INITIALIZATION IN FRACTIONAL ORDER SYSTEMS

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Abstract

This paper proves the requirement of a time-varying initialization for fractional differential equations. This then requires a new definition for the fractional differintegral that includes the initialization and a new form of the Laplace transform of the fractional differintegral. An initialized fractional system theory is developed.

1 Introduction

Increasingly many physical processes are found to be best described using fractional differential equations. These processes include; viscoelasticity, rheology, electrochemistry, fractal processes and many diffusion processes.

To effectively apply the fractional calculus to the problems of science and engineering requires a well-organized fractional system theory. One fundamentally-important aspect to achieving such a theory is the proper incorporation of the system history, i.e. initialization.

It has recently been shown [1] that the proper initializations of fractional differintegrals are non-constant (time varying) functions. This generalizes the integer order case, where integrals and derivatives are initialized by constants, which represent the effect of the past. The impact of this non-constant initialization is profound, as it affects the fundamental definitions of the fractional integral and derivative as well as their Laplace transforms. This paper will derive the time varying initialization and develop the ramifications of it.

2 Background

The issue of initialization has been an essentially unrecognized problem in the development of the fractional calculus. Liouville's definition ([2], p.21) for the fractional integral with lower limit of $-\infty$ and Riemann's ([2], p.21) choice of the lower limit of *c*, were in fact related to the issue of initialization. Ross [3],[4] recognizes that to satisfy composition of the fractional differintegral, that the integrated function and its (integer order) derivatives must be zero for times up to and including the start of fractional differintegration. Ross provides a history of the fractional calculus (see [5], p8.) in which he quotes A. Cayley referring to Riemann's paper "The greatest difficulty in Riemann's theory, it appears to me, is the interpretation of the complementary function…," Ross continues, "The question of the existence of a complementary function caused much confusion. Liouville and Peacock were led into error, and Riemann became inextricably entangled in his concept of a complementary function." In retrospect the difficulties of Riemann over the role of the complementary function, which has been abandoned in this mathematics, may in fact have been related to the issue of initialization. The complementary function issue is raised here because an initialization function, which accounts for the effect of history, for fractional integrals and derivatives, will appear in the definitions presented. Its form is similar to Riemann's complementary function, however, the meaning and use of this function is different.

In the solution of fractional differential equations with an assumed history, it has been implicitly inferred by many authors ([2], [3], [5], [6], and others), that an initializing constant, or set of constants, representing the value(s) of the fractional differintegrals (at $t = 0$) will provide an adequate representation for the effects of the past for each differintegral. That this is **not true** will be demonstrated in this paper.

The paper proceeds as follows: The time–varying initialization drives the need to create new, augmented, definitions for the fractional calculus. This in turn demands proving anew the basic properties (criteria) of the fractional calculus. Because the constant initialization of the past is insufficiently general, the widely used contemporary equation for the Laplace transform of the differintegral ([5], p.135 for example) that is based on that assumption, also lacks sufficient generality. An alternative generalized Laplace transform is presented. These elements then become the basis of new system and control theories that apply simultaneously to fractional and integer order systems.

Proof of Non-Constant Initialization

Consider the following *q*th order fractional integrals of $f(t)$, the first starting at time $t = a$, and the second starting at time $t = c$ where $c > a$.

$$ {}_a d_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} f(\tau)\, d\tau \tag{1a} $$

and

$$ {}_c d_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_c^t (t-\tau)^{q-1} f(\tau)\, d\tau . \tag{1b} $$

Assume that $f(t)$ is zero for all $t \le a$, then the time period between $t = a$ and $t = c$ may be considered to be the "history" of the fractional integral ${}_c d_t^{-q} f(t)$. Then, we should expect that when this integral, is properly initialized it should function as a continuation of the integral starting at $t = a$. To achieve this, an initialization must be addended to ${}_c d_t^{-q} f(t)$ so that the resulting fractional integration starting at $t = c$ should be identical to the result starting at $t = a$, for $t > c$. Thus, calling ψ the unknown initialization we have that

$$ {}_c d_t^{-q} f(t) + \psi = {}_a d_t^{-q} f(t), \quad t > c \tag{2}$$

Then

$$\psi = {}_a d_t^{-q} f(t) - {}_c d_t^{-q} f(t), \quad t > c.$$

$$\psi = \frac{1}{\Gamma(q)} \int_a^c (t-\tau)^{q-1} f(\tau)\, d\tau \equiv {}_a d_c^{-q} f(t), \quad t > c. \tag{3}$$

