A COMPARISON OF NON-ORTHOGONAL AND ORTHOGONAL FRACTAL DECODING

Ming Hong Pi, Anup Basu, Mrinal Mandal[1] and Hua Li[2]

Dept. of Computing Science, [1] Dept. of Electrical and Computer Engineering,
University of Alberta, Edmonton AB, T6G 2E8, Canada

[2] Dept. of Mathematics and Computer Science, University of Lethbridge
Lethbridge, AB, T1K 3M4
Email: {minghong,anup}@cs.ualberta.ca, mandal@ee.ualberta.ca, huali@cs.uleth.edu

ABSTRACT

The model coefficients for the case of the non-orthogonal basis in Jacquin's mappings are contrast scaling and luminance offset. After orthogonalization, the model coefficients become range block mean and contrast scaling. These two fractal coding algorithms have the same encoding procedure except that luminance offset is replaced by range block mean, and however, their decoding algorithms are different. In this paper, we prove that the orthogonal decoding algorithm converges faster than the non-orthogonal algorithm, while the two decoding algorithms produce the same iteration series if the initial image is the range-averaged image and the step size equals the side-length of range block.

1. INTRODUCTION

In 1988, Barnsley and Sloan [1] originated fractal image compression where an image is represented by an attractor of a set of affine transforms. Later, Jacquin implemented a block-based fractal compression algorithm by Partition Iterated Function System (PIFS), and formed the standard algorithm for fractal block coding [2]. Jacquin's method is based on the hypothesis that real world images exhibit affine redundancy. Under this hypothesis, an image is first segmented into range blocks, and each range block is then encoded by finding the best affine map to minimize the distortion between the range block and the candidate domain blocks by searching a global or partial domain block pool (similar to the codebook for vector quantization). As a result, the fractal code of each range block consists of a scaling parameter and a luminance offset, and the location of the best matched domain block. When decoding, the image is reconstructed by iteration on an arbitrary initial image.

Many researches on fractal coding are focused on speedup of fractal encoding [3-10] and fractal decoding [11, 12]. A detailed investigation on fractal coding can be found in [4, 13]. The affine transform in most of fractal coding methods [2-8] is defined by a scaling parameter and a luminance offset, and we refer to fractal coding as the non-orthogonal fractal coding. Øien and Lepsøy [11] derived the orthogonal fractal coding through orthogonalization, where the affine transform is defined by the range block mean and contrast scaling [9, 14]. Øien and Lepsøy proved that the orthogonal decoding algorithm reaches the attractor in a finite number of iterations if the step-size equals the side-length of range block. Moon et al. [12] proved that the range-averaged image is the optimal initial image for the non-orthogonal decoding algorithm. However, what is the difference between orthogonal and non-orthogonal fractal coding algorithms? Which decoding algorithm converges faster?

In the paper, we prove that the orthogonal decoding algorithm converges faster than the non-orthogonal decoding algorithm, while two decoding algorithms generate the same iteration series if the initial image is the range-averaged image and the step-size equals the side-length of range block.

The remainder of the paper is organized as follows. Section 2 introduces the orthogonal and non-orthogonal fractal coding. In Section 3, we prove the orthogonal decoding algorithm converges faster than the non-orthogonal counterpart, followed by conclusions.

2. ORTHOGONAL AND NON-ORTHOGONAL FRACTAL CODING

Jacquin-styled fractal image compression is a block-based compression scheme [1-8]. An image is first segmented into range blocks (of size $B \times B$). A domain block pool Ω is a set of domain blocks (of size $2B \times 2B$), generated by sliding $2B \times 2B$ window in the step-size δ within the image [3]. Given a range block $R = \{r_{ij}\}$, a domain block $D = \{d_{ij}\}$, and an affine transform $\tau(D) = s\rho(D) + gU$ (s is a scaling parameter and g is a luminance offset), the affine parameters s and g can be obtained using the least square method.

$$(s, g) = \arg\min_{s,g} E(R, D) = \arg\min_{s,g} \left\| R - s\rho(D) - gU \right\|^2 \quad (1)$$

