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An H_∞ approach to fault detection for sampled-data systems

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Abstract: This paper studies fault detection problems for sampled-data (SD) systems. At first a discrete-time residual generator is constructed in two steps. With the help of introduced operators, the residual is analyzed quantitatively. Following the principle of integrated design, the design of fault detection system is then formulated as an optimization problem. Finally, two design procedures using H_∞ optimization technique are provided and full decoupling problem is discussed.

Keyword: Fault detection, sampled-data system, H_∞ optimization, frequency domain, robustness

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1 Introduction and problem formulation

In this contribution, fault detection and isolation (FDI) problems are studied for SD systems, which are described by the following three sub-models:

The plant model:

$$y(s) = G_u(s)u(s) + G_d(s)d(s) + G_f(s)f(s) \qquad (1)$$

where $u \in \mathbf{R}^{k_u}$ denotes the vector of control signals, $y \in \mathbf{R}^m$ the vector of plant outputs, $d \in \mathbf{R}^{k_d}$ the vector of unknown but bounded disturbances with $\sup \|d\|_2 \leq \delta_d$ and $f \in \mathbf{R}^{k_f}$ the vector of faults to be detected. $G_u(s), G_d(s)$ and $G_f(s)$ are the transfer function matrices from u, d, f to y respectively.

The model of A/D converter [1]:

$$\psi(e^{j\omega h}) = \frac{1}{h}\sum_{k=-\infty}^{+\infty} y(j\omega + jk\omega_s) \qquad (2)$$

where ψ is the sampled output signal, h the sampling period, $\omega_s = \frac{2\pi}{h}$ the sampling frequency.

The model of D/A converter [1]:

$$u(j\omega) = h\phi(j\omega)v(e^{j\omega h}),\ \phi(j\omega) = e^{-j\omega\frac{h}{2}}\frac{\sin\omega\frac{h}{2}}{\omega\frac{h}{2}} \qquad (3)$$

For the SD system (1)-(3), our task is to design a discrete-time fault detection (FD) system which is sensitive to faults $f(t)$ and simultaneously robust to unknown disturbances $d(t)$.

During the last decade, many encouraging results have been achieved in sampled-data control [1], [2]. The basic problem is how to suitably describe and analyze the intersample behavior since the sampling period cannot be arbitrarily specified. In dealing with FDI, we indeed face the similar problem. Moreover, different from the controller design, the central problem in designing an FDI system for SD systems is to analyze the influence of continuous-time signal on the so-called residual signal, a discrete-time signal, and the main objective of designing an FDI system consists in finding a suitable compromise between enhancing the sensitivity to the faults and increasing the robustness to the unknown disturbances [3], [4], [5]. Recently, [6] and [7] demonstrated that the use of indirect FDI design methods for sampled-data systems, *analog design and SD implementation* or *discrete-time design based on the discretization of the plant model*, may cause some problems and cannot achieve the desired performance. The main objective of this paper is to study the H_∞ optimal design of FD system for SD systems using the direct design scheme.

For our purpose, we first briefly review the fault detection schemes in purely continuous- or discrete-time systems based on the integrated design principle. Consider the input-output description of purely continuous- or discrete-time LTI systems given by:

$$y(p) = G_u(p)u(p) + G_d(p)d(p) + G_f(p)f(p) \qquad (4)$$

where p denote either the complex variable of Laplace transformation for continuous-time signals or the complex variable of z-transformation for discrete-time signals. Generally, an FD system consists of a residual generator and a residual evaluator. It is well known that all LTI residual generators for system (4) can be described by [8]

$$r(p) = R(p)(\hat{M}_u(p)y(p) - \hat{N}_u(p)u(p)) \tag{5}$$

where $R(p)$ is an arbitrarily selectable parametrization matrix belonging to RH_∞, $\hat{M}_u(p)$ and $\hat{N}_u(p)$ are a left coprime factorization of $G_u(p)$. The dynamics of the residual generator (5) with respect to d and f is governed by

$$r(p) = R(p)\hat{M}_u(p)(G_d(p)d(p) + G_f(p)f(p)) \tag{6}$$

To evaluate the residual, the 2-norm of the residual signal r is used. The decision logic is

$$\|r\|_2 > J_{th} \Longrightarrow fault \tag{7}$$

otherwise no fault, where J_{th} is the threshold selected as the maximal possible 2-norm of the residual r in the fault-free case, i.e.