Here ψ is seen to be a function of the independent variable *t*, completing the proof. We see that ψ is a generalization of the case for the ordinary integral ($q = 1$), where

$$\psi = \int_a^c f(\tau)\, d\tau = \text{constant.} \tag{4}$$

Having now recognized the need for a more general initialization, it must be decided if it is prudent to proceed as is done in the ordinary (integer order) calculus. That is, to append the initialization (constant of integration or constant initialization terms of ordinary differential equations) when required in an ad hoc manner or to formalize the process.

Because of the increased complexity of the initialization relative to the integer order calculus case it is prudent to formalize the initialization, that is, to formally include an initialization term into the definitions for the fundamental operators. The rest of this paper; establishes the initialized fractional calculus definition set, considers the 'Ross criteria" [4] for a calculus, presents a generalized (corrected) form for the Laplace transform of a differintegral, demonstrates the solution of properly initialized fractional differential equations and develops an initialized fractional order system theory.

3 Initialized Fractional Calculus Defined

Two types of initializations are possible [7], 'terminal initialization (terminal charging)", where it is assumed that the differintegral operator can only be initialized ("charged") by effectively differintegrating prior to the "start" time, $t = c$, and "side initialization", where a fully arbitrary initialization may be applied to the differintegral operator at time $t = c$. Only terminal initialization results will be presented in this paper. For discussion purposes, it is assumed that *t* (time) is the independent variable associated with the fractional differintegration and $f(t)$ is the integrand.

The terminal initialization assumption requires that the fractional differintegration of interest "start" at $t = c$. An initialization period (or space) is defined as the region $a \le t \le c$. This period is the "history" required for the differintegration. It is further assumed that $f(t) = 0$ for all $t \le a$, and the fractional differintegration of interest takes place for $t > c \ge a$.

The Riemann-Liouville form of the fractional calculus will be the only basis considered here. A consideration of a Grünwald based initialized fractional calculus may be found in [7]. In the development that follows, attention is restricted to real values of the order, q, of the differintegrals.

The following definition set represents the kernal of the initialized fractional calculus;

The ***generalized fractional integral***, for v arbitrary, real, and non-negative:

$$ {}_c D_t^{-v} f(t) \equiv {}_c d_t^{-v} f(t) + \psi(f,-v,a,c,t), \quad v \ge 0 \tag{5}$$

$t > c \ge a$ and $f(t) = 0, \ \forall \ t \le a$.

Here ${}_c d_t^{-v} f(t) = \left(1/\Gamma(v)\right) \int_c^t (t-\tau)^{v-1} f(\tau)\, d\tau$ is the uninitialized fractional integral, it is the contemporary Riemann-Liouville fractional integral and $\psi(f,-v,a,c,t)$, $t > c$, is called the initialization function.

The ***generalized integer derivative*** is defined as

$$ {}_c D_t^m f(t) \equiv \frac{d^m}{dt^m} f(t) + \psi(f,m,a,c,t) \tag{6}$$

The ***generalized fractional derivative***, for q and p real:

$$ {}_c D_t^q \equiv {}_c D_t^m \, {}_c D_t^{-p} f(t)$$
$$= \frac{d^m}{dt^m} {}_c d_t^{-p} f(t) + \frac{d^m}{dt^m} \psi(f,p,a,c,t) + \psi(h,m,a,c,t), \tag{7}$$

where $t > c$, $h = {}_a D_t^{-p} f(t)$, $q = (m-p) > 0$, $p > 0$, and m is integer.

