On the Detectability and Observability of Discrete-Time Markov Jump Linear Systems [1]

Eduardo F. Costa and João B. R. do Val
UNICAMP - Fac. de Engenharia Elétrica e de Computação
Depto. de Telemática, C.P. 6101, CEP 13081-970, Campinas, SP, Brazil
email: jbosco@dt.fee.unicamp.br

Abstract

This paper presents a new detectability concept for discrete-time Markov jump linear systems with finite Markov state, which generalizes the MS-detectability concept found in the literature. The new sense of detectability can similarly assure that the solution of the coupled algebraic Riccati equation associated to the quadratic control problem is a stabilizing solution. In addition, the paper introduces a related observability concept which also generalizes previous concepts. Tests for detectability or observability are derived from the corresponding definitions, that can be perfomed in a finite number of steps. An illustrative example is included to show that a system may be detectable in the new sense but not in the MS sense.

1 Introduction

The paper considers the discrete-time Markov jump linear system (MJLS)

$$\Phi: \begin{cases} x(k+1) = A_{\theta(k)}x(k), & k \geq 0, \\ y(k) = C_{\theta(k)}x(k), & x(0) = x_0, \quad \theta(0) \sim \mu_0 \end{cases} \tag{1}$$

where x and y are the state and the output variables, respectively. The mode θ is the state of an underlying discrete-time homogeneous Markov chain $\Theta = \{\theta(k); k \geq 0\}$ having $\mathcal{S} = \{1,\dots,S\}$ as state space and $P = [p_{ij}]$, $i,j = 1,\dots,S$ as the transition probability matrix. The initial distribution of Θ is determined by $\mu_{0,i} = P(\theta(0) = i)$, $i = 1,\dots,S$. Matrices A_i and C_i, $i = 1,\dots,S$ belong to the collections of S real matrices: $A = (A_1,\dots,A_S)$, $\dim(A_i) = s \times s$, and $C = (C_1,\dots,C_S)$, $\dim(C_i) = q \times s$.

The theory of Markov jump linear systems is fairly complete nowadays, e.g. see [2], [4], [5], [7], [9] and [11], and it parallels the theory of deterministic linear systems in many aspects. One important similarity is the role of the stochastic concepts of stabilizability and detectability to guarantee existence, uniqueness and stability of closed-loop solutions of quadratic infinite horizon control problems, see [4] or [11].

It is widely known for deterministic linear systems that the following assertions are equivalent: (i) there exists a matrix G for the system (A,C) such that the spectral radius of $(A+GC)$ is less than one; (ii) if the output process has "very low" energy, then the state trajectory is "fast decaying", i.e., every non-observed state corresponds to stable modes of the system. Assertions (i) and (ii) give rise to the same detectability concept for deterministic systems. On the other hand, the single detectability concept available in the MJLS theory parallels the concept in assertion (i). The extension for the second concept has not been investigated yet and it is not known up to this date if a similar equivalence holds.

In this paper we fill this gap: we derive a detectability concept which relates the energy carried out by the output and the trajectory decay. We call this new concept W-detectability, and as a matter of fact, we show that it generalizes MS-detectability, the existing concept in the literature, see [4] and [10]. At one stage, a simple example shows that W-detectability does not imply MS-detectability, and the paper also demonstrates that the W-detectability plays the same role as MS-detectability in the theory of coupled algebraic Riccati equations (CARE): it guarantees that the solution to the CARE is stabilizing.

The related concept of W-observability is derived from the W-detectability. It is also shown that the requirements for observability which appear in the MJLS literature, such as in [10] and [12], are more demanding than those for W-observability. Tests that allow computation of the structural properties of W-detectability and W-observability are presented in the paper.

2 Notation, Concepts and Basic Results

Let $\mathbb{R}^s$ represent the linear space formed by all s-vectors. Let $\mathcal{R}^{r,s}$ (respectively, $\mathcal{R}^r$) represent the normed linear space formed by all $r \times s$ real matrices (respectively, $r \times r$) and $\mathcal{R}^{r0}$ ($\mathcal{R}^{r+}$) the closed convex cone $\{U \in \mathcal{R}^r : U = U' \geq 0\}$ (the open cone $\{U \in \mathcal{R}^r : U = U' > 0\}$) where

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U' denotes the transpose of U; $U \geq V$ ($U > V$) signifies that $U - V \in \mathcal{R}^{r0}$ ($U - V \in \mathcal{R}^{r+}$). Let $\mathcal{M}^{r,s}$ denote the linear space formed by a number S of matrices such that $\mathcal{M}^{r,s} = \{U = (U_1, \ldots, U_S) : U_i \in \mathcal{R}^{r,s}, i = 1, \ldots, S\}$; also, $\mathcal{M}^r \equiv \mathcal{M}^{r,r}$. We denote by $\mathcal{M}^{r0}$ ($\mathcal{M}^{r+}$) the set $\mathcal{M}^r$ when it is made up by $U_i \in \mathcal{R}^{r0}$ ($U_i \in \mathcal{R}^{r+}$) for all $i = 1, \ldots, S$. It is known that $\mathcal{M}^{r,s}$ equipped with the inner product

$$< U, V >= \sum_{i=1}^{S} tr\{U_i' V_i\}$$

forms a Hilbert space. We define the norm $\|U\|^2 =< U, U >$.

