n_k}) w_0|^2 \, d\tau = 0.$$

Thus, $w_0 = 0$ and $\phi_{n_k}(t_{n_k}, t_{n_k}, x_{n_k}^0) = x_{n_k}(0) \to 0$. This implies the equality (5) holds. The global stability follows from Theorem 2. □

Example 2 is taken from the book of Narendra et al. [11]. The original proof is based on the estimation of Lyapunov function. On the contrast, our proof is based on the verification of the weak zero-state detectability. Our approach is closed to the method of invariance principles, see [5] and [8].

3. *Uniformly asymptotical stability based on the direct Lyapunov method*

To simplify our discussion, only the uniformly asymptotical stability is studied. The globally uniformly asymptotical stability can be treated similarly. The following result is a consequence of Theorem 2 in terms of Lyapunov functions. And it will be used to prove a series of stability criterions.

Theorem 3: Consider an equation of the form (1). Suppose there exist a continuously differential decrescent lpdf *V(t,x)* defined on $\Re_+ \times \Re^p$ and a continuous nonnegative real-valued function *W(t,x)* defined on $\Re_+ \times \Re^p$ such that the inequality

$$\dot{V}(t,x) \leq -W(t,x) \tag{12}$$

holds for all $t \geq 0$ and all x in an open neighborhood of the origin. Define a virtual output map $y = W^{1/2}$. Then, the origin is uniformly asymptotical stable if the system is weakly zero-state detectable on an open neighborhood of the origin.

Proof: Since the inequality $\dot{V}(t,x) \leq -W(t,x) \leq 0$ holds locally, it is well known that the origin is uniformly Lyapunov stable, see [14]. By the definition of the output map and integrating the two-side of the inequality (12), the following inequality

$$\int_{t_0}^{\infty} |y|^2 \, dt \leq V(t_0, x_0) \tag{13}$$

can be derived for all $t_0 \geq 0$ and x_0 in an open neighborhood of the origin. Since V is decrescent, there is a function $\beta(x)$ of class K such that $V(t,x) \leq \beta(x)$ for all $t \geq 0$ and all x in an open neighborhood of the origin.

Thus, (A3) holds locally in view of the inequality (13). The theorem follows from Theorem 2 by choosing a small and suitable neighborhood X of the origin. □

Remark 2: Note that *W(t,x)*=0 is equal to $W^{1/2}(t,x) = 0$. Thus, the theorem also true if we define the output *y*=*W(t,x)* by the definition of the weak detectability.

In the following, let us use Theorem 3 to derive a series of stability criterions using the Lyapunov direct method.

Corollary 1: (Khalil, Theorem 4.5 in [7]) Consider an equation of the form (1). Suppose there exist a continuously differential decrescent lpdf *V(t,x)*, a constant $\delta > 0$, a constant $\gamma > 0$ and a function α of class K such that the inequality $\dot{V}(t,x) \leq 0$ holds and the inequality

$$\int_t^{t+\delta} \dot{V}(\tau, \phi(\tau,t,x)) d\tau \leq -\alpha(|x|) \tag{14}$$

holds for all $t \geq 0$ and all $|x| < \gamma$. Then, the origin is uniformly asymptotically stable.

Proof: Let $W(t,x) = -\dot{V}(t,x)$. We only need to check the weak zero-state detectability condition by Theorem 3. Let $\{\phi_n(t,t_n,x_n)\}$ be a sequence of solution such that the equality

$$\lim_{n\to\infty} \dot{V}(t+t_n, \phi_n(t+t_n, t_n, x_n)) = 0$$

holds for almost all t in $[0,\infty)$. Then, the following inequality

$$\lim_{n\to\infty} \alpha(|x_n|) \leq -\int_0^{\delta} \lim_{n\to\infty} \dot{V}(t+t_n, \phi_n(t+t_n,t_n,x_n))dt = 0$$

holds. This implies that $\inf\{|x_n| \, | n \in \aleph\} = 0$ since α is a function of class K. Thus, the equality (5) holds and the corollary follows from Theorem 3. □

In the following, a result modified from a theorem of Aeyles [1] is given based on Theorem 3. First, let us say that a function *V(t,x)* defined on $\Re_+ \times X$ is continuous at x uniformly in t if for any $\varepsilon > 0$, there is a $\delta > 0$ (independent on *t*) such that $|V(t,x) - V(t,y)| < \varepsilon$ for all $t \geq 0$, and all y with $|y - x| < \delta$.