Under the **t**erminal initialization assumption for the generalized fractional integral it can be shown [7] that

$$\psi(f,-v,a,c,t) = {}_a d_c^{-v} f(t) = \left(1/\Gamma(v)\right) \int_a^c (t-\tau)^{v-1} f(\tau)\, d\tau \tag{8}$$

and for the integer derivative $\psi(h,m,a,c,t) = 0$. It is noted that for some physical systems this is not necessarily the case. Figure 1 shows fractional integrals of $f(t) = (t-a)\, U(t-a)$.

4 Criteria

Because the basic definitions of the fractional calculus have been modified, it is necessary to prove the fundamental properties of a fractional calculus using the modified definitions. A set of such properties, criteria, has been established by Ross [4]. Since the proofs [7] are lengthy, in the material that follows, the criteria and the requirements that are associated with them are outlined briefly.

Backward Compatibility with Integer Order Calculus

This criterion requires that for *n* = integer we have

$$ {}_c D_t^n f(t) = \begin{cases} \dfrac{d^n f(t)}{dt^n} & n > 0, \\ 0 & n = 0, \\ \displaystyle\int_c^t \cdots \int_c^{t_{n-2}} f(t_{n-1})\, dt_{n-1} \ldots dt_1 & n < 0. \end{cases} \tag{9}$$

Backward compatibility is achieved when

$$\begin{aligned} &\psi(f,n,a,c,t)=0, && n=1,2,3\ldots \\ &\psi(f,-1,a,c,t)=\text{constant}, && n=-1 \\ &\psi(f,n,a,c,t)=b_0+b_1t+b_2t^2+\ldots, && n=-1,-2,-3\ldots\,. \end{aligned} \tag{10}$$

Zero Property

The operation of order zero leaves the function unchanged:

$${}_cD_t^0\, f(t)=\lim_{q\to 0}\,{}_cD_t^q\, f(t)=\lim_{p\to 1}\,{}_cD_t^1\,{}_cD_t^{-p}\, f(t)=0. \tag{11}$$

This condition holds for terminal initialization, equation (8), and requires $\psi(h,1,a,c,t)=0$, where $h={}_aD_t^{-p}\,f(t)$.

Linearity

The fractional operators must be linear, that is

$${}_cD_t^{-q}\left[bf(t)+kg(t)\right]=b\;{}_cD_t^{-q}\,f(t)+k\;{}_cD_t^{-q}\,g(t),\quad t>c. \tag{12}$$

Linearity of the fractional integral requires, for $t>c$,

$$\psi(bf+kg,-v,a,c,t)=b\psi(f,-v,a,c,t)+k\psi(g,-v,a,c,t). \tag{13}$$

Linearity of the fractional derivative requires

$$\begin{aligned} {}_cD_t^r\,(bf(t)+kg(t)) &= {}_cD_t^m\,{}_cD_t^{-v}\,(bf(t)+kg(t)) = \\ & b\,{}_cD_t^r\,f(t)+k\,{}_cD_t^r\,g(t),\qquad r>0,\ t>c. \end{aligned} \tag{14}$$

where m is an integer and $m>v\geq 0$ and $r=m-v$. This leads to the following initialization function requirement:

$$\psi(h,m,a,c,t)=b\psi(p,m,a,c,t)+k\psi(l,m,a,c,t),\ \ t>c, \tag{15}$$

where $p(t)={}_ad_t^{-v}\,(f(t))$, $l(t)={}_ad_t^{-v}\,(g(t))$ and $h(t)={}_ad_t^{-v}\,f(t)$.
Conditions (13) and (15) hold for terminal initialization.

Index Law

The composition or the index law requires that

$${}_cD_t^u\,{}_cD_t^v\,f(t)={}_cD_t^v\,{}_cD_t^u\,f(t)={}_cD_t^{u+v}\,f(t),\qquad t>c. \tag{16}$$

For fractional differintegrals, under terminal initialization (only), no additional requirements are placed on the initialization functions. Note that side initialization may impose additional constraints on the initialization functions.

5 Laplace Transform of the Fractional Differintegral

The redefinition of the fractional differintegral to include the time-varying initialization function renders the contemporary form of its Laplace transform incorrect.

Forms for the Laplace transform of fractional differintegrals are derived that generalize contemporary fractional and integer derivative forms and include initialization effects.

