Graph Constrained Switching Differential Games

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Abstract

In this paper we discuss a two-person zero-sum differential game of finite horizon with graph-constrained control strategies. We prove the existence of value and state optimality conditions in terms of value functions that solve a coupled system of quasi-variational inequalities.

Keywords: Graph games, differential games, switching strategies.

1 Introduction

The problem of controlling an ordinary differential equation subject to positive switching costs was addressed in [1], where it was proved that value functions are viscosity solutions to dynamic programming quasi-variational inequalities. The corresponding differential game formulation was introduced in [3], as a zero-sum differential game where both players select their switching controls independently and incur positive switching costs. Here, we extend these results by considering, not only a different set of more easily verifiable hypotheses, but also graph constrained switching strategies.

We consider a zero-sum differential game between players A and B having, respectively, at their disposal piecewise constant functions, $u(\cdot)$ and $v(\cdot)$, to control the dynamic system:

$$\dot{x}(t) = f(t, x(t), u(t), v(t)), \qquad [0, T]\text{-a.e.}, x(0) = x_0$$

where $x \in X$ and X is a finite-dimensional Euclidean space; $u(t) \in U = \{u_1, u_2, ..., u_N\}$ and $v(t) \in V = \{v_1, v_2, ..., v_M\}$; the switching times for $u(\cdot)$ and $v(\cdot)$ are respectively $\{\tau_i\}$ and $\{\sigma_j\}$. $T \geq \tau_i > \tau_{i-1} \geq 0$, $i = 1, \ldots, \bar{N}$ for some $\bar{N} < \infty$; $T \geq \sigma_j > \sigma_{j-1} \geq 0$, $j = 1, \ldots, \bar{M}$ for some $\bar{M} < \infty$; $u(t)$ and $v(t)$ are selected from the control graph.

Definition 1 (Control graph $\mathcal{G}$) *$\mathcal{G} := (\mathcal{N}, \mathcal{E})$, where: the set of nodes is $\mathcal{N} := \{n_p : p = 1, ...P\}$ with $n_p := (u_k, v_l)$ for some $k = 1, ..., N$; and $l = 1, ..., M$, and, the arcs $\mathcal{E}$ define the allowable control switches. The set of arcs is generated by the set-valued function $e : \mathcal{N} \to 2^{\mathcal{N}}$, where the edges out of the vertex (u, v) are all the pairs of the form $((u, v), (\bar{u}, \bar{v}))$ where $(\bar{u}, \bar{v}) \in e(u, v)$.*

The payoff function for this zero-sum game is:

$$J_\eta(u(\cdot), v(\cdot)) = h(x(T)) + \int_s^T g(t, x(t), u(t), v(t))dt + \sum_{i \geq 1} k(u_{i-1}, u_i) - \sum_{i \geq 1} l(v_{i-1}, v_i)$$

where: $\eta := (\bar{u}, \bar{v}, s, \bar{x}) = (u(s), v(s), s, x(s))$ and $k(a, b)$, where $a, b \in U$, and $l(c, d)$, where $c, d \in V$, are the switching cost functions.

We consider the following hypotheses:

(H1) h is Lipschitz continuous with constant K_h.

(H2) f and g are continuous in u and v and Lipschitz continuous in x with constants K_f and K_g respectively. f is measurable in t and g continuous in t.

(H3) U and V are finite bounded sets.

(H4) k and l are bounded $\forall (u, v) \in U \times V$.

(H5) Let $n = (u, v), \bar{n} = (\bar{u}, \bar{v}), \tilde{n} = (\tilde{u}, \tilde{v})$. Then, $\forall (n, \tilde{n}), (n, \bar{n}), (\bar{n}, \tilde{n}) \in \mathcal{E}$, the following holds:

- $k(u, \tilde{u}) \leq k(u, \bar{u}) + k(\bar{u}, \tilde{u})$,
- $l(v, \tilde{v}) \leq l(v, \bar{v}) + l(\bar{v}, \tilde{v})$,
- $k(u, \tilde{u}) \geq 0$, $l(v, \tilde{v}) \geq 0$, with $k(u, \tilde{u}) = 0$ and $l(v, \tilde{v}) = 0$ only if $u = \tilde{u}$ and $v = \tilde{v}$, respectively.

(H6) There are constants $\bar{K}$ and $\bar{L}$ such that, for any integer J, $J \geq 1$ and feasible sequence $\{e_{i+j}\}_{j=0}^{J}$, the following inequalities are satisfied:

$$\sum_{j=0}^{J-1} k(u_{n_{i+j}}, u_{n_{i+j+1}}) \leq \bar{K}(\tau_{n_i} - \tau_{n_{i+J}})$$

$$\sum_{j=0}^{J-1} l(v_{m_{i+j}}, v_{m_{i+j+1}}) \leq \bar{L}(\sigma_{m_i} - \sigma_{m_{i+J}})$$

(H7) The players select controls one at a time and in turns. This means that for each i, either $u_{n_i} = u_{n_{i+1}}$ or $v_{m_i} = v_{m_{i+1}}$.

