Study of Lasers as Nonlinear Dynamical Systems

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Abstract

The rate equations of a large class of lasers are considered. These equations represent the evolution of photon and carrier densities in the laser, where the laser output is proportional to the photon density. By applying techniques from the theory of dynamical systems to the rate equations, four important properties of the lasers are rigorously proved. These properties are: (P1) for nonnegative and bounded inputs, the laser outputs are bounded; (P2) for a positive constant input, the laser output settles at a positive steady-state value; (P3) for positive constant inputs, the laser does not exhibit a limit cycle behavior; (P4) for positive constant inputs, the relaxation oscillations in the laser output can be attenuated if the coefficient of the spontaneous emission is increased.

1. Introduction

Lasers were invented in the early 1960s. Since then the science and technology of lasers have made phenomenal advances. In the past few decades, optical devices and components have been becoming an integral part of electronic devices, offering many advantages. Presently, lasers are used in most branches of science, technology, medicine, and perhaps more importantly, in the fiber-optic communication systems. The literature on the analysis and applications of lasers is enormous (see, e.g., [1], [2], [4], [7], [8], [9], [12], and the references therein).

One approach to study lasers is to present the evolution of photon and carrier densities in the active media of lasers by the following set of nonlinear ordinary differential equations, known as the rate equations (see, e.g., [1], [2], [4], [7], [8], [9], [12]):

$$\dot{P}(t) = -\left(\frac{1}{\tau_p} + A\ N_t\right) P(t) + \frac{\beta N(t)}{\tau_n} + A\ P(t) N(t)\,,$$

$$P(0) =: P_0 \geq 0\,, \qquad (1a)$$

$$\dot{N}(t) = A\ N_t\ P(t) - \frac{N(t)}{\tau_n} - A\ P(t) N(t) + \frac{I(t)}{e}\,,$$

$$N(0) =: N_0 \geq 0\,, \qquad (1b)$$

for all $t \geq 0$. In (1), the states $t \mapsto P(t) \in \mathbb{R}$ and $t \mapsto N(t) \in \mathbb{R}$, respectively, denote the photon and carrier densities in the active medium of the laser, P_0 and N_0 denote the initial conditions of the states, and $t \mapsto I(t) \in \mathbb{R}$ denotes the applied current density. The output of the laser is proportional to $P(\cdot)$. The constant parameters in (1) are as follow: $\tau_p > 0$ denotes the photon life time, $\tau_n > 0$ denotes the carrier life time, $0 < \beta \ll 1$ denotes the spontaneous emission factor, $A > 0$ denotes the optical gain factor (differential gain), $N_t > 0$ denotes the carrier density at the transparency, and $e = 1.60218 \times 10^{-19}$ is the electronic charge. Typical values of the parameters are $\tau_p = 2$ p sec, $\tau_n = 3$ n sec, $\beta \in [\,10^{-6}, 10^{-2}]$, $A = 1.7 \times 10^{3}\ \text{sec}^{-1}$, and $N_t = 1.5 \times 10^{8}$.

The laser represented by (1) has the following properties:

(P1) If the input $I(\cdot)$ is nonnegative and bounded, then so are the state $N(\cdot)$ and the output $P(\cdot)$. That is, the laser is the bounded-input bounded-state (BIBS) and the bounded-input bounded-output (BIBO) stable.

(P2) If $I(\cdot)$ is a positive constant input, then $P(\cdot)$ and $N(\cdot)$ settle at positive constant values, denoted by P_e^+ and N_e^+, respectively (see, e.g., [4, p. 315], [5], [8, p. 216], [11], [12, p. 957]).

(P3) The laser does not exhibit a limit cycle (periodic) behavior when $I(\cdot)$ is a positive constant input (see, e.g., [12, p. 961]).

(P4) If $I(\cdot)$ is a positive constant input, then $P(\cdot)$ initially oscillates at high frequencies with large amplitudes before it settles at the steady-state value $P_e^+ > 0$ (see, e.g., [4, p. 315], [5], [8, p. 216], [11], [12, p. 957]). These oscillations are called the *relaxation oscillations*. The relaxation oscillations are attenuated if the spontaneous emission factor β increases (see, e.g., [5], [11]).

Although properties (P1)-(P4) are well known, they have not been proved rigorously. It is the goal of this paper to present such proofs.