$$J_{th} = \sup_{f=0,d} \|r\|_2 = \left\|R(p)\hat{M}_u(p)G_d(p)\right\|_\infty \delta_d \tag{8}$$

A full decoupling of $r(p)$ from $d(p)$, i.e. $\exists\ R(p)$ such that

$$R(p)\hat{M}_u(p)G_d(p) \equiv 0 \text{ and } R(p)\hat{M}_u(p)G_f(p) \neq 0 \tag{9}$$

is an ideal case which is achievable if and only if [8]

$$rank\ [G_d(p)\ \ G_f(p)] > rank\,[G_d(p)] \tag{10}$$

However, (10) is difficult to be satisfied in practice. Bearing in mind that the main objective of designing residual generators is to enhance the robustness of the FD system to disturbances without loss of the sensitivity to faults and considering the residual evaluation process, the selection of the post-filter $R(s)$ can be formulated as an optimization problem [9]

$$\min_{R(p)} J = \min_{R(p)} \frac{\|R(p)M_u(p)G_d(p)\|_\infty}{\|R(p)M_u(p)G_f(p)\|_\infty} \tag{11}$$

In the continuous-time system, we have the following solution. Under the assumption that $G_d(s)$ has no zeros on the imaginary axis or at infinity, a solution to the optimization problem (11) was derived in [9] to be

$$R(s) = G_{do}^{-1}(s) \tag{12}$$

where $G_{do}(s)$, $G_{di}(s)$ are the co-inner-outer factorization (CIOF) of $\hat{M}_u(s)G_d(s)$, i.e.

$$\hat{M}_u(s)G_d(s) = G_{do}(s)G_{di}(s) \tag{13}$$

and $G_{di}(j\omega)G_{di}^T(-j\omega) = I$, $G_{do}(s)$ is left invertible.

In the discrete-time systems, (11) can be written as

$$\min_{R(z)} J = \min_{R(z)} \frac{\left\|R(z)\hat{M}_u(z)G_d(z)\right\|_\infty}{\left\|R(z)\hat{M}_u(z)G_f(z)\right\|_\infty} \tag{14}$$

Analogous to the method used in [9], a solution to the optimization problem (14) can be obtained as follows. Assume that $G_d(z)$ has no zeros on the unit circle and that $\hat{M}_u(z)\bar{G}_d(z)$ has a CIOF as

$$\hat{M}_u(z)G_d(z) = G_{do}(z)G_{di}(z) \tag{15}$$

with $G_{di}(z)$ as co-inner satisfying $G_{di}(e^{j\omega h})G_{di}^*(e^{j\omega h}) = I$, $G_{do}(z)$ as co-outer having a left inverse $G_{do}^{-1}(z)$ in RH_∞. Then

$$\frac{\left\|R(z)\hat{M}_u(z)G_d(z)\right\|_\infty}{\left\|R(z)\hat{M}_u(z)G_f(z)\right\|_\infty} = \frac{\|R(z)G_{do}(z)\|_\infty}{\left\|R(z)\hat{M}_u(z)G_f(z)\right\|_\infty} \tag{16}$$

Let $R(z) = R_0(z)G_{do}^{-1}(z)$. It turns out

$$\frac{\|R(z)G_{do}(z)\|_\infty}{\left\|R(z)\hat{M}_u(z)G_f(z)\right\|_\infty} \geq \frac{1}{\left\|G_{do}^{-1}(z)\hat{M}_u(z)G_f(z)\right\|_\infty}$$

Since setting $R_0(z) = I$ can make the equality achieved, $R(z) = G_{do}^{-1}(z)$ is the optimal solution to the optimization problem (14).