Consider system Φ; for $i = 1, \ldots, S$ we define

$$X_i(t) = E_{x_0,\mu_0}\{x(t)x(t)' 1_{\theta(t)=i}\} \tag{2}$$

and, for instance, with this notation we can write $E_{x_0,\mu_0}\{|x(t)|^2\} =< X, I >= \|X(t)^{1/2}\|^2$. We also define the operators $\mathcal{L} : \mathcal{M}^{s0} \to \mathcal{M}^{s0}$, and the dual $\mathcal{T} : \mathcal{M}^{s0} \to \mathcal{M}^{s0}$ as

$$\mathcal{L}_i(U) = \sum_{j=1}^{S} p_{ji} A_j U_j A_j' \tag{3}$$

$$\mathcal{T}_i(U) = A_i' \{\sum_{j=1}^{S} p_{ij} U_j\} A_i, \quad i = 1, \ldots, S$$

We denote $\mathcal{T}^0(U) = U$, and for $t \geq 1$, we can define $\mathcal{T}^t(U)$ recursively by $\mathcal{T}^t(U) = \mathcal{T}(\mathcal{T}^{t-1}(U))$. Notice that $\mathcal{T}$ and $\mathcal{L}$ are linear and continuous on their arguments and the set $\mathbb{C} = \{U \in \mathcal{M}^{s0} : \|U\| = 1, \|\mathcal{T}^t(U)\| \geq \delta\}$ for some $t \geq 0$ and $\delta \geq 0$, is a compact set.

For $N \geq 0$, let us introduce the functional

$$W^N(X) = \sum_{t=0}^{N-1} < X(t), C'C > \tag{4}$$

whenever $X(0) = X$. We consider the sequence $O(n)$, $n \geq 0$ on $\mathcal{M}^{s0}$, defined recursively as

$$O_i(n) := C_i' C_i + \mathcal{T}_i(O(n-1)), \quad n = 1, 2, \ldots \tag{5}$$

for each $i = 1, \ldots, S$, with $O(0) = 0$. Notice that $O_i(n+1) \geq O_i(n)$, $i = 1, \ldots, S$.

The following results are adapted from [3] and [6]; the proof is omitted.

Lemma 1. *Consider system Φ. Then,*

$$\begin{cases} X_i(t+1) = \mathcal{L}_i(X(t)), & t \geq 0 \\ X_i(0) = X_i = x_0 x_0' \mu_{0,i}, & i = 1, \ldots, S \end{cases} \tag{6}$$

and $W^N(X) =< X, O(N) >$.

2.1 MS-Stability, MS-Detectability and W-Detectability

Definition 1. The system Φ is mean square stable (MS-stable) if for each x_0 and μ_0,

$$\lim_{k \to \infty} E_{x_0,\mu_0}\{|x(k)|^2\} = 0$$

Remark 1. Ji et al. in [11] have shown that MS-stability concept is equivalent to others second moment stability concepts, such as exponential stability. It is also known that: (i) if the system is not MS-stable, then there exists $X(0) \in \mathcal{M}^{s0}$ such that $\|X(t)\| \geq \rho \xi^t \|X(0)\|$ for some $0 < \rho \leq 1$ and $\xi \geq 1$; (ii) if $\lim_{t \to \infty} \|X(t)\| = 0$, then $\|X(t)\| \leq \beta \zeta^t \|X(0)\|$ for some $\beta \geq 1$ and $0 < \zeta < 1$.

Definition 2. Consider system Φ. We say that (A, C) is MS-detectable when there exists $L \in \mathcal{M}^{r,s}$ for which the system $z(k+1) = (A_{\theta(k)} - L_{\theta(k)} C_{\theta(k)}) z(k)$, $z(0) = z_0$, $k \geq 0$, is MS-stable.

Notice that the functional in (4) has the physical interpretation of the accumulated energy of the output process y on the interval $0 \leq t \leq N-1$. Indeed,

$$\begin{aligned} W^N(X) &= E_{x_0,\mu_0}\left\{\sum_{t=0}^{N-1} x(t)' C_{\theta(t)}' C_{\theta(t)} x(t)\right\} \\ &= E_{x_0,\mu_0}\left\{\sum_{t=0}^{N-1} |y(t)|^2\right\} \end{aligned}$$

The W-detectability concept relates the energy of the output and the trajectory decay, as follows.