2. Mathematical Justifications of Laser Properties

In this section, we first present a useful result regarding the dynamics of the laser represented by (1). Then, we establish (P1)-(P4).

2.1. Invariance of the Nonnegative Quadrant

The state space of the system (1) is $\mathbb{R}^2$. Let the state vector of the system (1) be denoted by $X(t) := [\,P(t) \quad N(t)\,]^T \in \mathbb{R}^2$ for all $t \geq 0$. Let the solution of the system starting at $t = 0$ from the initial vector $X_0 := X(0) = [\,P_0 \quad N_0\,]^T$ be denoted by the vector $X(t, 0, X_0)$ for all $t \geq 0$.

The useful result to be established is that for nonnegative inputs the nonnegative quadrant $\mathbb{R}_+^2$ is an invariant set of the system (1). That is, if the system starts from any initial vector $X_0 \in \mathbb{R}_+^2$, then its solution vector $X(t, 0, X_0) \in \mathbb{R}_+^2$ for all $t \geq 0$.

Lemma 2.1: If the input $I(\cdot)$ is nonnegative, then the nonnegative quadrant $\mathbb{R}_+^2$ is an invariant set of the system (1).

Proof: First, we consider the subspaces

$$\Sigma_N := \{ (P, N) \in \mathbb{R}^2 \mid P = 0 \}, \tag{2a}$$

$$\Sigma_P := \{ (P, N) \in \mathbb{R}^2 \mid N = 0 \}, \tag{2b}$$

which are the N-axis and P-axis in the PN-plane, respectively. From (1a) (respectively, (1b)), we conclude that for any point in Σ_N (Σ_P), $P(t) \geq 0$ ($N(t) \geq 0$) for all $t \geq 0$ if $N(t) \geq 0$ ($P(t) \geq 0$).

Knowing these properties of the subspaces Σ_N and Σ_P, we conclude that any trajectory of the system (1) starting in $\mathbb{R}_+^2$ cannot traverse into regions of $\mathbb{R}^2$ for which $P < 0$ or $N < 0$. □

The invariance of $\mathbb{R}_+^2$ will be used in many instances in the paper.

2.2. Bounded States and Output

The input to the laser represented by (1) is the current density $t \mapsto I(t)$, which is a bounded function of time. This current density is commonly a constant positive bias plus a modulation current density (see, e.g., [1], [2], [4], [7], [8], [9], [12]). In this section, our goal is to show the boundedness of $P(\cdot)$ and $N(\cdot)$ when nonnegative and bounded inputs $I(\cdot)$ are applied to the system (1). We achieve this goal by obtaining upper bounds on the L_∞-norms of $P(\cdot)$ and $N(\cdot)$.

First, an upper bound on $\| N \|_\infty$ is obtained.

Theorem 2.2: If the input $I(\cdot)$ is nonnegative and bounded, then the state $N(\cdot)$ of the system (1) is bounded. More precisely,

$$\| N \|_\infty \leq \max \{ N_0, N_t, \tau_n \| I \|_\infty / e \} < \infty. \tag{3}$$

Proof: By Lemma 2.1, the states $P(\cdot)$ and $N(\cdot)$ are nonnegative. Using this fact, from (1b), we conclude that if

$$N(t) > \frac{A\, N_t\, P(t) + \| I \|_\infty / e}{A\, P(t) + 1/\tau_n}, \tag{4}$$

for all $t \geq 0$, then $\dot{N}(t) < 0$. Therefore, any trajectory of the system (1) starting from the initial point $(P_0, N_0) \in \mathbb{R}_+^2$ traverses towards the curve

$$C_N := \{ (P, N) \in \mathbb{R}_+^2 \mid N = \frac{A\, N_t\, P + \| I \|_\infty / e}{A\, P + 1/\tau_n} \}, \tag{5}$$

shown in Figure 1. Since for any point on C_N, the second coordinate N is finite, we conclude that $N(t) < \infty$ for all $t \geq 0$. From Figure 1, we note that if a trajectory starts from an initial point for which $N_0 > \max \{ N_t, \tau_n \| I \|_\infty / e \}$, then $\| N \|_\infty = N_0$, where as for any other initial point either $\| N \|_\infty \leq N_t$ (when $N_t > \tau_n \| I \|_\infty / e$, i.e., when C_N is monotonically increasing) or $\| N \|_\infty \leq \tau_n \| I \|_\infty / e$ (when $N_t < \tau_n \| I \|_\infty / e$, i.e., when C_N is monotonically decreasing). Thus, (3) holds. □

In order to establish the boundedness of $P(\cdot)$, we define a scalar-valued function of time $t \mapsto V(t)$, which consists of the system states $P(\cdot)$ and $N(\cdot)$. We prove that when $I(\cdot)$ is nonnegative and bounded, the function $V(\cdot)$ along the solution of the system (1) is bounded. Then, having $V(\cdot)$ bounded, the boundedness of $P(\cdot)$ follows.