2 Main results

In this section, we shall first construct a discrete-time FD system for SD systems (1)-(3). Based on the analysis of the influence of continuous-time disturbances and faults on the discrete-time residual signal, the design of the FD system will then be formulated as an optimization problem and two solution algorithms will be presented. Finally we shall discuss the full decoupling problem in SD systems.

2.1 Construction of the FD system

Let $y_u(s) = G_u(s)u(s)$, $y_{df}(s) = G_d(s)d(s) + G_f(s)f(s)$, then (1) can be written into $y(s) = y_u(s) + y_{df}(s)$. Assume that $G_u(s)$ in (1) has a minimal state space realization (A_u, B_u, C_u, O). The characteristics of the SD system at the sampling instants is exactly described by

$$x(k+1) = A_{ud}x(k) + B_{ud}v(k),\ \psi(k) = \psi_u(k) + \psi_{df}(k)$$
$$\psi_u(k) = C_u x(k), \psi_{df}(k) = y_{df}(t)\mid_{t=kh}, x(k) = x(t)\mid_{t=kh}$$
$$y_{df}(t) = \Theta^{-1}(y_{df}(s)) = \Theta^{-1}(G_d(s)d(s) + G_f(s)f(s))$$
$$A_{ud} = e^{A_u h},\ \ B_{ud} = \int_0^h e^{A_u t}B_u dt \tag{17}$$

and Θ^{-1} denotes the inverse Laplace transform. The residual generator is constructed in two steps. At first, construct a discrete-time system for (1)-(3) as follows

$$\begin{aligned}\hat{x}(k+1) &= A_{ud}\hat{x}(k) + B_{ud}v(k) + L(\psi(k) - \hat{\psi}(k)) \\ \hat{\psi}(k) &= C_u\hat{x}(k), \xi(k) = \psi(k) - \hat{\psi}(k) \end{aligned} \tag{18}$$

Define $e(k) \triangleq x(k) - \hat{x}(k)$. The dynamics of the discrete-time system (18) is governed by

$$\begin{aligned} e(k+1) &= (A_{ud} - LC_u)e(k) - L\psi_{df}(k) \\ \xi(k) &= C_u e(k) + \psi_{df}(k) \end{aligned} \tag{19}$$

(18) is internally stable if the gain matrix L is suitably selected so that all the eigenvalues of $A_{ud} - LC_u$ are inside the unit circle. Recalling the importance of the application of post-filter in improving the performance of an FD system [10], we then use a post-filter $R(z)$ to filter the output estimation error signal ξ and generate residual signal $r \in \mathbf{R}^q$

$$r(z) = R(z)\xi(z) \tag{20}$$

In summary, the construction of the residual generator consists of the discrete-time system (18) and the post-filter (20).

To evaluate the residual, the 2-norm of the discrete-time residual signal r is used and the decision logic is selected as (7). The threshold J_{th} is defined as the maximal possible 2-norm of the residual r in the fault-free case

$$J_{th} = \sup_{f=0,d} \|r\|_2 \tag{21}$$

2.2 Optimization of the FD system

We first consider dynamics of the residual generator. From (18), we have

$$\begin{aligned}\xi(z) &= \psi(z) - \hat{\psi}(z) \\ &= [I - C_u(zI - A_{ud} + LC_u)^{-1}L]\psi(z) \\ &\quad - C_u(zI - A_{ud} + LC_u)^{-1}B_{ud}v(z)\end{aligned} \tag{22}$$