0-7803-8554-3/04/$20.00 ©2004 IEEE.

where U is a matrix whose elements are all ones, ρ is a contractive operator which shrinks D into the same size with R [2], and $\|.\|$ is the L_2-norm. As a result,

$$s = < R - \overline{R}U, \rho(D - \overline{D}U) > \Big/ \left\|\rho(D - \overline{D}U)\right\|^2, \ g = \overline{R} - s\overline{D}$$

where $\overline{R}$ and $\overline{D}$ are the average intensity of the range block and domain block, respectively, $< .,. >$ is the inner product. Note that the matrices are converted into the vector by concatenating adjacent rows before the inner product and L_2 -norm are computed. By minimizing $E(R, D)$ over $D \in \Omega$, the fractal code of the range block R is obtained as:

$$(s, g, i_D, j_D) = \arg\min_{D\in\Omega} E(R, D)$$

where (i_D, j_D) is the top-left corner coordinate of the best matched domain block. In place of g with $\overline{R} - s\overline{D}$, we derive

$$\begin{aligned}\tau(D) &= s\rho(D) + gU = \overline{R}U + s\rho(D) - s\overline{D}U \\ &= \overline{R}U + s\rho(D - \overline{D}U)\end{aligned} \qquad (2)$$

As a result, due to $\left\langle U, \rho(D - \overline{D}U)\right\rangle = 0$, τ is converted into the orthogonal form [11]. The affine transform of the orthogonal form is determined by $\overline{R}$ and s, only s is obtained by the least square's method. For a given range block R and a given domain block D, the distortions of the two affine transforms are equal, and hence, the best matched domain blocks of R obtained by the orthogonal and non-orthogonal forms are the same. As a result, there is no difference between the encoding procedures of the two forms except that the fractal code of R of the orthogonal form is $\{s, \overline{R}, i_D, j_D\}$. However, their decoding algorithms are different.

The non-orthogonal decoding algorithm (s , g) is

$$R_{old}^{(n)} = s\rho\left(D^{(n-1)}\right) + gU = \overline{R}U + s\rho\left(D^{(n-1)} - \overline{D}U\right) \quad (3)$$

Whereas, the orthogonal decoding algorithm ($\overline{R}$, s) is:

$$R_{new}^{(n)} = \overline{R}U + s\rho\left(D^{(n-1)} - \overline{D^{(n-1)}}U \right) \qquad (4)$$

$\overline{D}$ in (3) is constant, while $\overline{D^{(n-1)}}$ in (4) is the average of the (n-1)-th iteration domain block, and it is a variable that is updated as the iteration proceeds.

Øien and Lepsøy [11] revealed the significance of orthogonalization for fast decoding, and proved the orthogonal decoding algorithm reaches the attractor in a finite number of iterations under some conditions. The choice of initial image of the iteration procedure has been discussed for speeding up convergence. Whether for non-orthogonal or for orthogonal decoding algorithms, the range-averaged image (produced by replacing each pixel of the range block with mean of the range block) is the best initial image [11, 12].

3. FAST CONVERGENCE

Øien and Lepsøy have proved that, for $2^b \times 2^b$ range block and $2^{b+1} \times 2^{b+1}$ domain block, the orthogonal decoding algorithm exactly reaches its attractor in $b+1$ iterations for arbitrary initial image and in b iterations if initial image is the range-averaged image. They have also shown by the experiments that the orthogonal decoding algorithm converges at least as fast as the non-orthogonal decoding algorithm, however, they have not proved it. In this paper, we prove that the orthogonal decoding algorithm converges faster than the non-orthogonal counterpart, and two decoding algorithms produce the same iteration series, if the iteration started from the range-averaged image. First, we state Lemma 1 and 2, and then we prove the fast convergence based on these two Lemmas.

Lemma 1: If $X = \{x_1, x_2, \cdots, x_n\}$ is a n-dimensional vector, and $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the mean of X, then

$$\left\|X - \overline{X}U\right\| \le \left\|X - aU\right\| \quad \text{For } a \in R^1$$

The equality holds if and only if $a = \overline{X}$.