We define

$$V(t) := P(t) + N(t), \tag{6}$$

for all $t \geq 0$, where $P(\cdot)$ and $N(\cdot)$ satisfy (1). By Lemma 2.1, $V(t) \geq 0$ for all $t \geq 0$. At $t = 0$,

$$V(0) = P_0 + N_0 \geq 0. \tag{7}$$

We now prove the boundedness of $V(\cdot)$.

Lemma 2.3: If the input $I(\cdot)$ is nonnegative and bounded, then the function $V(\cdot)$ along the solution of the system (1) is bounded. More precisely,

$$\| V \|_\infty \leq P_0 + N_0 + \frac{\| I \|_\infty}{e \min\{ 1/\tau_p, (1 - \beta)/\tau_n \}} < \infty. \tag{8}$$

Proof: From (6), we obtain

$$\dot{V}(t) = \dot{P}(t) + \dot{N}(t), \tag{9}$$

for all $t \geq 0$. Substituting $\dot{P}(\cdot)$ and $\dot{N}(\cdot)$ from (1) into (9), we obtain

$$\dot{V}(t) = - P(t)/\tau_p - (1 - \beta)\, N(t)/\tau_n + I(t)/e, \tag{10}$$

for all $t \geq 0$. By Lemma 2.1, the states $N(\cdot)$ and $P(\cdot)$ are nonnegative. Using this fact and that $\beta < 1$, we can thus obtain the linear differential inequality

$$\dot{V}(t) \leq - \xi\, V(t) + I(t)/e, \tag{11}$$

for all $t \geq 0$, where $\xi := \min\{ 1/\tau_p, (1 - \beta)/\tau_n \}$. By a comparison theorem given in [3, p. 2] or [10, p. 3], we conclude that $V(\cdot)$ in (11) satisfies

$$V(t) \leq \exp(-\xi\, t)\, V(0) + \frac{1}{e} \int_0^t \exp(-\xi\, (t - \tau))\, I(\tau)\, d\tau, \tag{12}$$

for all $t \geq 0$. Using (7) and the boundedness of $I(\cdot)$, we obtain

$$V(t) \leq P_0 + N_0 + \| I \|_\infty / (e\, \xi) < \infty, \tag{13}$$

for all $t \geq 0$. That is, $V(\cdot)$ is bounded and its norm satisfies (8). □

Now, the boundedness of the laser output $P(\cdot)$ follows.

Theorem 2.4: If the input $I(\cdot)$ is nonnegative bounded, then the output $P(\cdot)$ of the system (1) is bounded. More precisely,

$$\| P \|_\infty \le P_0 + N_0 + \frac{\| I \|_\infty}{e \, \min\{ 1/\tau_p \, , (1-\beta)/\tau_n \}} < \infty . \quad (14)$$

Proof: By Lemma 2.1, the states $P(\cdot)$ and $N(\cdot)$ are nonnegative. Using this fact, from (6), we conclude that

$$P(t) \le V(t) , \quad (15)$$

for all $t \ge 0$. Thus, $\| P \|_\infty \le \| V \|_\infty$ and (14) follows. □

Thus far, we have proved property (P1).

2.3. Equilibrium Points

In this section, we are interested to know if the output of the laser represented by (1) will settle at a positive constant value when it is pumped by a positive constant current. Our first task is thus to locate the equilibrium points of the system (1).

Assertion 2.5: If the input $I(t) = \bar{I} > 0$ for all $t \ge 0$, then the system (1) has a unique equilibrium point $E^+ := (P_e^+, N_e^+) \in \mathbb{R}_+^2$.