(H8) For any loop $\{q_1, ..., q_S\}$ in the graph, it holds that

$$\sum_{j=1}^{S} k(u_j, u_{j+1}) - \sum_{j=1}^{S} l(v_j, v_{j+1}) \neq 0, \forall s \in [0, T]$$

Under the above hypotheses it can be proved [2] that: 1) there exists a unique solution $x(\cdot)$ to the problem $\dot{x}(t) = f(t, x(t), u(t), v(t)), [s, T], x(s) = x_0$ where $(\bar{u}, \bar{v}) \in \mathcal{N}, x \in X$ and u(.),v(.) are compatible control functions; 2) the payoff functional is well defined; 3) the minimum time interval between two consecutive jumps is strictly positive. Hence, the number of switches for each player in each finite time interval is finite but not defined apriori, i.e., it will follow from the choice of the control strategy.

2 Optimality conditions

We have adapted the proof techniques from [1] and [3] in order to take into account the graph constrained switching strategies. Hypothesis H6 was crucial in the process of adapting these proofs, namely in what concerns uniqueness of jumps and to prevent the occurrence of instantaneous loops in the control graph [2].

Let $x_{s,\bar{x}}(t)$ denote the state trajectory at time $t > s$ when $x(s) = \bar{x}$ and $\mathcal{U}^{\bar{u},s}$ and $\mathcal{V}^{\bar{v},s}$ denote the feasible control for players A and B, respectively. $\Gamma^u[s,T]$ (where $\Gamma^u[T,T] = \{u\}$) and $\Gamma^v[s,T]$ (where $\Gamma^v[T,T] = \{v\}$) are the sets of all admissible strategies for players A and B respectively.

Definition 2 (Lower and upper value functions)

$$V(\eta) = \inf_{\alpha \in \Gamma^{\bar{u}}[s,T]} \sup_{v(\cdot) \in \mathcal{V}^{\bar{v},s}} J_\eta(\alpha(\cdot), v(\cdot))$$

$$U(\eta) = \sup_{\beta \in \Gamma^{\bar{v}}[s,T]} \inf_{u(\cdot) \in \mathcal{U}^{\bar{u},s}} J_\eta(u(\cdot), \beta(\cdot))$$

$$J_\eta|_{s=T}(u(\cdot), v(\cdot)) = V(\eta)|_{s=T} = U(\eta)|_{s=T} = h(\bar{x}).$$

Straightforward arguments reveal that both the lower and the upper value functions satisfy an optimality principle.

Theorem 1 *For any feasible pair $(\bar{u}, \bar{v}) \in \mathcal{N}$, for all $x \in X$ and $0 \le s < \tilde{s} \le T$, we have*

$$V(\bar{u}, \bar{v}, s, \bar{x}) =$$
$$\inf_{\alpha \in \Gamma^{\bar{u}}[s,T]} \sup_{v(\cdot) \in \mathcal{V}^{\bar{v},s}} \Big\{ \int_s^{\tilde{s}} g(t, x_{s,\bar{x}}(t), \alpha[v(\cdot)](t), v(t)) dt$$
$$+ \sum_{i \ge 1} k(u_{i-1}, u_i) - \sum_{j \ge 1]} l(v_{j-1}, v_j)$$
$$+ V(\alpha[v(\cdot)](\tilde{s}^+), v(\tilde{s}^+), \tilde{s}, x_{s,\bar{x}}(\tilde{s})) \Big\}.$$

In order to state the main results it is convenient to introduce the following operators.

Definition 3 (Obstacle operators)

$$M^+[V](u, v, s, x) =$$
$$\min_{\overline{u} \neq u, \overline{u} \in Proj_U(e(u,v))} \{V(\overline{u}, v, s, x) + k(u, \overline{u})\}$$
$$M^-[V](u, v, s, x) =$$
$$\max_{\overline{v} \neq v, \overline{v} \in Proj_V(e(u,v))} \{V(u, \overline{v}, s, x) - l(v, \overline{v})\}$$

$Proj_U(n_p), Proj_V(n_p)$ are the projection operators, where $n_p = (u_k, v_l)$ and $Proj_U(n_p) = u_k$ and $Proj_V(n_p) = v_l$.

Let $H(\bar{u}, \bar{v}, s, x, p) = \langle p, f(s, x, \bar{u}, \bar{v}) \rangle + g(s, x, \bar{u}, \bar{v})$

Theorem 2 *The lower value function $V(\cdot)$ and the upper value function $U(\cdot)$ are viscosity solutions of the quasi-variational inequality with bilateral obstacles:*

$$M^-[V](\eta) \le V(\eta) \le M^+[V](\eta)$$

$$\text{on the set} \quad \{(s,x) \in [0,T] \times X : M^-[V](\eta) < V(\eta)\},$$
$$V_s(\eta) + H(\eta, V_x(\eta)) \ge 0$$
$$\text{on the set} \quad \{(s,x) \in [0,T] \times X \ : M^+[V](\eta) > V(\eta)\},$$
$$V_s(\eta) + H(\eta, V_x(\eta)) \le 0$$
$$\text{with terminal condition} \quad V(\bar{u}, \bar{v}, T, x) = h(x)$$

Theorem 3 *Under the above hypotheses the upper and lower value functions coincide and the value of the differential game exists.*

The optimal controls can be derived from these optimality conditions.

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