In the material that follows, considering discontinuities at $t=0$, evaluations stated as $t=0$ are to be evaluated at $t=0+$ consistent with conventional definitions for the Laplace transform.

Laplace Transform of Fractional Integrals

For simplicity the starting point is taken as $c=0$. Thus it is desired to evaluate

$$\begin{aligned} & L\left\{{}_0D_t^{-q}\,f(t)\right\}= \\ & \int_0^\infty e^{-st}\left(\int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)}\,f(\tau)d\tau+\psi(f,-q,a,0,t)\right)dt\quad q>0,t>0. \end{aligned} \tag{17}$$

The Laplace transform convolution theorem is applied. If $H(s)$ and $G(s)$ are transforms of $h(t)$ and $g(t)$ which are sectionally continuous functions and are of exponential order as $t\to\infty$, then the convolution $h(t)*g(t)$ exists and

$$L(h(t)*g(t))=H(s)G(s)=L\left(\int_0^t h(\tau)g(t-\tau)d\tau\right). \tag{18}$$

Then taking $h(t)=f(t)$ and $g(t)=\dfrac{t^{q-1}}{\Gamma(q)}$ the convolution thm. gives

$$F(s)G(s)=L(f(t))\,L\left(\frac{t^{q-1}}{\Gamma(q)}\right)=\frac{1}{s^q}\,L(f(t)),\ \ q>0. \tag{19}$$

This yields the general result,

$$\boxed{L\left({}_0D_t^{-q}\,f(t)\right)=\frac{1}{s^q}\,L(f(t))+L(\psi(f,-q,a,0,t)),\quad q>0.} \tag{20}$$

Now in equation (20), $\psi(f,-q,a,0,t)$ may be thought of as the composed (equivalent) effect of the initializations of the mathematical elements used to create ${}_0D_t^{-q}\,f(t)$. For the fractional order case there are infinitely many ways ${}_0D_t^{-q}\,f(t)$ may be composed, as opposed to the integer order calculus case where only combinations of integer order integrations are possible. For example, in the integer order calculus, the familiar Laplace transform,

$$L\left\{f^{(n)}(t)\right\}=s^nF(s)-\sum_{k=0}^{n-1}s^{n-1-k}\,f^{(k)}(0+) \tag{21}$$

infers that $f^{(n)}(t)$ is composed of (or decomposed into) n separate differentiations each of order 1.

Integer Order Decomposition of the Fractional Integral

It is clear that the composition law must be satisfied to achieve equivalent initialization functions. Further, so long as fractional integration is considered, composition is satisfied.

As an example decomposition, for $q>1$, and with $\psi^{(1)}=d\psi/dt$, using integration by parts,

$$\begin{aligned} L(\psi(f,-q,a,0,t)) &= \int_0^\infty e^{-st}\psi(f,-q,a,0,t)dt= \\ &= 0+\frac{1}{s}\psi(f,-q,a,0,t)\Big|_{t=0}+\frac{1}{s}L\left(\psi^{(1)}(f,-q,a,0,t)\right). \end{aligned} \tag{22}$$

Repeating the process a total of n times, where n is an integer such that $n+1>q>n$ yields

$$\begin{aligned} L\left({}_0D_t^{-q}\,f(t)\right) &= \frac{1}{s^q}L(f(t))+\frac{1}{s^n}L\left(\psi^{(n)}(f,-q,a,0,t)\right)+ \\ & \sum_{j=1}^{n}\frac{1}{s^j}\psi^{(j-1)}(f,-q,a,0,t)\Big|_{t=0} \end{aligned} \tag{23}$$

The inference of this equation is that the q th differintegral is composed of n order 1 integer integrations and a fractional integration of order $q-n$ (see figure 2). Further, the order 1 integrations are each initialized by a constant ($\psi^{(j-1)}\big|_{t=0}$ terms in the summation). Backward compatibility of this form with the Laplace transform for repeated integration from the integer order calculus, i.e.

$$L\left\{\int_0^t\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{n-1}} f(t_n)\,dt_n\,dt_{n-1}\ldots dt_2\,dt_1\right\}= \frac{1}{s^n}L\{f(t)\}+\sum_{i=1}^{n}\frac{c_i}{s^i}\quad n=1,2,3.... \tag{24}$$

is seen by taking $q=n=1,2,3...$ and properly selecting ψ in equation (23) above. In equation (23) the summation term is the result of repeated application of integration by parts and the term represents a redundancy relative to the preceding term for the generalized calculus.