Definition 3 (W-detectability). Consider system Φ. We say that (A, C) is W-detectable when there exist integers N_d, $t_d \geq 0$ and scalars $0 \leq \delta < 1$, $\gamma > 0$ such that $W^{N_d}(X) \geq \gamma \|X\|$ whenever $\|X(t_d)\| \geq \delta \|X\|$.

Remark 2. The above concept of detectability is based in standard concepts of detectability for linear time-varying systems, see e.g. [1] or [8]. As we shall see in Lemma 3, the concept retrieves the idea that every non-observed state corresponds to stable modes of the system. Notice that every MS-stable MJLS is W-detectable with N_d and γ arbitrary and δ and t_d such that $\delta = \beta \zeta^{t_d} < 1$, where β and ζ are as in Remark 1.

3 W-Detectability Properties

We start this section presenting some properties of the functional $W^N(X)$ and the associated sequence $O(n)$ in (4) and (5) respectively.

A deterministic linear representation for MJLS with finite Markov state space was proposed in [3]; see Appendix A for details. The associated dimension is

$$\ell = s^2 S \tag{7}$$

We apply the aforementioned representation and the controllability theorem for linear time-invariant systems, to obtain the results in the lemma.

Lemma 2. *(i) If for some X, $W^\ell(X)=0$, then $W^n(X)=0$ for all $n\geq 0$;*

(ii) If there exists n, $0\leq n\leq \ell$, such that $W^\ell(X(n))=0$, then $W^\ell(X(t))=0$ for every $t\geq n$.

See Appendix A for the proof. Now we are ready to present the main result of the section.

Lemma 3. *Consider system Φ. (A,C) is W-detectable if and only if whenever $W^\ell(X)=0$ one has that $||X(t)||\to 0$ as $t\to\infty$.*

Proof: Necessity. We shall show that under W-detectability of (A,C), $||X(t)||\to 0$ as $t\to\infty$ when $W^\ell(X)=0$. Since $W^\ell(X)=0$, from Lemma 2 (ii) we have that $W^\ell(X(t))=0$ for every $t\geq 0$. Then, in view of the W-detectability of (A,C), we have that $||X(t+t_d)||<\delta||X(t)||$ for every $t\geq 0$ and some $t_d\geq 0$ and $0\leq\delta<1$; consequently, $||X(t+nt_d)||<\delta^n||X(t)||$ and, hence,

$$\lim_{n\to\infty}\max_{0\leq t\leq t_d-1}||X(t+nt_d)||\leq\lim_{n\to\infty}\delta^n\max_{0\leq t\leq t_d-1}||X(t)||=0$$

and the result follows in a straightforward manner.

Sufficiency. Let us consider the set

$$\mathbb{Z}=\{Z:||Z||=1,<Z,O(\ell)>=0\}\qquad(8)$$

and let us denote as $Z(t)$ the trajectory corresponding to an initial condition $Z\in\mathbb{Z}$. By hypothesis, $||Z(t)||\to 0$ as $t\to\infty$ and we can write in a similar fashion to Remark 1 that there exist $0<\zeta<1$ and $\beta\geq 1$ such that $||Z(t)||\leq\beta\zeta^t$, $\forall Z\in\mathbb{Z}$. Consequently, there exist $t_d\geq 0$ and $0\leq\delta<1$ such that $||Z(t_d)||<\delta$, $\forall Z\in\mathbb{Z}$, and we can write

$$\mathbb{Z}\subset\bar{\mathbb{C}}=\{Z:||Z||=1,||Z(t_d)||<\delta\}$$

In this proof we shall demonstrate that there exists $\gamma>0$ such that whenever $||X(t_d)||\geq\delta$ then $W^\ell(X)=<X,O(\ell)>\geq\gamma||X||$ and consequently (A,C) is W-detectable. Let us deny the assertion and suppose that for every $\gamma>0$ there exists X, $||X||=1$ such that $<X,O(\ell)><\gamma$ and $||X(t_d)||\geq\delta$, i.e., $X\in\mathbb{C}$ where $\mathbb{C}$ is the compact set

$$\mathbb{C}=\{X:||X||=1,||X(t_d)||\geq\delta\}$$

which is the complement of set $\bar{\mathbb{C}}$. Let us take a sequence $X_n\in\mathbb{C}$ such that $<X_n,O(\ell)><\gamma_n$ and $\gamma_n\to 0$ as $n\to\infty$. Hence, from the compacity of $\mathbb{C}$ there exists a subsequence X_m which converges to some $\widehat{X}\in\mathbb{C}$ and from the continuity of $W^\ell(X)=<X,O(\ell)>$,

$$\lim_{m\to\infty}(W^\ell(X_m))=W^\ell(\widehat{X})=0$$

In view of (8), $\widehat{X}\in\mathbb{Z}\subset\bar{\mathbb{C}}$, which completes the proof by contradiction. □