Proof: Let (P_e, N_e) denote an equilibrium point of the system (1). By setting the right-hand sides of (1) equal to zero and solving the resulting set of nonlinear algebraic equations for P_e and N_e, we can conclude that they satisfy

$$\frac{P_e}{\tau_p} + \frac{(1-\beta)\, N_e}{\tau_n} = \frac{\bar{I}}{e} , \quad (16a)$$

$$P_e^2 - \left(\frac{A \ \tau_p \ \tau_n \ \bar{I} - e \ [\, 1 + A \ N_t \ \tau_p \ (1-\beta) \,]}{e \ A \ \tau_n} \right) P_e - \frac{\beta \, \tau_p \ \bar{I}}{e \ A \ \tau_n} = 0 , \quad (16b)$$

$$N_e^2 - \left(\frac{A \ \tau_p \ \tau_n \ \bar{I} + e \ [\, 1 + A \ N_t \ \tau_p \ (1-\beta) \,]}{e \ A \ \tau_p \ (1-\beta)} \right) N_e + \frac{(1 + A \ N_t \ \tau_p)\, \tau_n \ \bar{I}}{e \ A \ \tau_p \ (1-\beta)} = 0 . \quad (16c)$$

Let P_{e1} and P_{e2} denote the solution of the quadratic equation (16b). Clearly, $P_{e1} P_{e2} = - \beta \, \tau_p \ \bar{I} / (e \ A \ \tau_n) < 0$. Thus, both P_{e1} and P_{e2} are real numbers, and one of them, say $P_{e1} = P_e^+ > 0$, and the other one $P_{e2} < 0$. By (16a), N_{e1} and N_{e2}, which correspond to P_{e1} and P_{e2}, respectively, are real numbers as well. From (16c), it is clear that $N_{e1} + N_{e2} > 0$ and $N_{e1} N_{e2} > 0$. Since N_{e1} and N_{e2} are both real, we conclude that $N_{e1} = N_e^+ > 0$ and $N_{e2} > 0$. Thus, there exists only one equilibrium point $E^+ = (P_e^+, N_e^+) \in \mathbb{R}_+^2$. □

The global asymptotic stability of the equilibrium point $E^+ = (P_e^+, N_e^+)$ of the system (1) should be established. This stability result implies that for a positive constant current, the laser output settles at the steady-state value $P_e^+ > 0$. One way value $P_e^+ > 0$. One way to establish the stability of E^+ is to use the Lyapunov technique, which depends on a suitably devised Lyapunov function. We, however, have not been able to devise such a Lyapunov function for the system (1). Therefore, we will establish the stability of E^+ by a different means later in the paper.

2.4. Nonexistence of Limit Cycles and Output Stability

For positive constant inputs, the states of the system (1) are bounded and evolve in $\mathbb{R}_+^2$. In this section, we prove that trajectories of the system (1) in $\mathbb{R}_+^2$ do not converge to (respectively, diverge from) a stable (an unstable) limit cycle. That is, we prove property (P3).

Theorem 2.6: The system (1) does not exhibit a limit cycle (periodic) behavior when the input $I(t) = I > 0$ for all $t \ge 0$.

Proof: We rewrite the system (1) as

$$\dot{P}(t) = F(P(t), N(t)) , \quad P(0) =: P_0 \ge 0 , \quad (17a)$$

$$\dot{N}(t) = G(P(t), N(t), \bar{I}) , \quad N(0) =: N_0 \ge 0 , \quad (17b)$$

for all $t \ge 0$, where F and G are, respectively, the right-hand sides of (1a) and (1b). We use a result [6, Corollary 1.3] or [13, Corollary on p. 18] to establish the nonexistence of limit cycles in the system (17). This result is applied as follows. Choose nonnegative and differentiable functions $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}_+$ and $\tilde{G} : \mathbb{R}^2 \to \mathbb{R}_+$, denoted by $\tilde{F}(P,N)$ and $\tilde{G}(P,N)$, respectively, and a differentiable function $B : \mathbb{R}^2 \to \mathbb{R}$, denoted by $B(P,N)$, and compute

$$\Delta(P,N) := \frac{\partial(\tilde{F} \, F)}{\partial N} - \frac{\partial(\tilde{G} \, G)}{\partial P} + \frac{\partial(B \ F)}{\partial P} + \frac{\partial(B \ G)}{\partial N} . \quad (18)$$

If $\Delta(P,N) \ge 0$ (respectively, $\Delta(P,N) \le 0$) in a simply connected region R of $\mathbb{R}_+^2$, and the set of points (P,N) at which $\Delta(P,N) = 0$ do not completely fill any subregion of R, then there does not exist any positively (negatively) oriented limit cycle in R.