It is clear that infinitely many more possibilities exist than this form. As a further example the integer order integrations in figure 2 could just as well be replaced by fractional integrations each also of order 1. For this case, ${}_0D_t^{-q}f(t)$ is decomposed into n generalized order 1 integrations and a fractional integration of order $-q+n$. Thus, n is the greatest integer less than q. Then in similar manner [7] we have

$$L\left\{{}_0D_t^{-q}f(t)\right\}=\frac{1}{s^q}L\{f(t)\}+\frac{1}{s^n}L\{\psi(f,-q+n,a,0,t)\}+ \sum_{j=1}^{n}\frac{1}{s^{(j-1)}}L\left\{\psi_j\left(x_j,-1,a,0,t\right)\right\}. \tag{25}$$

The difference between this equation and equation (23) is the generalizing effect of initializing here by a function of time instead of a constant as in equation (23).

Fractional Order Decomposition of Fractional Integral

A much more general decomposition of ${}_0D_t^{-q}f(t)$ can be obtained that is not limited to integer integral elements. Consider the mathematics associated with the decomposition indicated in figure 3. Since only fractional integrations are considered, composition holds for the case shown in this diagram. The result, after algebraic reduction of the block diagram, is

$$L\left\{{}_0D_t^{-q}f(t)\right\}=L\left\{{}_0D_t^{-B_1}x_1(t)\right\}=L\{x_{n+1}\}=s^{-B_1}L\{x_1\} +\psi_n(s)+\sum_{j=1}^{n-1}s^{-B_{j+1}}\psi_j(s) \tag{26}$$

where $B_a=\sum_{i=a}^{n}q_i\quad 1\le a\le n\quad q_k\ge 0\ \forall k.$

The attraction of this form is the fact that the q_i can be integer, non-integer or mixed. Further, although this analysis was done for $q>0$, that is for fractional integrals, the basic form can also be shown [7] to hold for fractional derivatives, provided composition is satisfied. The effective initialization then for this case is

$$L\{\psi_{effect}\}=\psi_n(s)+\sum_{j=1}^{n-1}s^{-B_{j+1}}\psi_j(s). \tag{27}$$

Laplace Transform of Fractional Derivatives

Here it is desired to evaluate

$$L\left\{{}_0D_t^{u}f(t)\right\}=L\left\{{}_0D_t^{m}\,{}_0D_t^{-p}f(t)\right\},\qquad u>0, \tag{28}$$

where m is the least integer $>u$ such that $u=m-p$. Then substituting the definitions and expanding, gives

$$L\left\{{}_0D_t^{u}f(t)\right\}=L\left\{{}_0D_t^{m}\,{}_0D_t^{-p}f(t)\right\}= L\left\{\frac{d^{\mathrm{m}}}{dt^m}\left(\int_0^t\frac{(t-\tau)^{p-1}}{\Gamma(p)}f(\tau)d\tau\right)\right\}+ L\left\{\frac{d^{\mathrm{m}}}{dt^m}\psi(f,-p,a,0,t)\right\}+L\{\psi(h,m,a,0,t)\},\qquad t>0, \tag{29}$$

where $h(t)={}_ad_t^{-p}f(t)$. The last two terms on the right hand side become an equivalent ψ for ${}_0D_t^{u}f(t)$, thus

$$\psi_{eq}(f,u,a,0,t)=\frac{d^m}{dt^m}\psi(f,-p,a,0,t)+\psi(h,m,a,0,t). \tag{30}$$

The first term of the right hand side of equation (29),
is found to be

$$L\left\{\frac{d^m}{dt^m}\int_0^t\frac{(t-\tau)^{p-1}}{\Gamma(p)}f(\tau)d\tau\right\}=s^m L\left\{{}_0d_t^{-p}f(t)\right\}. \tag{31}$$