Remark 3. A test of W-detectability can be produced by first identifying a base $\mathcal{V}=\{v_i^1,\dots,v_i^r\}$, $v_i^j\in\mathbb{R}^s$, $r\leq s$, for $\mathcal{N}(O_i(\ell))$, the null space of each matrix $O_i(\ell)$ in (5). For each $i=1,\dots,S$, define V_i by setting $V_i=\sum_{j=1}^r v_i^j v_i^{j\prime}$, and check if the following set of equations has a solution P in $\mathcal{M}^{s0}$

$$P_i=V_i+\mathcal{L}_i(P),\quad i=1,\dots,S\qquad(9)$$

In the positive case, one has that $\lim_{t\to\infty}||X(t)||=0$ when $X(0)=V$, since $\sum_{t=0}^\infty\mathcal{L}_i^t(V)\leq P_i$ for each i; in this situation, it is not difficult to check that $\lim_{t\to\infty}||X(t)||=0$ whenever $W^\ell(X(0))=0$ and from Lemma 3 we have that (A,C) is W-detectable. In the negative case, (A,C) is not W-detectable; in fact, if we deny this assertion and assume that (A,C) is W-detectable, Lemma 3 leads to $\lim_{t\to\infty}||\mathcal{L}^t(V)||=0$, since $W^\ell(V)=0$, and due to Remark 1 we can write $P_i=\sum_{t=0}^\infty\mathcal{L}_i^t(V)\in\mathcal{M}^{s0}$, which satisfies (9).

3.1 W-detectability is Weaker than MS-detectability

In this section we show that MS-detectability implies W-detectability, but the reverse implication fails.

Lemma 4. *Suppose system Φ is MS-stable. Then $(A-GD,D)$ is W-detectable for every $G\in\mathcal{M}^{s,q}$ and $D\in\mathcal{M}^{q,s}$.*

Proof: In this proof $X(\cdot)$ and $\mathcal{T}(\cdot)$ refers to the system Φ and $\widehat{X}(\cdot)$ and $\widehat{\mathcal{T}}(\cdot)$ refers to the system

$$\widehat{\Phi}:\begin{cases}x(k+1)=(A_{\theta(k)}-G_{\theta(k)}D_{\theta(k)})x(k), & k\geq 0\\ y(k)=D_{\theta(k)}x(k), \qquad x(0)=x_0, & \theta(0)\sim\mu_0\end{cases}$$

We shall show that $||\widehat{X}(t)||\to 0$ as $t\to\infty$, provided that $W^\ell(\widehat{X})=0$. From Lemma 2 (i) we can write

$$0=W^n(\widehat{X})=<\sum_{t=0}^{n-1}\widehat{X}(t),D'D>$$

for every $n\geq 0$. Then, we have that $<\widehat{X}(t),D'D>=<D\widehat{X}(t)^{1/2},D\widehat{X}(t)^{1/2}>=0$ which implies that $D_i\widehat{X}_i(t)=0$ for all $t\geq 0$ and i. Now, we set $\widehat{A}_i=(A_i-G_iD_i)$ for every $i=1,\dots,S$ and we obtain

$$\begin{aligned}\widehat{\mathcal{L}}_i(\widehat{X}(t))&=\sum_{i=1}^S p_{ij}\widehat{A}_i\widehat{X}_i(t)\widehat{A}_i'\\&=\sum_{i=1}^S p_{ij}(A_i-G_iD_i)X_i(t)(A_i-G_iD_i)'\\&=\sum_{i=1}^S p_{ij}A_i\widehat{X}_i(t)A_i'=\mathcal{L}_i(\widehat{X}(t))\end{aligned}$$

Thus, provided that $\widehat{X}(0)=X(0)$, one has that $\widehat{X}(t)=X(t)$ for all $t\geq 1$,and since Φ is MS-stable, we conclude that $||\widehat{X}(t)||=||X(t)||\to 0$ as $t\to\infty$. Finally, from Lemma 3 we have that $(A-GD,D)$ is W-detectable. □