Let $\hat{M}_u(z) = I - C_u(zI - A_{ud} + LC_u)^{-1}L$, $\hat{N}_u(z) = C_u(zI - A_{ud} + LC_u)^{-1}B_{ud}$. It is well-known that $\hat{M}_u(z)$ and $\hat{N}_u(z)$ are just a left coprime factorization of $G_{ud}(z) = C_u(zI - A_{ud})^{-1}B_{ud}$. Thus (22) turns out to be

$$\xi(z) = \hat{M}_u(z)(\psi(z) - \psi_u(z)) = \hat{M}_u(z)\psi_{df}(z)$$

where $\psi_{df}(z) = \mathcal{Z}(\psi_{df}(k))$. Considering (20), we have

$$r(z) = R(z)\hat{M}_u(z)\psi_{df}(z) \tag{23}$$

Due to the following relationship between $y_{df}(j\omega)$ and $\psi_{df}(e^{j\omega h})$ [1]

$$\begin{aligned}\psi_{df}(e^{j\omega h}) &= \frac{1}{h}\sum_{k=-\infty}^{+\infty} y_{df}(j\omega + jk\omega_s) \\ &= \frac{1}{h}\sum_{k=-\infty}^{+\infty} G_d(j\omega + jk\omega_s)d(j\omega + jk\omega_s) \\ &\quad + \frac{1}{h}\sum_{k=-\infty}^{+\infty} G_f(j\omega + jk\omega_s)f(j\omega + jk\omega_s)\end{aligned}$$

the dynamics of the residual generator constructed can then be described in the frequency domain by

$$\begin{aligned} r(e^{j\omega h}) &= R(e^{j\omega h})\hat{M}_u(e^{j\omega h})\psi_{df}(e^{j\omega h}) \\ &= r_d(e^{j\omega h}) + r_f(e^{j\omega h}) \qquad (24) \\ r_d(e^{j\omega h}) &= \frac{1}{h}R(e^{j\omega h})\hat{M}_u(e^{j\omega h}) \\ &\quad \sum_{k=-\infty}^{+\infty} G_d(j\omega + jk\omega_s)d(j\omega + jk\omega_s) \\ r_f(e^{j\omega h}) &= \frac{1}{h}R(e^{j\omega h})\hat{M}_u(e^{j\omega h}) \\ &\quad \sum_{k=-\infty}^{+\infty} G_f(j\omega + jk\omega_s)f(j\omega + jk\omega_s)\end{aligned}$$

Therefore, it is clear that the dynamics of discrete-time residual r is influenced by the continuous-time unknown disturbances $d(t)$ and faults $f(t)$.

To measure such influence, the operators $\Gamma_{G_d,R\hat{M}_u}$ and $\Gamma_{G_f,R\hat{M}_u}$ are introduced and defined as

$$\Gamma_{G_d,R\hat{M}_u} : L_2(j\mathbf{R}, \mathbf{C}^{k_d}) \mapsto L_2(\mathbf{\Omega}, \mathbf{C}^q) \tag{25}$$

$$r_d(e^{j\omega h}) = (\Gamma_{G_d,R\hat{M}_u}d)(e^{j\omega h}) \tag{26}$$

$$\Gamma_{G_f,R\hat{M}_u} : L_2(j\mathbf{R}, \mathbf{C}^{k_f}) \mapsto L_2(\mathbf{\Omega}, \mathbf{C}^q) \tag{27}$$

$$r_f(e^{j\omega h}) = (\Gamma_{G_f,R\hat{M}_u}f)(e^{j\omega h}) \tag{28}$$

where $L_2(j\mathbf{R}, \mathbf{C}^k)$, $k = k_d, k_f$, denotes the space of all vector-valued functions $\alpha(j\omega) \in \mathbf{C}^k$ for $j\omega \in j\mathbf{R}$ with inner product

$$< \alpha(j\omega), \zeta(j\omega) >= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \alpha^*(j\omega)\zeta(j\omega)d\omega \tag{29}$$

$\mathbf{\Omega}$ denotes the unit circle in the complex plane and $L_2(\mathbf{\Omega}, \mathbf{C}^q)$ denotes the space of all vector-valued functions $\beta(e^{j\omega h}) \in \mathbf{C}^q$ for $e^{j\omega h} \in \mathbf{\Omega}$ with inner product

$$< \beta(e^{j\omega h}), \varphi(e^{j\omega h}) >= \frac{h}{2\pi}\int_0^{\omega_s} \beta^*(e^{j\omega h})\varphi(e^{j\omega h})d\omega \tag{30}$$