We choose $\tilde{F} = \tilde{G} = 1$ and $B = -1$. We then obtain

$$\Delta(P,N) = 2A \ P + 1/\tau_p + (1+\beta)/\tau_n , \quad (19)$$

which is positive in any simply connected region of $\mathbb{R}_+^2$. Thus, the system (1) does not have any limit cycle in $\mathbb{R}_+^2$. Since by Lemma 2.1 for $\bar{I} > 0$, trajectories of the system (1) traverse exclusively in $\mathbb{R}_+^2$, we conclude that the system does not exhibit a limit cycle behavior. □

Having (P3) established, the stability of the equilibrium point E^+ follows.

Corollary 2.7: The unique equilibrium point $E^+ = (P_e^+, N_e^+) \in \mathbb{R}_+^2$ of the system (1), due to the input $I(t) = \bar{I} > 0$ for all $t \ge 0$, is globally asymptotically stable.

Proof: For a positive constant input: (i) the planar system (1) is time-invariant; (ii) by Theorem 2.4, the states of the system are bounded; (iii) by Theorem 2.6, the system does not have any limit cycle in $\mathbb{R}_+^2$. Having (i)-(iii), it is clear that any trajectory of the system (1) in $\mathbb{R}_+^2$, starting from the initial point $(P_0, N_0) \in \mathbb{R}_+^2$, has no choice except converging to E^+ as $t \rightarrow \infty$. □

Thus, properties (P2) is established.

2.5. Relaxation Oscillations

The results we have established so far imply that the output of the laser represented by (1) settles at a positive constant value when it is pumped by a positive constant current. It turns out that the output of the laser initially oscillates at high frequencies with large amplitudes before it settles at the steady-state value (see, e.g., [4, p. 315], [5], [8, p. 216], [11], [12, p. 957]). These oscillations are known as the relaxation oscillations and are usually undesirable. It is well known that the relaxation oscillations can be attenuated when β increases (see, e.g., [5], [11]). In this section, we prove this fact.

We rewrite the system (1) as

$$\frac{dP(\tau)}{d\tau} = -\frac{1}{\beta}\left(\frac{1}{\tau_p} + A\ N_t\right) P(\tau) + \frac{N(\tau)}{\tau_n} + \frac{A\ P(\tau)\, N(\tau)}{\beta}\,,$$

$$P(0) =: P_0 \geq 0\,, \qquad (20a)$$

$$\frac{dN(\tau)}{d\tau} = \frac{A\ N_t\ P(\tau)}{\beta} - \frac{N(\tau)}{\beta\, \tau_n} - \frac{A\ P(\tau)\, N(\tau)}{\beta} + \frac{\bar{I}}{\beta\, e}\,,$$

$$N(0) =: N_0 \geq 0\,, \qquad (20b)$$

for all $\tau \geq 0$, where $\tau = \beta\, t$ and $\bar{I} > 0$ is a constant input. Applying the Laplace transform to the system (20), we obtain

$$P(s) = H_{11}(s)\, P_0 + H_{12}(s)\, N_0 + \left[\frac{H_{12}(s)}{\beta}\right] \left(\frac{\bar{I}}{e\ s}\right) + \left[\frac{A\ [\, H_{11}(s) - H_{12}(s)\,]}{\beta}\right] Z(s)\,, \qquad (21a)$$

$$N(s) = H_{21}(s)\, P_0 + H_{22}(s)\, N_0 + \left[\frac{H_{22}(s)}{\beta}\right] \left(\frac{\bar{I}}{e\ s}\right) + \left[\frac{A\ [\, H_{21}(s) - H_{22}(s)\,]}{\beta}\right] Z(s)\,, \qquad (21b)$$

where $P(s)$, $N(s)$, and $Z(s)$, $s \in \mathbb{C}$, respectively, denote the Laplace transforms of $\tau \mapsto P(\tau)$, $\tau \mapsto N(\tau)$, and $\tau \mapsto Z(\tau) := P(\tau)\, N(\tau)$, and

$$H_{11}(s) := \frac{s + 1/(\beta\, \tau_n)}{D(s)}\,, \quad H_{12}(s) := \frac{1/\tau_n}{D(s)}\,, \qquad (22a)$$

$$H_{21}(s) := \frac{A\ N_t/\beta}{D(s)}\,, \quad H_{22}(s) := \frac{s + (1/\tau_p + A\ N_t)/\beta}{D(s)}\,, \qquad (22b)$$

with

$$D(s) := s^2 + [\, (1/\tau_p + A\ N_t + 1/\tau_n)/\beta\,]\, s + [\, 1/\tau_p + A\ N_t\ (1 - \beta)\,]/(\beta^2\, \tau_n)\,. \qquad (23)$$