Theorem 1. *If (A,C) is MS-detectable then (A,C) is W-detectable.*

Proof: Since (A,C) is MS-detectable, from definition there exists L such that the system

$$\widetilde{\Phi}: \begin{cases} x(k+1) = (A_{\theta(k)} - L_{\theta(k)}C_{\theta(k)})x(k), & k \geq 0 \\ y(k) = D_{\theta(k)}x(k), & x(0) = x_0, \quad \theta(0) \sim \mu_0 \end{cases}$$

is MS-stable. Let $\widetilde{A}_i = A_i - L_iC_i$ for all i and, from Lemma 4, $(\widetilde{A} - GD, D)$ is W-detectable for every $G \in \mathcal{M}^{s,q}$ and $D \in \mathcal{M}^{q,s}$. The proof is completed by retrieving the original system Φ with the choice $D = C$ and $G = -L$. □

We present here a simple example showing that the converse of Theorem 1 is not necessarily true, and hence that W-detectability generalizes the MS-detectability concept. Let $S = 2$ and set

$$A_1 = A_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}; C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \; C_2 = 0; \; P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \tag{10}$$

From (7) we have that $\ell = 8$ and from (5) and Lemma 1, we can evaluate $W^8(X) = < X, O^8 >> 98.69\,\|X\|$. Hence, $W^8(X) = 0$ only if $X = 0$ and we conclude from Lemma 3 that (A,C) is W-detectable. On the other hand, one can check by implementing it that the Riccati difference equation:

$$\begin{aligned} P_i(n) =& A_i\mathcal{E}_i(P(n-1))\{I - C_i'(R_i + C_i\mathcal{E}_i(P(n-1))C_i')^{-1} \\ & C_i\mathcal{E}_i(P(n-1))\}A_i' + I \\ P_i(0) =& 0, \quad i = 1, \ldots, S \end{aligned}$$

does not converge as $n \to \infty$, and from [4, proposition 6] we have that there is no $L \in \mathcal{M}^{r,s}$ such that the system $z(k+1) = (A'_{\theta(k)} - C'_{\theta(k)}L'_{\theta(k)})z(k)$ is MS-stable and hence, (A,C) is not MS-detectable.

4 W-Detectability and the Linear Quadratic Control

Consider the closed loop version of system Φ,

$$\Phi_c \; : \; x(k+1) = (A_{\theta(k)} - B_{\theta(k)}G_{\theta(k)})x(k) \tag{11}$$

where $B \in \mathcal{M}^{s,r}$ and $G \in \mathcal{M}^{r,s}$ can be regarded as a linear state feedback control; consider also the associated cost functional

$$\begin{aligned} J^N(X) &= E_{x_0,\mu_0}\left\{ \sum_{t=0}^{N-1} x(t)'(Q_{\theta(t)} + G'_{\theta(k)}R_{\theta(t)}G'_{\theta(k)})x(t) \right\} \\ &= \sum_{t=0}^{N-1} < X(t), Q + G'RG > \end{aligned} \tag{12}$$

where $Q \in \mathcal{M}^{s0}$ and $R \in \mathcal{M}^{s+}$ are weighting matrices. Notice that the functionals J^N and W^N are closed related. In fact, it is easy to check that when $C = (Q + G'RG)^{1/2}$, the expressions for $W^N(X)$ and $J^N(X)$ are identical.

Consider the coupled equation in the unknown $P \in \mathcal{M}^{s0}$

$$P_i = \widehat{\mathcal{T}}_i(P) + G_i'R_iG_i + Q_i, \quad i = 1, \ldots, S, \tag{13}$$

where $\widehat{\mathcal{T}}_i(U) = \widehat{A}_i'\mathcal{E}_i(U)\widehat{A}_i$ and $\widehat{A}_i = A_i - B_iG_i$ for each i. It can be shown from [4] or [11] that when G_i, $i = 1, \ldots, S$ satisfies (13), then

$$J^\infty(X) = \lim_{N\to\infty} J^N(X) \leq < X, P > \tag{14}$$

In this section we shall show that system Φ_c is MS-stable provided that G satisfies (13) and $(A, Q^{1/2})$ is W-detectable. In particular, we conclude that the solution to the CARE arising in optimal linear quadratic control problems is a stabilizing solution under the W-detectability assumption, in such a manner that W-detectability plays the same role as the MS-detectability concept in optimal linear quadratic problems, cf. [4] and [7].