Therefore, one form of equation (29) is;

$$L\left\{{}_0D_t^{u}f(t)\right\}=s^m L\left\{{}_0d_t^{-p}f(t)\right\}+L\{\psi(f,u,a,0,t)\}, \tag{32}$$

where $\psi_{eq}(f,u,a,c,t)$ is written as $\psi(f,u,a,c,t)$ to allow the generalization that follows. Applying the results for the fractional integral, equation (29), gives

$$L\left\{{}_0D_t^{u}f(t)\right\}=s^{m-p}L\{f(t)\}+L\{\psi(f,u,a,0,t)\}$$

$$\boxed{L\left\{{}_0D_t^{u}f(t)\right\}=s^{u}L\{f(t)\}+L\{\psi(f,u,a,0,t)\}.} \tag{33}$$

These simple forms are the most general forms for the Laplace transform of ${}_0D_t^{u}f(t)$. It is noted that this form is the same as that of equation (20), thus in equation (33) u may take on any real value.

Fractional Differential Equations

Proper initialization is crucial in the solution and understanding of fractional differential equations. The application of the initialized fractional calculus to the solution of initialized fractional differential equations will be illustrated with the following example.

Example

Podlubny ([6], p. 138) and Oldham and Spanier ([5], p. 157) consider the following fractional differential equation

$${}_0D_t^{1/2}f(t)+b\,f(t)=0,\quad t>0;\ {}_0D_t^{-1/2}f(t)\Big|_{t=0}=C. \tag{34}$$

The notation ${}_0D_t^{1/2}f(t)$ is that of Podlubny, and, refers to the uninitialized derivative. Applying the contemporary Laplace transform of the fractional derivative Podlubny obtains

$$F(s)=C/\left(s^{1/2}+b\right). \tag{35}$$

The inverse transform is then given in terms of a two-parameter Mittag-Leffler series expansion

$$f(t)=C\,t^{-1/2}E_{\frac{1}{2},\frac{1}{2}}\left(-b\sqrt{t}\right). \tag{36}$$

$$\text{For } b=1,\quad f(t)=C\left(\frac{1}{\sqrt{\pi t}}-e^t erfc\left(\sqrt{t}\right)\right), \tag{37}$$

which agrees with the result of Oldham and Spanier [5].

This is now contrasted with the following approach using the results from the initialized fractional calculus.

We now solve equation (34) again, but now ${}_0D_t^{1/2} f(t)$ is interpreted as an initialized fractional derivative as defined in section 3 of this paper. Thus, we have

$${}_0D_t^{1/2} f(t) + bf(t) = 0,\ t > 0,\ \psi(f, 1/2, a, 0, t) \text{ is arbitrary.} \quad (38)$$

This may be rewritten as

$$\begin{gathered} {}_0d_t^{1/2} f(t) + \psi(f, 1/2, a, 0, t) + bf(t) = 0, \quad t > 0\ , \\ \psi(f, 1/2, a, 0, t) \text{ is arbitrary.} \end{gathered} \quad (39)$$

The Laplace transform of equation (38) using equation (33) is

$$F(s) = \frac{-\psi(f, 1/2, a, 0, s)}{s^{1/2} + b} = \frac{-\psi(s)}{s^{1/2} + b}\ . \quad (40)$$

This equation should be contrasted with equation (35) above, they are only the same when $\psi(t) = -C\delta(t)$, that is, when an impulse at $t = 0$ is used to initialize the fractional differential equation!