The operators $\Gamma_{G_d,R\hat{M}_u}$ and $\Gamma_{G_f,R\hat{M}_u}$ map, respectively, the continuous-time disturbances d and faults f to the discrete-time signal r_d and r_f through a so-called continuous-discrete system, which is composed of a strictly proper continuous LTI system, an ideal A/D converter and a discrete LTI system. It can be easily derived that the adjoint operators of $\Gamma_{G_d,R\hat{M}_u}$ and $\Gamma_{G_f,R\hat{M}_u}$ are respectively

$$\Gamma^*_{G_d,R\hat{M}_u} : L_2(\mathbf{\Omega}, \mathbf{C}^q) \mapsto L_2(j\mathbf{R}, \mathbf{C}^{k_d}) \tag{31}$$

$$(\Gamma^*_{G_d,R\hat{M}_u} r_d)(j\omega) = G^*_d(j\omega)\hat{M}^*_u(e^{j\omega h})R^*(e^{j\omega h})r_d(e^{j\omega h}) \tag{32}$$

$$\Gamma^*_{G_f,R\hat{M}_u} : L_2(\mathbf{\Omega}, \mathbf{C}^q) \mapsto L_2(j\mathbf{R}, \mathbf{C}^{k_f}) \tag{33}$$

$$(\Gamma^*_{G_f,R\hat{M}_u} r_f)(j\omega) = G^*_f(j\omega)\hat{M}^*_u(e^{j\omega h})R^*(e^{j\omega h})r_f(e^{j\omega h}) \tag{34}$$

where $*$ represents adjoint and conjugate transpose for operator and matrix respectively.

The norms of the operators $\Gamma_{G_d,R\hat{M}_u}$ and $\Gamma_{G_f,R\hat{M}_u}$ are defined by

$$\left\|\Gamma_{G_d,R\hat{M}_u}\right\| = \sup_{d\neq 0} \frac{\|r_d\|_2}{\|d\|_2} \tag{35}$$

$$\left\|\Gamma_{G_f,R\hat{M}_u}\right\| = \sup_{f\neq 0} \frac{\|r_f\|_2}{\|f\|_2} \tag{36}$$

which measure the maximal possible influence of continuous-time disturbances d and faults f on discrete-time signals r_d and r_f respectively. Furthermore, the norms of the operators $\Gamma_{G_d,R\hat{M}_u}$ and $\Gamma_{G_f,R\hat{M}_u}$ can be calculated by the H_∞ norm of associated discrete-time systems [1], [11], i.e.

$$\left\|\Gamma_{G_d,R\hat{M}_u}\right\| = \left\|R(z)\hat{M}_u(z)\bar{G}_d(z)\right\|_\infty \tag{37}$$

$$\left\|\Gamma_{G_f,R\hat{M}_u}\right\| = \left\|R(z)\hat{M}_u(z)\bar{G}_f(z)\right\|_\infty \tag{38}$$

with the following definitions

$$\bar{G}_d(z) = C_d(zI - A_{dd})^{-1}\bar{B}_d,\ \bar{G}_f(z) = C_f(zI - A_{fd})^{-1}\bar{B}_f \tag{39}$$

$$A_{dd} = e^{A_d h},\ \bar{B}_d\bar{B}_d^T = \int_0^h e^{A_d t}B_d B_d^T e^{A_d^T t}dt \tag{40}$$

$$A_{fd} = e^{A_f h},\ \bar{B}_f\bar{B}_f^T = \int_0^h e^{A_f t}B_f B_f^T e^{A_f^T t}dt \tag{41}$$

where (A_d, B_d, C_d, O) and (A_f, B_f, C_f, O) are the state space realizations of $G_d(s)$ and $G_f(s)$ respectively. Using operators $\Gamma_{G_d,R\hat{M}_u}$ and $\Gamma_{G_f,R\hat{M}_u}$, the frequency dynamics (24) of the residual generator can be expressed as

$$r(e^{j\omega h}) = (\Gamma_{G_d,R\hat{M}_u} d)(e^{j\omega h}) + (\Gamma_{G_f,R\hat{M}_u} f)(e^{j\omega h}) \tag{42}$$