From (21), we obtain

$$P(\tau) = L^{-1}[\, H_{11}(s)\,]\, P_0 + L^{-1}[\, H_{12}(s)\,]\, N_0 + L^{-1}\left[\frac{H_{12}(s)}{\beta}\right] * \left(\frac{\bar{I}}{e}\right) + L^{-1}\left[\frac{A\ [\, H_{11}(s) - H_{12}(s)\,]}{\beta}\right] * P(\tau)\, N(\tau)\,, \qquad (24a)$$

$$N(\tau) = L^{-1}[\, H_{21}(s)\,]\, P_0 + L^{-1}[\, H_{22}(s)\,]\, N_0 + L^{-1}\left[\frac{H_{22}(s)}{\beta}\right] * \left(\frac{\bar{I}}{e}\right) + L^{-1}\left[\frac{A\ [\, H_{21}(s) - H_{22}(s)\,]}{\beta}\right] * P(\tau)\, N(\tau)\,, \qquad (24b)$$

for all $\tau \geq 0$, where $L^{-1}[\cdot]$ denotes the inverse Laplace transform of a function and $*$ denotes the convolution of two functions of time. The block diagram of the system in (24) is shown in Figure 2. In this figure, $\delta(\cdot)$ denotes the delta function.

With this preliminary setup, we can prove property (P4). In the following, by a function of time with less oscillations we mean a function with negligible high frequency contents.

Theorem 2.8: Let the input to the system (1) be $I(t) = \bar{I} > 0$ for all $t \geq 0$. The system output $P(\cdot)$ converges to the steady-state output $P_e^+ > 0$ with less oscillations if β increases.

Proof: The states $\tau \mapsto P(\tau)$ and $\tau \mapsto N(\tau)$ of the system (20) are given by (24). Using the fact that $\beta \ll 1$, we obtain the terms in (24a) as

$$L^{-1}[\, H_{11}(s)\,]\, P_0 \approx L^{-1}\left[\frac{1}{s + (1/\tau p + A\ N_t)/\beta}\right] P_0$$

$$= \exp(-\, (1/\tau p + A\ N_t)\, \tau/\beta)\, P_0\,, \qquad (25a.1)$$

$$L^{-1}[\, H_{12}(s)\,]\, N_0 \approx$$

$$L^{-1}\left[\frac{1/\tau_n}{[\, s + 1/(\beta\, \tau_n)\,]\, [\, s + (1/\tau_p + A\ N_t)/\beta\,]}\right] N_0 =$$

$$\left[\frac{\beta/\tau_n}{1/\tau_p + A\ N_t - 1/\tau_n}\right] \times [\, \exp(-\, \tau/(\beta\, \tau_n)) -$$

$$\exp(-\, (1/\tau_p + A\ N_t)\, \tau/\beta)\,]\, N_0\,, \qquad (25a.2)$$

$$L^{-1}\left[\frac{H_{12}(s)}{\beta}\right] * \left(\frac{\bar{I}}{e}\right) \approx$$

$$L^{-1}\left[\frac{1/(\beta\, \tau_n)}{[\, s + 1/(\beta\, \tau_n)\,]\, [\, s + (1/\tau_p + A\ N_t)/\beta\,]}\right] * \left(\frac{\bar{I}}{e}\right), \qquad (25a.3)$$

$$L^{-1}\left[\frac{A\,[\,H_{11}(s)-H_{12}(s)\,]}{\beta}\right]*P(\tau)\,N(\tau)\approx$$

$$L^{-1}\left[\frac{A/\beta}{s+(1/\tau_p+A\,N_t)/\beta}\right]*P(\tau)\,N(\tau)\,, \qquad (25a.4)$$