Lemma 2:

$$\overline{\rho(\overline{D}U)} = \overline{D} = \overline{\rho(D)} = \overline{\rho^2(D)} = \cdots$$

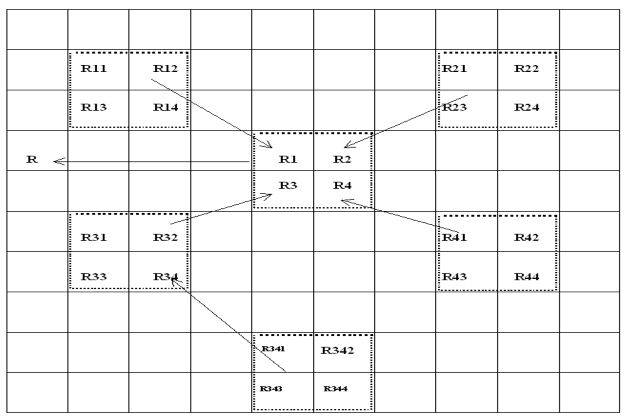


Figure 1: Mapping between range block and domain block

Lemma 1 has been proven in [14], Lemma 2 is easy to prove, because of the relation between the average operator and the spatial contractive operator ρ. For simplicity, we assume $\delta = B$, that is, the step-size equals the side-length of range block. As a result, each domain block consists of four range blocks (see Fig. 1). The following notations are introduced for the proof of theorem 1 and 2. Let $D=\begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}$ denote the best matched domain block of R, and $D^{(n)}=\begin{pmatrix} R_1^{(n)} & R_2^{(n)} \\ R_3^{(n)} & R_4^{(n)} \end{pmatrix}$ $(n=0,1,2,\cdots)$ denote the nth iteration of D. Let $D_i=\begin{pmatrix} R_{i1} & R_{i2} \\ R_{i3} & R_{i4} \end{pmatrix}$ denote the best matched domain block of R_i, and $D_i^{(n)}=\begin{pmatrix} R_{i1}^{(n)} & R_{i2}^{(n)} \\ R_{i3}^{(n)} & R_{i4}^{(n)} \end{pmatrix}$ denote the nth iteration of D_i. Fig. 1 shows the range blocks and domain blocks and their relations. Next, let us begin to prove theorem 1.

Theorem 1: The orthogonal decoding algorithm (4) converges faster than the non-orthogonal decoding algorithm (3), that is,

$$\left\|R-R_{new}^{(n)}\right\| \le \left\|R-R_{old}^{(n)}\right\| \qquad n\ge 1$$

Proof: Based on PIFS theory, the original image is approximated by the attractor of fractal transforms. Since (3) and (4) have the same collage error, collage error is ignored, range block R equals

$$R=s\rho(D)+gU=\overline{R}U+s\rho(D-\overline{D}U)$$

when $n=1$, Theorem 1 has been proven in [14], that is,

$$\left\|R-R_{new}^{(1)}\right\| \le \left\|R-R_{old}^{(1)}\right\|$$

Let us proceed to the second iteration. For the non-orthogonal decoding algorithm

$$R-R_{old}^{(2)}=s\rho\left(D-D_{old}^{(1)}\right)$$

$$=s\rho\begin{pmatrix} R_1-R_{1,old}^{(1)} & R_2-R_{2,old}^{(1)} \\ R_3-R_{3,old}^{(1)} & R_4-R_{4,old}^{(1)} \end{pmatrix}$$

$$=s\rho\begin{pmatrix} s_1\rho(D_1-D_1^{(0)}) & s_2\rho(D_2-D_2^{(0)}) \\ s_3\rho(D_3-D_3^{(0)}) & s_4\rho(D_4-D_4^{(0)}) \end{pmatrix}$$

$$=s\begin{pmatrix} s_1\rho^2(D_1-D_1^{(0)}) & s_2\rho^2(D_2-D_2^{(0)}) \\ s_3\rho^2(D_3-D_3^{(0)}) & s_4\rho^2(D_4-D_4^{(0)}) \end{pmatrix}$$