Similar to (11), in consideration of the sensitivity of the FD system to faults $f(t)$ and simultaneously its robustness to unknown disturbances $d(t)$, the optimization problem can be formulated as

$$\min_{R(z)\in RH_\infty} J = \min_{R(z)\in RH_\infty} \frac{\left\|\Gamma_{G_d,R\hat{M}_u}\right\|}{\left\|\Gamma_{G_f,R\hat{M}_u}\right\|} \tag{43}$$

According to (37) and (38), the optimization problem (43) is equivalent to

$$\min_{R(z)\in RH_\infty} J = \min_{R(z)\in RH_\infty} \frac{\left\|R(z)\hat{M}_u(z)\bar{G}_d(z)\right\|_\infty}{\left\|R(z)\hat{M}_u(z)\bar{G}_f(z)\right\|_\infty} \tag{44}$$

due to which the design approach proposed in this contribution is also called H_∞ approach. Note that the optimization problem (44) is solvable by the method described in Section 1. Therefore, the optimal solution to the original optimization problem (43) is

$$R(z) = \bar{G}_{do}^{-1}(z) \tag{45}$$

where $\bar{G}_{do}^{-1}(z)$ is the left inverse of the co-outer matrix of $\hat{M}_u(z)\bar{G}_d(z)$, i.e.

$$\hat{M}_u(z)\bar{G}_d(z) = \bar{G}_{do}(z)\bar{G}_{di}(z) \tag{46}$$

and $\bar{G}_d(z)$ is given by (39). Considering (21), the threshold should be set to

$$J_{th} = \left\|R(z)\hat{M}_u(z)\bar{G}_d(z)\right\|_\infty \delta_d = \delta_d \tag{47}$$

Remark: Since $\hat{M}_u(z)$ is the denominator of the left coprime factorization of $G_{ud}(z)$, $\hat{M}_u(z)$ is a full rank square matrix for any z. Thus, $\forall z$, $rank\, \hat{M}_u(z)\bar{G}_d(z) = rank\, \bar{G}_d(z)$. The co-inner-outer factorization (46), where the co-outer matrix $\bar{G}_{do}(z)$ is left RH_∞ invertible, can always be done as long as $\bar{G}_d(z)$ doesn't have zeros on the unit circle [12].

Remark: The optimal post-filter $R(z)$ is not unique in the sense that, depending on different CIOF approaches, e.g. [12], [13], different $R(z)$ may be obtained. Taking account of the strictly proper character of $\hat{M}_u(z)\bar{G}_d(z)$, Lemma 1 is provided below for the computation of the optimal post-filter $R(z)$ based on the approach to the dual problem inner-outer factorization given by [12].

Lemma 1 *Suppose that $\hat{M}_u(z)\bar{G}_d(z)$ has no zeros on the unit circle and that (A, B, C, O) is its detectable state space realization and has no unreachable null modes and no unobservable modes on the unit circle. Then*

$$R(z) = Q + QC(zI - A - FC)^{-1}F \tag{48}$$

where Q is the right inverse of H which is an appropriate full row rank matrix satisfying $HH^T = CXC^T$,

$F = F_o^T$ *and* (X, F_o) *is the stabilizing solution to the discrete-time algebraic Riccati system (DTARS)*

$$\begin{bmatrix} AXA^T - X + BB^T & AXC^T \\ CXA^T & CXC^T \end{bmatrix} \begin{bmatrix} I \\ F_o \end{bmatrix} = 0$$

From the above analysis, the optimal FD system can be designed as follows.