and the terms in (24b) as

$$L^{-1}[\,H_{21}(s)\,]\,P_0\approx$$

$$L^{-1}\left[\frac{A\,N_t/\beta}{[\,s+1/(\beta\,\tau_n)\,]\,[\,s+(1/\tau_p+A\,N_t)/\beta\,]}\right]P_0=$$

$$[\,\frac{A\,N_t}{1/\tau_p+A\,N_t-1/\tau_n}\,]\times[\,\exp(-\tau/(\beta\,\tau_n))\,-$$

$$\exp(-(1/\tau_p+A\,N_t)\,\tau/\beta)\,]\,P_0\,, \qquad (25b.1)$$

$$L^{-1}[\,H_{22}(s)\,]\,N_0\approx$$

$$L^{-1}\left[\frac{1}{s+1/(\beta\,\tau_n)}\right]N_0=\exp(-\tau/(\beta\,\tau_n))\,N_0\,, \qquad (25b.2)$$

$$L^{-1}\left[\frac{H_{22}(s)}{\beta}\right]*(\frac{\bar{I}}{e})\approx L^{-1}\left[\frac{1/\beta}{s+1/(\beta\,\tau_n)}\right]*(\frac{\bar{I}}{e})=$$

$$(\tau_n\bar{I}/e)\,[\,(1-\exp(-\tau/(\beta\,\tau_n)\,)\,]\,, \qquad (25b.3)$$

$$L^{-1}\left[\frac{A\,[\,H_{21}(s)-H_{22}(s)\,]}{\beta}\right]*P(\tau)\,N(\tau)\approx$$

$$-L^{-1}\left[\frac{(A/\beta)\,[\,s+1/(\beta\,\tau_p)\,]}{[\,s+1/(\beta\,\tau_n)\,]\,[\,s+(1/\tau_p+A\,N_t)/\beta\,]}\right]*$$

$$P(\tau)\,N(\tau)\,, \qquad (25b.4)$$

for all $\tau\geq 0$.

The time functions in (25a.1), (25a.2), (25b.1), and (25b.2) are nonnegative for all $\tau\geq 0$ and any $\beta>0$ and $\tau_p>0$. These functions tend to zero as $\tau\longrightarrow\infty$ without an oscillatory behavior.

The time function on the right-hand side of (25a.3) can be easily derived. However, there is no need for such a derivation. We observe that the transfer function of the system through which $\bar{I}/e$ passes is an overdamped second order system. Therefore, the time function in (25a.3) starts from zero and monotonically increases. This function converges to the steady-state value $\beta\,\bar{I}/(e\,(1/\tau_p+A\,N_t))$ as $\tau\longrightarrow\infty$ without an overshoot (oscillatory behavior).

The time function in (25b.3) starts from zero and monotonically increases. This function converges to the steady-state value $\tau_n\bar{I}/e$ as $\tau\longrightarrow\infty$ without an oscillatory behavior.

The oscillatory behavior of $\tau\mapsto P(\tau)$ and $\tau\mapsto N(\tau)$ can be due to the time functions in (25a.4) and (25b.4), respectively, which are resulted by the feedback of $\tau\mapsto P(\tau)\,N(\tau)$ (see Figure 2). The time functions $\tau\mapsto P(\tau)$ and $\tau\mapsto N(\tau)$ will be less oscillatory if the time functions in (25a.4) and (25b.4) have negligible high frequency contents. This requirement can be fulfilled if the transfer functions

$$G_p(s):=\frac{A/\beta}{s+(1/\tau_p+A\,N_t)/\beta}\,, \qquad (26a)$$

$$G_n(s):=\frac{(A/\beta)\,[\,s+1/(\beta\,\tau_p)\,]}{[\,s+1/(\beta\,\tau_n)\,]\,[\,s+(1/\tau_p+A\,N_t)/\beta\,]}\,, \qquad (26b)$$

which appear in (25a.4) and (25b.4), respectively, are low-pass filters. The frequency responses of $G_p(s)$ and $G_n(s)$ are sketched in Figure 3. From this figure, it is evident that if β increases, then $G_p(s)$ and $G_n(s)$ will become low-pass filters, i.e., will have low cut-off frequencies. □

3. Conclusions

In this paper, we considered a large class of lasers that are represented by a widely used set of nonlinear ordinary differential equations, known as the rate equations. By applying techniques from the theory of dynamical systems to the rate equations, we rigorously proved four important properties of the lasers. These properties are: (P1) the boundedness of the laser output for nonnegative and bounded inputs; (P2) the stability of positive steady-state laser outputs due to positive constant inputs; (P3) the nonexistence of a limit cycle behavior in the laser when it is pumped by positive constant inputs; (P4) the attenuation of the relaxation oscillations in the laser output by increasing the spontaneous emission factor.