Corollary 2: Consider an equation of the form (1) where *f(t,x)* is locally Lipschitz continuous with a bounded Lipschitz constant $l_x(t)$. Suppose there exist a continuously differential decrescent lpdf *V(t,x)* which is continuous at x uniformly in t, a positive definite function $\mu(x)$ and a positive constant γ such that $\dot{V}(t,x) \leq 0$ and

$$\limsup_{t\to\infty} \int_t^{t+\mu(x)} \dot{V}(\tau, \phi(\tau,t,x)) d\tau < 0 \tag{15}$$

for all $t \geq 0$ and $0 < |x| < \gamma$. Then, the origin is uniformly asymptotically stable.

Proof: Let $W(t,x) = -\dot{V}(t,x)$ and let us show that the weak zero-state detectability condition holds locally. Let $\{\phi_n(t,t_n,x_n)\}$, lying within a compact subset of $\{x \,|\, |x| < r\}$, be any sequence of solutions. Then, there is a subsequence $\{n_k\}$ of $\{n\}$ such that $\{x_{n_k}\}$ converges to a x_0 by the compactness. Let $\{\bar{\phi}(t,t_n,x_0)\}$ be a sequence of solutions of (1) starting at $t=t_n$. By the Lipschitz continuality of *f(t,x)*, there exists a positive constant *L* such that

$$
\begin{aligned}
&|\phi_n(t+t_{n_k},t_{n_k},x_{n_k})-\bar{\phi}(t+t_{n_k},t_{n_k},x_0)| \\
&\le e^{Lt}\,|x_{n_k}-x_0|
\end{aligned}
\tag{16}
$$

for all $t\ge 0$ and all *k* in $\aleph$.

Since *V(t,x)* is continuous at *x* uniformly in *t* and $\lim_{k\to\infty} x_{n_k}=x_0$, we have

$$
\begin{aligned}
&\lim_{k\to\infty} V(t+t_{n_k},\phi_{n_k}(t+t_{n_k},t_{n_k},x_{n_k}))- \\
&V(t+t_{n_k},\bar{\phi}(t+t_{n_k},t_{n_k},x_0))=0
\end{aligned}
\tag{17}
$$

for all $t\in[0,\mu(x_0)]$ by the inequality (16). Now, suppose the equation (4) holds, i.e., the equality

$$\lim_{n\to\infty}\dot{V}(t+t_n,\phi_n(t+t_n,t_n,x_n))=0$$

holds for almost all $t\in[0,\infty)$. In particular,

$$
\begin{aligned}
&\lim_{k\to\infty} V(\mu(x_0)+t_{n_k},\phi_{n_k}(\mu(x_0)+t_{n_k},t_{n_k},x_{n_k}))-V(t_{n_k},x_{n_k}) \\
&=\int_{t_{n_k}}^{t_{n_k}+\mu(x_0)}\lim_{k\to\infty}\dot{V}(\tau,\phi_{n_k}(\tau,t_{n_k},x_{n_k}))d\tau=0.
\end{aligned}
$$

This implies,

$$\lim_{k\to\infty} V(\mu(x_0)+t_{n_k},\bar{\phi}(\mu(x_0)+t_{n_k},t_{n_k},x_0))-V(t_{n_k},x_0)=0$$

in view of the equality (17) and hence

$$
\begin{aligned}
&\limsup_{t\to\infty}\int_t^{t+\mu(x_0)}\dot{V}(\tau,\bar{\phi}(\tau,t,x_0))d\tau \\
&\ge \lim_{k\to\infty}\int_{t_{n_k}}^{t_{n_k}+\mu(x_0)}\dot{V}(\tau,\bar{\phi}(\tau+t_{n_k},t_{n_k},x_0))d\tau=0.
\end{aligned}
$$

This implies $x_0=0$ by the assumption of corollary (the inequality (15)). The equality (5) follows from the inequality

$$\inf\{|\phi_n(t,t_n,x_n)|\,|\forall t\in[t_n,t_n+s_n],\forall n\in\aleph\}\le\lim_{k\to\infty}|x_{n_k}|=0.$$

Thus, the weak zero-state detectability condition holds and the corollary follows from Theorem 3. □

Remark 3: In the original paper of Aeyles [1], the asymptotic stability rather than the uniformly asymptotic stability is guaranteed. Moreover, the Lyapunov function *V(t,x)* is assumed to be time-invariant. However, Corollary 3 shows that the same result holds for time-varying Lyapunov function based on our approach.