Algorithm 2 *H_∞ optimal design of fault detection system for SD systems (1)-(3)*

- *Find (A_u, B_u, C_u, O) of $G_u(s)$.*
- *Compute A_{ud} and B_{ud} using (17).*
- *Determine L to ensure the stability of $A_{ud} - LC_u$ and construct the discrete-time system (18).*
- *Calculate $\hat{M}_u(z)$, $\bar{G}_d(z)$ using (22), (39).*
- *Find a minimal state space realization of $\hat{M}_u(z)\bar{G}_d(z)$.*
- *Compute $R(z)$ according to Lemma 1.*
- *Construct the residual generator according to (18) and (20).*
- *Set the threshold to $J_{th} = \delta_d$.*

2.3 Discussion

In this subsection, we shall present at first an alternative design procedure of the optimal FD system and then address the full disturbance decoupling problem.

An alternative design procedure

In fact, if the discrete-time system (18) is constructed based on the state space realization of $[G_u(s) \;\; G_d(s)]$ instead of only on $G_u(s)$, the computation of the optimal post-filter can be simplified. Assume that one minimal state space realization of $[G_u(s) \;\; G_d(s)]$ is

$$[G_u(s) \;\; G_d(s)] = C(sI - A)^{-1}B \tag{49}$$

and partition the matrix B into $[B_u \;\; B_d]$, i.e. $G_u(s) = C(sI - A)^{-1}B_u$, $G_d(s) = C(sI - A)^{-1}B_d$. Then

$$\begin{aligned} G_{ud}(z) &= C(zI - A_d)^{-1}B_{ud} \\ \bar{G}_d(z) &= C(zI - A_d)^{-1}\bar{B}_d, A_d = e^{Ah} \\ B_{ud} &= \int_0^h e^{At}B_u dt, \bar{B}_d\bar{B}_d^T = \int_0^h e^{At}B_dB_d^Te^{A^Tt}dt \end{aligned} \tag{50}$$

Note that $\hat{M}_u(z) = I - C(zI - A_d + LC)^{-1}L$ and one of the left coprime factorizations of $\bar{G}_d(z)$ is given by

$$\hat{M}_d(z) = I - C(zI - A_d + LC)^{-1}L \tag{51}$$

$$\hat{N}_d(z) = C(zI - A_d + LC)^{-1}\bar{B}_d \tag{52}$$

It turns out that $\hat{M}_u(z)\bar{G}_d(z) = C(zI - A_d + LC)^{-1}\bar{B}_d$. According to Lemma 1, Algorithm 3 is thus obtained, which can be applied to design the optimal FD system as an alternative to Algorithm 2.

Algorithm 3 *H_∞ optimal design of fault detection system for SD systems (1)-(3)*

- *Find $(A, [B_u \;\; B_d], C, O)$ of $G_u(s)$.*
- *Compute A_d, B_{ud} and $\bar{B}_d$ using (50).*
- *LetbL e any constant matrix stabilizing $A_d - LC$.*
- *Compute $R(z)$, which is given by*

  $$R(z) = \tilde{Q} + \tilde{Q}C(zI - A_d + \tilde{L}C)^{-1}(L - \tilde{L}) \tag{53}$$

  where $\tilde{Q}$ is the right inverse of $\tilde{H}$ which is an appropriate full row rank matrix satisfying $\tilde{H}\tilde{H}^T = CXC^T$, $\tilde{L} = -L_o^T$ and (X, L_o) is the stabilizing solution to DTARS

  $$\begin{bmatrix} A_dXA_d^T - X + \bar{B}_d\bar{B}_d^T & A_dXC^T \\ CXA_d^T & CXC^T \end{bmatrix} \begin{bmatrix} I \\ L_o \end{bmatrix} = 0$$

- *Construct the optimal residual generator as $r(z) = R(z)\xi(z)$ with*

  $$\begin{aligned} \hat{x}(k+1) &= A_d\hat{x}(k) + B_{ud}v(k) + L(\psi(k) - \hat{\psi}(k)) \\ \hat{\psi}(k) &= C\hat{x}(k), \xi(k) = \psi(k) - \hat{\psi}(k) \end{aligned}$$

  and set the threshold $J_{th} = \delta_d$.