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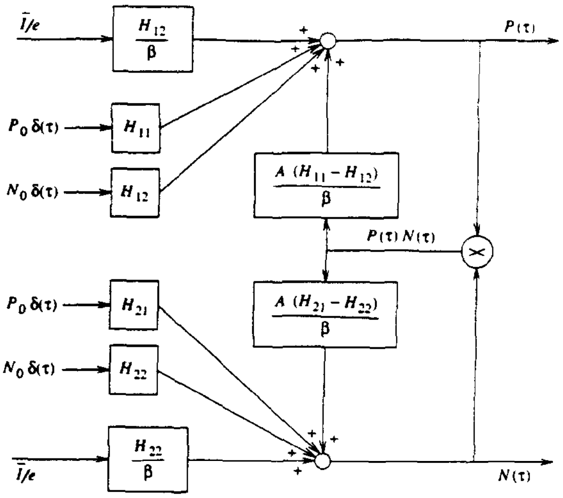

Figure 2. The block diagram of the system (32). The nonlinear feedback of $\tau \mapsto P(\tau)N(\tau)$ affects $\tau \mapsto P(\tau)$ and $\tau \mapsto N(\tau)$.

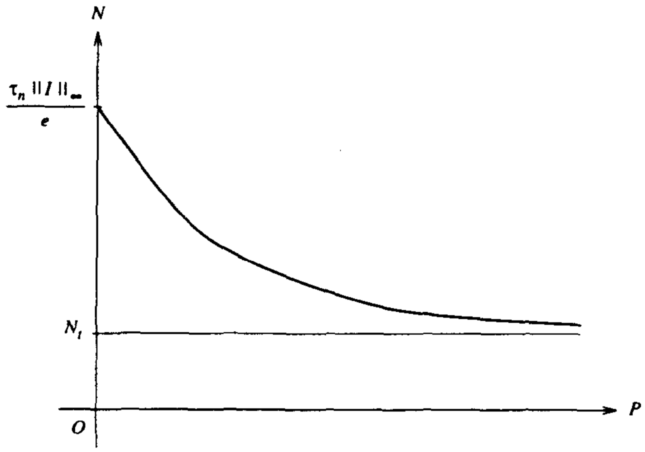

Figure 1. The curve C_N in (6).

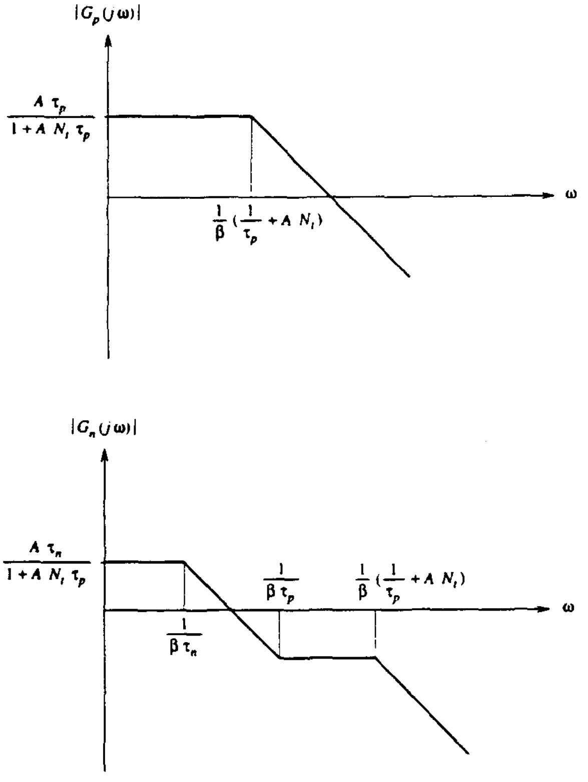

Figure 3. Sketches of the frequency responses of the transfer functions $G_p(s)$ and $G_n(s)$ in (34), designated by (a) and (b), respectively. It is evident that if β increases, then $G_p(s)$ and $G_n(s)$ will become low-pass filters.