Lemma 5. *Assume $(A, Q^{1/2})$ is W-detectable. Then, $(A - BG, (Q + G'RG)^{1/2})$ is W-detectable for any $G \in \mathcal{M}^{r,s}$.*

Proof: In this proof, $\mathcal{L}$ and W refer to the system Φ, and $\widehat{\mathcal{L}}$ and $\widehat{W}$ refer to Φ_c; $X(\cdot)$ is the trajectory of system Φ_c. We show that $\|X(t)\| \to 0$ as $t \to \infty$ whenever $\widehat{W}^\ell(X) = 0$. From Lemma 2 (i) we can write for every $n \geq 0$, that

$$\begin{aligned} 0 = \widehat{W}^n(X) &= \sum_{t=0}^{n-1} < X(t), (Q + G'RG)'^{1/2}(Q + G'RG)^{1/2} > \\ &= \sum_{t=0}^{n-1} < X(t), Q + G'RG > \geq < \sum_{t=0}^{n-1} X(t), G'RG > \end{aligned} \tag{15}$$

Hence, we can evaluate $< X(t), G'RG > = < R^{1/2}GX(t)^{1/2}, R^{1/2}GX(t)^{1/2} > = 0$ and since $R_i > 0$, we get that $G_iX_i(t) = 0$, for each $t \geq 0$ and i. We set $\widehat{A}_i = (A_i - B_iG_i)$ for $i = 1, \ldots, S$ and we obtain

$$\begin{aligned} \widehat{\mathcal{L}}_i(X(t)) &= \sum_{i=1}^{S} p_{ij}\widehat{A}_iX_i(t)\widehat{A}_i' \\ &= \sum_{i=1}^{S} p_{ij}A_i(X_i(t))A_i' = \mathcal{L}_i(X(t)) \end{aligned}$$

for every $t \geq 0$, meaning that the trajectories of systems Φ_c and Φ coincide whenever the initial conditions coincide. We conclude from (15) that

$$W^\ell(X) = \sum_{t=0}^{\ell-1} < X(t), Q > \leq \widehat{W}^\ell(X) = 0$$

and the detectability of $(A, Q^{1/2})$ assures that $\|X(t)\| \to 0$ as $t \to \infty$. □

Theorem 2. *Consider the system Φ_c and assume that $(A, Q^{1/2})$ is W-detectable and G satisfies (13) in the unknown $P \in \mathcal{M}^{s0}$. Then, the system is MS-stable.*

Proof: In this proof $X(\cdot)$ refers to the trajectory of system Φ_c. Notice that, from Lemma 5 and the assumption in the theorem, we have that $(\hat{A}, \hat{C})$ is W-detectable, where

$\widehat{A} = (A - BG)$ and $\widehat{C} = (Q + G'RG)^{1/2}$; let N_d, t_d, δ, γ be as in Definition 3. For sake of argument we assume that the system is not MS-stable. In this situation, there exists $X(0) \neq 0$ such that

$$||X(t)|| \geq \rho\xi^t \, ||X(0)|| \tag{16}$$

for some $0 < \rho \leq 1$ and $\xi \geq 1$, see Remark 1. Let us define the sequence $\mathcal{N} = \{n_1, n_2, \ldots\}$ where each n_i is such that

$$||X((n_i + 1)t_d)|| \geq \delta ||X(n_i t_d)||$$

holds. If the number of elements of $\mathcal{N}$ is finite, one can check that

$$\lim_{n\to\infty} ||X(nt_d)|| = 0$$

which contradicts (16) and we conclude that the number of elements of $\mathcal{N}$ is not finite. Then we can take a subsequence from $\mathcal{N}$ with infinetly many elements $\mathcal{N}' = \{n_{i_1}, n_{i_2}, \ldots\}$, where each i_k is such that $n_{i_{k+1}} \geq n_{i_k} + \max\{1, (N_d/t_d)\}$, and we can write

$$\begin{aligned} J^\infty(X) &= \sum_{n=0}^{\infty} < X(n), Q + G'RG > \\ &\geq \sum_{k=0}^{\infty}\sum_{j=0}^{N_d} < X(n_{i_k}t_d + j), Q + G'RG > \\ &\geq \sum_{k=0}^{\infty} \gamma \|X(n_{i_k}t_d)\| \geq \gamma \sum_{k=0}^{\infty} \rho\xi^{(n_{i_k}t_d)} \, ||X(0)|| \\ &\geq \gamma \sum_{k=0}^{\infty} \rho\xi^{(n_{i_0}t_d)} \, ||X(0)|| = \infty \end{aligned}$$

Thus, there exists no $P \in \mathcal{M}^{s0}$ satisfying (14), contradicting the hipothesis that there exists $P \in \mathcal{M}^{s0}$ satisfying (13). □

Corollary 1. *Consider the system*

$$x(k+1) = A_{\theta(k)}x(k) + B_{\theta(k)}u(k)$$

and the associated infinite-horizon linear quadratic cost $J^\infty(X)$. *If* $(A, Q^{1/2})$ *is W-detectable and there exists a solution* $P \in \mathcal{M}^{s0}$ *to the CARE*

$$P_i = A_i'\mathcal{E}_i(P)\{I - B_i(R_i + B_i'\mathcal{E}_i(P)B_i)^{-1}B_i'\mathcal{E}_i(P)\}A_i + Q_i \tag{17}$$

then the system is MS-stable with the feedback gain

$$u(k) = -(R_i + B_i'\mathcal{E}_i(P)B_i)^{-1}B_i'\mathcal{E}_i(P)A_i x(k) \tag{18}$$

defined whenever $\theta(k) = i$.