In the remainder of this section, let us give an alternative proof of well known Krasovskii-LaSalle theorem.

Corollary 3: Consider an equation of the form (1) where *f(t,x)* is continuous and periodic. Suppose there exiest a positive constant γ and a continuously differentially lpdf *V* having th same period as the function *f* such that $\dot{V}(t,x)\le 0$ for all $t\ge 0$ and all $|x|<\gamma$. Let $S=\{x\,|\,|x|<\gamma \text{ and } \dot{V}(t,x)=0 \text{ for some } t\ge 0\}$. Then the origin is uniformly asymptotically stable if *S* does not contain any trajectories other than the trivial trajectory.

Proof: We only need to check the weak detectability in view of Theorem 3. Let $\{\phi_n(t,t_n,x_n)\}$, lying within a compact subset of $\{x\,|\,|x|<r\}$, be any sequence of solutions. Suppose the equation (4) holds, i.e., the equality

$$\lim_{n\to\infty}\dot{V}(t+t_n,\phi_n(t+t_n,t_n,x_n))=0 \tag{18}$$

holds for almost all *t* in $[0,\infty)$. Let $s_n\in[0,T_0]$ and $\lambda_n\in\aleph$ such that $t_n=s_n+\lambda_n T_0$ where T_0 is a period of the function *f*. Then, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\{s_{n_k}\}$ converges to a constant t_0 in $[0,T_0]$. Let $f_n(t,x)=f(t+t_n,x)$, $\forall t\ge 0$, $\forall |x|<\gamma$. Then, $\{f_{n_k}(t,x)\}$ converges uniformly to $f(t+t_0,x)$ on every compact subset of $\Re_+\times X$. From Lemma 1, there is also a subsequence $\{n_{k_m}\}$ of $\{n_k\}$ and a solution $\phi(t,t_0,x_0)$ of (1) such that $\{\phi_{n_{k_m}}(t+t_{n_{k_m}},t_{n_{k_m}},x_{n_{k_m}})\}$ converges uniformly to $\phi(t+t_0,t_0,x_0)$ on every compact subset of $[0,\infty)$. Then, the equation $\dot{V}(t+t_0,\phi(t+t_0,t_0,x_0))=0$, for almost all *t* in $[0,\infty)$, follows from equation (18). By the continuity of *V* and ϕ, we have $\dot{V}(t,\phi(t,t_0,x_0))=0$ for all $t\ge t_0$. In particular, $\phi(t,t_0,x_0)$ is a trajectory contained in *S*. Then, $x_0=0$ by the assumption and hence $x_{n_{k_m}}\to 0$. This implies the equality (5). So, the corollary follows from Theorem 3. □

4. Conclusion

The uniformly asymptotical stability of time-varying systems has been studied in terms of the integral inequalities involving the output map. A new stability criterion was proposed based on the weak zero-state detectability condition. Several well-known stability results using the Lyapunov direct method were deduced based on the proposed scheme. In particular, the criterion gives an unified approach for the stability analysis of nonlinear time-varying systems. The used methodology is the approach of point-set topology rather than Lyapunov function scheme. Thus, it is possible to extend the results in this paper to the study of more general dynamical systems. It was also observed that the weak zero-state detectability condition needs some modification to simplify the processing of verification. The concept of limit systems proposed in [10] may be served as a starting point on this direction.

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Appendix A: A proof of Theorem 1

Proof: In the following, let us denote the Lebesque measure of a set *S* as $|S|$. First, we claim that for any positive constant ε, any compact subset κ of *X* and any solution $\phi(t,t_0,x_0)$ of (1) lying within the compact set κ, there exists a positive constant *T* (only depending on