A very useful function in the fractional calculus is the R-function. It and its Laplace transform [8], are given by

$$\begin{gathered} R_{q,v}(\alpha, c, t) \equiv \sum_{n=0}^{\infty} \frac{(\alpha)^n (t-c)^{(n+1)q-v-1}}{\Gamma((n+1)q - v)} \Leftrightarrow \frac{s^v}{s^q - \alpha}\ , \\ \operatorname{Re}(q - v) > 0,\ \ \operatorname{Re}(s) > 0\ . \end{gathered} \quad (41)$$

The general inverse transform for equation (40), is obtained by applying Laplace convolution integral,

$$f(t) = -\int_0^t R_{1/2,0}(-b, 0, t-\tau)\, \psi(\tau)\, d\tau \qquad t > 0. \quad (42)$$

Thus, with arbitrary $\psi(t)$ this provides the ***most*** general solution to equation (34), or (38). If we take $\psi(t) = -C\delta(t)$ in equation (41), the result is

$$f(t) = C\, R_{1/2,0}(-b, 0, t), \quad (43)$$

which is identical with the result for equation (35). In the context of the initialized fractional calculus, it is clear that impulsive initialization seldom occurs in real application. A more useful result is obtained under the assumption of terminal initialization, that is apply equation (30) with $q = m - p,\ m = 1,\ p = 1/2$, and $\psi(h, m, a, 0, t) = 0$. Then

$$\begin{gathered} \psi(f, 1/2, a, 0, t) = \frac{d}{dt}\psi(f, -1/2, a, 0, t) = \\ \frac{d}{dt}\frac{1}{\Gamma(1/2)}\int_a^0 (t-\tau)^{-1/2} f(\tau) d\tau\ . \end{gathered} \quad (44)$$

In this evaluation ($f(t), t < 0$) need not be identical to ($f(t), t > 0$), if $f(t)$ is considered to be a composite function.

6 Initialized Fractional Systems

A useful representation for systems of fractional order differential equations is the vector space of fractional dynamic variables [9]. The number of fractional dynamic variables in a particular representation is arbitrary, as any sub-multiple basis value can be chosen for q. Once this is done, the vector representation, under terminal initialization, can be written as

$${}_cD_t^q\, x(t) = Ax(t) + Bu(t), \text{ and } \ y(t) = C\,x(t) + D\,u(t), \quad (45)$$

given $x(t)$ for $a \le t < c$, or $\psi(q, x, a, c, t)$ for $t > c$, and where $x(t)$, $y(t)$, $u(t)$, and $\psi(q, x, a, 0, t)$ are vectors of the appropriate dimension, and c will usually be chosen as zero. Equation (44) can be rewritten as

$${}_0d_t^q\, x(t) + \psi(q, x, a, 0, t) = Ax(t) + Bu(t)\ , \quad (46)$$

where it is now understood that the initialization function is the vector of initialization functions,

$$\psi(q, x, a, 0, t) = \begin{bmatrix} \psi(q, x_1, a, 0, t) \\ \psi(q, x_2, a, 0, t) \\ \vdots \\ \psi(q, x_n, a, 0, t) \end{bmatrix}. \quad (47)$$

This will, without loss of generality, be written as

$$\psi(q, x, a, 0, t) = \psi(t) = [\psi_1(t)\ \ \psi_2(t)\ \ldots\ \psi_n(t)]^T\ . \quad (48)$$

The fractional dynamic variables in the system of equations (45) and (46) are not states in the true sense of the name ”state” space. In the usual integer order system theory, the set of states of the system, $x(t)$, known at any given point in time, along with the system equations, are sufficient to predict the response of the system both forward or backward in time. The fractional dynamic variables in the fractional formulation do not represent the “state” of a system at any given time because the history is now carried in the initialization function vector.

The initialization problem can be solved for $x(t)$ by Laplace transforming equations (45) and (46)

$$s^q\, X(s) + \psi(s) = A\,X(s) + BU(s) \quad (49)$$

$$Y(s) = C\,X(s) + DU(s)\ . \quad (50)$$

Rearranging equation (49) yields

$$X(s)\ = (Is^q - A)^{-1} BU(s) - (Is^q - A)^{-1}\psi(s)\ . \quad (51)$$

Now inserting equation (51) into equation (50) gives the Laplace transform of the total system output response

$$Y(s) = \left\{C(Is^q - A)^{-1}B + D\right\}U(s) - C\,(Is^q - A)^{-1}\psi(s)\ . \quad (52)$$

Inverse transforming obtains the time response as

$$y(t) = C\int_0^t R_{q,0}[A, \tau]\{B\,u(t-\tau) - \psi(t-\tau)\}d\tau + D\,u(t)\ , \quad (53)$$

where the matrix $R-$ function is defined by equation (41) and A is now generally the square $n\,x\,n$ system matrix.