Proof: It is easy to check that the gain $G_i = (R_i + B_i'\mathcal{E}_i(P)B_i)^{-1}B_i'\mathcal{E}_i(P)A_i$ satisfies equation (13) in P, and the result follows immediately from Theorem 2. □

5 W-Observability Concept

A particularization of the W-detectability concept in Definition 3 is obtained by setting $\delta = 0$. This leads to the following observability concept.

Definition 4. Consider system Φ. We say that (A, C) is W-observable when there exist a positive integer N_d and a scalar $\gamma > 0$ such that $W^{N_d}(X) \geq \gamma ||X||$ for every $X \in \mathcal{M}^{s0}$.

An equivalent concept of W-observability which allows a simple computational test is presented in the sequel. It is obtained in a similar fashion to that in Lemma 2 (i); the proof is omitted.

Lemma 6. *Consider system* Φ. (A, C) *is W-observable if and only if* $O_i(\ell) > 0$ *for each* $i = 1, \ldots, S$.

The observability condition in Definition 4 or Lemma 6 is more general than similar conditions for MJLS in the literature. We can mention the concept found in [12], namely, observability in the deterministic sense of each pair $(p_{ii}^{1/2}A_i, C_i)$, $i = 1, \ldots, S$. This condition is sufficient for the W-observability to hold, since from (5) we can write

$$O_i(s) \geq C_i'C_i + p_{ii}A_i'C_i'C_iA_i + \cdots + p_{ii}^{s-1}A_i^{s-1\prime}C_i'C_iA_i^{s-1} > 0$$

In another comparison, Ji and Chizeck introduce in [10] the idea of sample path controllability, for which, there should exist a number N such that for each sample path $\theta(0), \ldots, \theta(N)$, the matrix $[C_{\theta(0)}' \;\; A_{\theta(1)}'C_{\theta(1)}' \cdots A_{\theta(N)}'C_{\theta(N)}']$ has a.s. full rank. It is simple to check that this condition assures observability in Definition 4.

6 Conclusions

The paper introduces the concept of W-detectability for MJLS, which generalizes the previous concept of MS-detectability, and plays the same role in the quest for stabilizing solutions of quadratic control problems. The corresponding concept of W-observability also generalizes previous ideas on observability encountered in the literature of discrete-time MJLS. The W-detectability concept carries along a computational test for this structural property as presented in Remark 3. The test of W-observability in Lemma 6 for MJLS is similar to the test for deterministic time-invariant linear systems with dimension ℓ.

A Appendix: Proof of Lemma 2

For $V \in \mathcal{R}^s$, let us identify the columns of $V = [v_1 \vdots v_2 \vdots \cdots \vdots v_s]$. For $U = (U_1, \ldots, U_S)$ and following [3], let us introduce the following linear and invertible operators

$$\varphi(V) = \begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix}, \qquad \widehat{\varphi}(U) = \begin{bmatrix} \varphi(U_1) \\ \vdots \\ \varphi(U_S) \end{bmatrix}$$

Proposition 4a) of [3] yields that

$$\widehat{\varphi}(\mathcal{T}(U)) = \mathcal{A}\widehat{\varphi}(U) \tag{19}$$

where $\mathcal{A} \in \mathcal{R}^{\ell}$ is the matrix defined by

$$\mathcal{A} := \mathrm{diag}(A_1' \otimes A_1', \ldots, A_S' \otimes A_S')(P \otimes I_{s^2})$$

where $V \otimes Z$ represents the Kronecker tensor product of matrices V and Z.

Let us consider $O(n) = (O_1(n), \ldots, O_S(n))$, $n \geq 0$ as in (5), and define $o^n \in \mathcal{R}^{\ell,1}$ and $q \in \mathcal{R}^{\ell,1}$ by

$$o^n = \widehat{\varphi}(O(n)), \quad q = \widehat{\varphi}(C'C)$$

and we write (5) equivalently as

$$\begin{aligned} o^n &= \widehat{\varphi}(C_1'C_1 + \mathcal{T}_1(O(n-1)), \ldots, C_S'C_S + \mathcal{T}_S(O(n-1))) \\ &= q + \widehat{\varphi}(\mathcal{T}_1(O(n-1)), \ldots, \mathcal{T}_S(O(n-1))) \end{aligned}$$

the last identity is provided by the linearity of $\widehat{\varphi}$. From (19), we have that

$$o^n = q + \mathcal{A}\widehat{\varphi}(O_1(n-1), \ldots, O_S(n-1)) = q + \mathcal{A}o^{n-1}, \quad n \geq 1 \tag{20}$$

and (20) is equivalent to

$$o^n = q + \mathcal{A}q + \cdots + \mathcal{A}^{n-1}q, \quad n \geq 1. \tag{21}$$

Now let us consider the expression $< X, O(n) >$. We denote $x_v = \widehat{\varphi}(X) \in \mathcal{R}^{\ell,1}$ and it is straightforward to check that

$$< X, O(n) > = x_v' \, o^n \tag{22}$$

and since $< X, \mathcal{T}^n(C'C) > \geq 0$, one has that $x_v'\mathcal{A}^n q \geq 0$.