ε and κ) such that $|\phi(t,t_0,x_0)|<\varepsilon$ for all $t\geq t_0+T$. If the claim is false, there exist a $\varepsilon_0>0$ and a compact subset κ_0 of X such that for each $n\in\aleph$, there exist a $\bar{s}_n\geq n$ and a solution $\bar{\phi}_n(t,\bar{t}_n,\bar{x}_n)$ of (1) lying within the compact set κ_0 such that $|\bar{\phi}_n(\bar{t}_n+\bar{s}_n,\bar{t}_n,\bar{x}_n)|\geq\varepsilon_0$ for each n in $\aleph$. Since the origin is uniformly Lyapunov stable, there exists a $\delta>0$ such that the inequality

$$|\bar{\phi}_n(t,\bar{t}_n,\bar{x}_n)|\geq\delta \qquad \text{(A1)}$$

holds for any n in $\aleph$ and any t in $[\bar{t}_n,\bar{t}_n+\bar{s}_n]$. For each k in $\aleph$, let $l=2^{2k}$. Then, there exist a positive constant $M_k>2^{2k+1}$ and a constant T_k with $2^{2k}\leq T_k\leq M_k-2^{2k}$, such thatb the inequality (6) holds. Choose a large positive integer $n_k>n_{k-1}+M_k$ inductively. Then, $\bar{s}_{n_k}\geq n_k\geq M_k$. Let $h_k(t)=h(t+\bar{t}_{n_k}+T_k,\bar{\phi}(t+\bar{t}_{n_k}+T_k,\bar{t}_{n_k},\bar{x}_{n_k}))$ and $E_k=\left\{t\in[0,2^{2k}]\,\middle|\,|h_k(t)|^2\geq\frac{1}{2^k}\right\}$. Then, we have

$$|E_k|\cdot\frac{1}{2^k}\leq\int_0^{2^{2k}}|h_k(t)|^2\,dt<\frac{1}{2^{2k}}$$

where the latter inequality follows from the inequality (6). This implies that the following inequality

$$|\bigcup_{k\geq m}E_k|\leq\sum_{k\geq m}^{\infty}\frac{1}{2^k}=\frac{1}{2^{m-1}}$$

holds for each m in $\aleph$. Thus, $\left|\bigcap_{m\geq1}\bigcup_{k\geq m}E_k\right|=\lim_{m\to\infty}\left|\bigcup_{k\geq m}E_k\right|=0$. Note that the following equality

$$\lim_{k\to\infty}h_k(t)=0 \qquad \text{(A2)}$$

holds for all $t\in[0,\infty)-\bigcap_{m\geq1}\bigcup_{k\geq m}E_k$ and hence the equality $\lim_{k\to\infty}h_k(t)=0$ holds for almost all $t\in[0,\infty)$.

Let $t_k=\bar{t}_{n_k}+T_k, x_k=\bar{\phi}_{n_k}(\bar{t}_{n_k}+T_k,\bar{t}_{n_k},\bar{x}_{n_k})$ and $\phi_k(t,t_k,x_k)=\bar{\phi}_{n_k}(t,\bar{t}_{n_k},\bar{x}_{n_k})$. Then, for each k in $\aleph$, ϕ_k is a solution of (1) starting at x_k at time $t=t_k$. Note that $h_k(t)=h(t+t_k,\phi_k(t,t_k,x_k))$ by the definition. Since the equality (A2) holds for almost all t in $[0,\infty)$, we have the equation (4). Let $s_k=\bar{s}_{n_k}-T_k$. Then, $s_k\geq M_k-T_k\geq 2^{2k}$. In particular, $s_k\to\infty$. Similarly, $t_k\geq T_k\geq 2^{2k}$ and hence $t_k\to\infty$. Thus,

$$\inf\{|\phi_k(t,t_k,x_k)|\,|\,\forall t\in[t_k,t_k+s_k],\forall k\in\aleph\}=0 \qquad \text{(A3)}$$

by the weak zero-state detectability condition. Since $[t_k,t_k+s_k]\subseteq[\bar{t}_{n_k},\bar{t}_{n_k}+\bar{s}_{n_k}]$ for each k in $\aleph$ from the definition of t_k and s_k,

$$\inf\{|\bar{\phi}_{n_k}(t,\bar{t}_{n_k},\bar{x}_{n_k})|\,|\,\forall t\in[\bar{t}_{n_k},\bar{t}_{n_k}+\bar{s}_{n_k}],\forall k\in\aleph\}=0$$

by the equality (A3). In view of the inequality (A1), we reach a contradiction. Thus, the claim is true. Now, let us check the uniformly asymptotical stability. Since the origin is uniformly Lyapunov stable, there is a $\delta_0>0$ such that for each $|x_0|<\delta_0$, every solution $\phi(t,t_0,x_0)$ of (1) will belong to a fixed compact subset of X. Thus, the origin is uniformly asymptotically stable in view of the claim. Similarly, since $X=\Re^n$ and solutions of (1) are globally uniformly bounded, the globally uniformly asymptotical stability follows from the claim. □

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