7 Summary

The effect of past history has a profound impact on fractional order system behavior. The major effect is time-varying initialization, which has been incorporated into the definitions for the fractional operators. This leads to more general solutions of fractional differential equations and to a generalization of the Laplace transform for the fractional differintegral.

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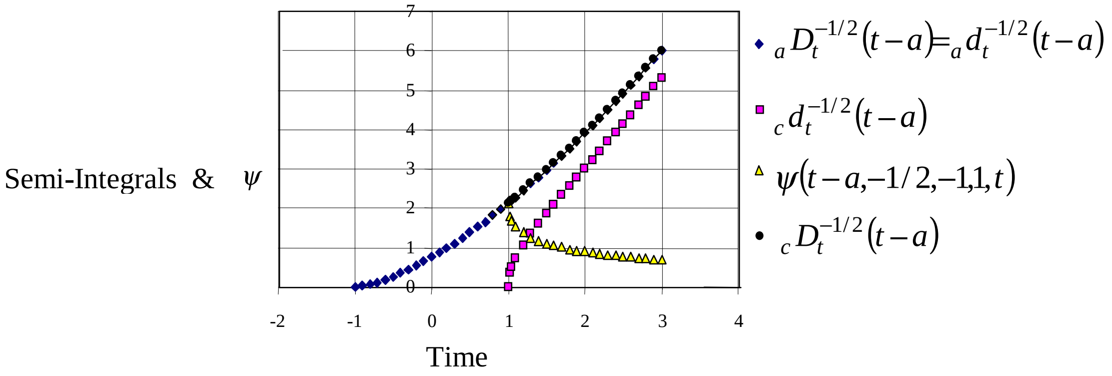

Figure 1 Semi-Integrals of $(t-a)U(t-a)$ vs Time, with $a=-1,\ c=1$

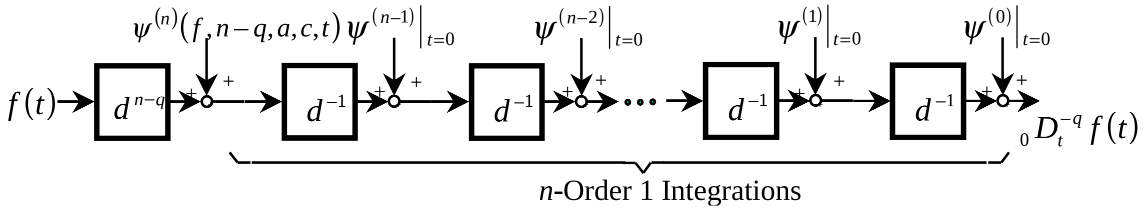

Figure 2 Block Diagram for Integer Order Decomposition of the Fractional Integral.

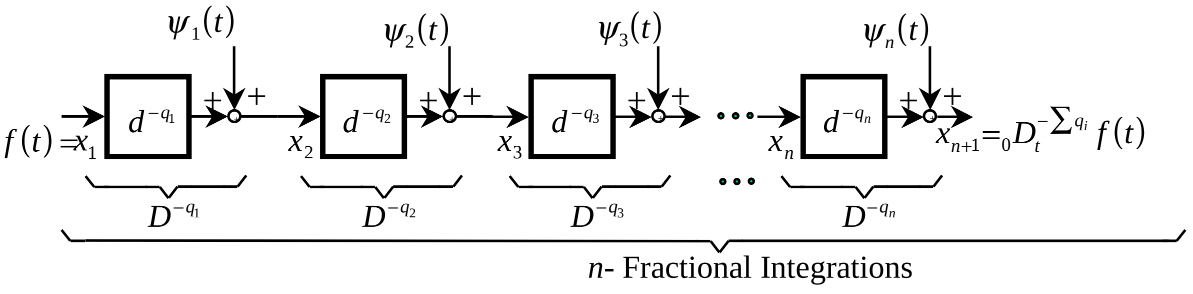

Figure 3 Block Diagram for Multiple Fractional Integrations.