It is interesting to note the following remarks.

Remark: Algorithm 2, which is based on the state space realization of $[G_u(s) \;\; G_d(s)]$, provides a more convenient computation approach of the optimal post-filter $R(z)$ due to the special selection of (51). On the other side, Algorithm 1, which is based on the state space realization of $G_u(s)$, is more intuitive and sheds more insight into the essence of the optimal design of FD system for SD systems.

Remark: If $L = \tilde{L}$ in (53), then $R(z)$ reduces to a constant matrix $\tilde{Q}$. In this case, the residual generator has the lowest order.

Remark: Apparently, the discrete-time system (18) can also be constructed based on the state space realization of $[G_u(s) \;\; G_d(s) \;\; G_f(s)]$

$$\begin{aligned} \dot{x}_p &= A_px_p + B_uu + B_dd + B_ff \\ y &= C_px_p \end{aligned}$$

instead of only on $[G_u(s) \;\; G_d(s)]$ or $G_u(s)$. However, note that the optimal solution (45) is indeed unrelated to $G_f(s)$. If the dynamics from $f(s)$ to $y(s)$ is different from the dynamics from $[u^T(s)\ d^T(s)]^T$ to $y(s)$, such a treatment will unnecessarily increase the order of the residual generator, which is undesirable in practice.

Full decoupling problem

The discrete-time residual r is said to be fully decoupled from the continuous-time disturbances d if the dynamics of the residual generator satisfies $r(e^{j\omega h}) = r_d(e^{j\omega h}) + r_f(e^{j\omega h}) \equiv r_f(e^{j\omega h})$, i.e. for any disturbance d,

$$r_d(e^{j\omega h}) = (\Gamma_{G_d,R\hat{M}_u} d)(e^{j\omega h}) \equiv 0, \quad \forall \omega \in [0, \omega_s] \quad (54)$$

which holds iff

$$\forall d, \quad \|r_d\|_2 = \frac{h}{2\pi} \int_0^{\omega_s} r_d^*(e^{j\omega h}) r_d(e^{j\omega h}) d\omega = 0 \quad (55)$$

From (35) and (37), (55) is equivalent to

$$\left\| \Gamma_{G_d,R\hat{M}_u} \right\| = \left\| R(z)\hat{M}_u(z)\bar{G}_d(z) \right\|_\infty = 0 \quad (56)$$

Because $\hat{M}_u(z)$ is a full rank square matrix, (56) holds iff

$$rank\, \bar{G}_d(z) < m \quad (57)$$

Thus for the SD system (1)-(3), there exists a linear discrete-time residual generator fully decoupled from disturbance d if and only if (57) holds.

Recall that in purely continuous-time systems described by (1), the condition for full decoupling of $r(t)$ from disturbance $d(t)$ is $rank\, G_d(s) < m$ and note that $rank\, \bar{G}_d(z) \geq rank\, G_d(s)$, because for the state space realization of $G_d(s) = C_d(sI - A_d)^{-1}B_d$ and corresponding $\bar{G}_d(z) = C_d(zI - A_{dd})^{-1}\bar{B}_d$ in (39),

$$\begin{aligned} rank\, \bar{B}_d &= rank\, \bar{B}_d\bar{B}_d^T = rank \int_0^h e^{A_d t} B_d B_d^T e^{A_d^T t} dt \\ &\geq rank B_d B_d^T = rank B_d \end{aligned}$$

Therefore, a full decoupling of the residual from the disturbance is more difficult in SD systems than in purely continuous-time systems.

3 Conclusion

In this paper, we proposed an H_∞-optimization technique based approach to the optimal design of discrete-time fault detection system for SD systems. With the help of the introduced operators, the influence of continuous-time disturbances and faults on the discrete-time residual can be quantitatively analyzed without any approximation. The optimization problem was then formulated following the principle of integrated design. Two algorithms were presented for the design of the optimal fault detection system. Further discussion shows that full decoupling of the residual from disturbances becomes more difficult in SD systems.

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