Proof of Lemma 2 (i).

From Lemma 1, we have that $W^{\ell}(X) = < X, O(\ell) >$. If there exists X such that $W^{\ell}(X) = 0$ then, from (21) and (22), we obtain

$$0 = x_v' o^{\ell} = x_v'(q + \mathcal{A}q + \cdots + \mathcal{A}^{\ell-1}q) \tag{23}$$

Since $x_v'\mathcal{A}^n q \geq 0$ for each $0 \leq n \leq \ell - 1$, we have that

$$x_v'\mathcal{A}^n q = 0, \quad \text{for each } 0 \leq n \leq \ell - 1$$

Thus, using the Cayley-Hamilton theorem, we can conclude that $x_v' o^k = 0$ for each $k \geq 0$, and from (22) it follows that

$$< X, O(k) > = 0, \quad k \geq 0. \quad \square$$

Proof of Lemma 2 (ii).

From Lemma 1, we have that $W^{\ell}(X(t)) = < X(t), O(\ell) >$. If $< X(n), O(\ell) > = 0$ for some $0 \leq n \leq \ell$, then from Lemma 2 (i) we have that $< X(n), O(\ell + t - n) > = 0$, for each $t \geq n$, and we can write,

$$\begin{aligned} < X(t), O(\ell) > &= < \mathcal{L}^{t-n}(X(n)), O(\ell) > \\ &= < X(n), \mathcal{T}^{t-n}(O(\ell)) > \\ &\leq < X(n), O(\ell + t - n) > = 0 \quad \square \end{aligned}$$

References

[1] B. D. O. Anderson and J. B. Moore. Detectability and stabilizability of time-varying discrete-time linear systems. *SIAM Journal on Control and Optimization*, 19(1):20–32, 1981.

[2] O. L. V. Costa, J. B. R. Do Val, and J. C. Geromel. A convex programming approach to h_2 control of discrete-time Markovian jump linear systems. *Int. J. Control*, 66(4):557–579, 1997.

[3] O. L.V. Costa and M. D. Fragoso. Stability results for discrete-time linear systems with Markovian jumping parameters. *J. Math. Analysis and Applications*, 179:154–178, 1993.

[4] O.L.V. Costa and M.D. Fragoso. Discrete-time LQ-optimal control problems for infinite markov jump parameter systems. *IEEE Trans. Auto. Control*, AC-40:2076–2088, 1995.

[5] D. P. de Farias, J. C. Geromel, J. B. R. do Val, and O. L. V. Costa. Output feedback control of Markov jump linear systems in continuous-time. *IEEE Trans. Auto. Control*, to appear.

[6] J.B.R. do Val and T. Başar. Receding horizon control of jump linear systems and a macroeconomic policy problem. *Journal of Economic Dynamics & Control*, 23:1099–1131, 1999.

[7] J.B.R. do Val, J.C. Geromel, and O.L.V. Costa. Solutions for the linear quadratic control problem of Markov jump linear systems. *Journal of Optimization Theory and Applications*, 103:283–311, 1999.

[8] W. W. Hager and L. L. Horowitz. Convergence and stability properties of the discrete Riccati operator equation and the associated optimal control and filtering problems. *SIAM Journal on Control and Optimization*, 14(2):295–312, 1976.

[9] Y. Ji and H. J. Chizeck. Controlability, stabilizability and continuous time Markovian jump linear quadratic control. *IEEE Trans. Auto. Control*, 35(7):777–788, 1990.

[10] Y. Ji and H. J. Chizeck. Jump linear quadratic gaussian control: Steady-state solution and testable conditions. *Control Theory and Advanced Technology*, 6(3):289–319, 1990.

[11] Y. Ji, H. J. Chizeck, X. Feng, and K. A. Loparo. Stability and control of discrete-time jump linear systems. *Control Theory and Advanced Tecnology*, 7(2):247–269, 1991.

[12] G. Pan and Y. Bar-Shalom. Stabilization of jump linear Gaussian systems without mode observations. *Int. J. Control*, 64(4):631–661, 1996.