Thus

$$\left\|R-R_{old}^{(2)}\right\|=\sum_{l=1}^{4} ss_l\left\|\rho^2(D_l-D_l^{(0)})\right\|$$

For the orthogonal decoding algorithm

$$\rho(D-D_{new}^{(1)})=\rho\begin{pmatrix} R_1-R_{1,new}^{(1)} & R_2-R_{2,new}^{(1)} \\ R_3-R_{3,new}^{(1)} & R_4-R_{4,new}^{(1)} \end{pmatrix}$$

$$=\rho\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}=\begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$$

where

$$F_l=s_l(\rho(D_l-D_l^{(0)})-\overline{\rho(D_l-D_l^{(0)})}U$$

$$H_l=\rho(F_l)=s_l(\rho^2(D_l-D_l^{(0)})-\overline{\rho^2(D_l-D_l^{(0)})}U)$$

Thus

$$\left\|R-R_{new}^{(2)}\right\|=s\left\|\rho(D-D_{new}^{(1)})-\overline{\rho(D-D_{new}^{(1)})}U\right\|$$

$$\le s\left\|\rho(D-D_{new}^{(1)})\right\| \quad \text{(Lemma 1)}$$

$$=\sum_{l=1}^{4} ss_l\left\|\rho^2(D_l-D_l^{(0)})-\overline{\rho^2(D_l-D_l^{(0)})}U\right\|$$

$$\le\sum_{l=1}^{4} ss_l\left\|\rho^2(D_l-D_l^{(0)})\right\|=\left\|R-R_{old}^{(2)}\right\| \quad \text{(Lemma 1)}$$

Repeat the process until the nth iteration, we have

$$\left\|R-R_{new}^{(n)}\right\|\le s\left\|\rho(D-D_{new}^{(n-1)})\right\|\le\sum_{l_1=1}^{4} ss_{l_1}\left\|\rho^2(D_{l_1}-D_{l_1}^{(n-2)})\right\|$$

$$\le\sum_{l_1=1,l_2=1}^{4} ss_{l_1}s_{l_1l_2}\left\|\rho^3(D_{l_1l_2}-D_{l_1l_2}^{(n-3)})\right\|\le\cdots$$

$$\le\sum_{l_1=1,l_2=1,\cdots,l_{n-1}=1}^{4} ss_{l_1}s_{l_1l_2}\cdots s_{l_1l_2\cdots l_{n-1}}\left\|\rho^n(D_{l_1l_2\cdots l_{n-1}}-D_{l_1l_2\cdots l_{n-1}}^{(0)})\right\|=\left\|R-R_{old}^{(n)}\right\|$$

Thus

$$\left\|R-R_{new}^{(n)}\right\|\le\left\|R-R_{old}^{(n)}\right\|$$

Theorem 2: if the initial image is the range-averaged image and $\delta=B$, then the orthogonal and non-orthogonal decoding algorithms produce the same iteration series, or in other words,

$$R_{new}^{(n)}=R_{old}^{(n)}$$

Proof: since the initial image is the range-average image**,** then $\overline{D}=\overline{D_0}$, furthermore,

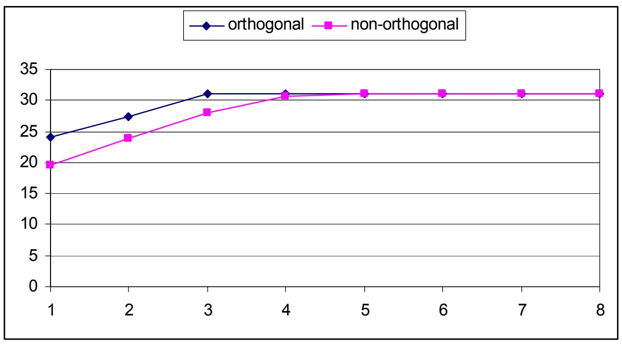
$$\overline{\rho(D-D^{(0)})}=\overline{\rho^2(D-D^{(0)})}=\cdots=0 \quad \text{(Lemma 2)}$$

Following the proof of Theorem 1, we easily obtain

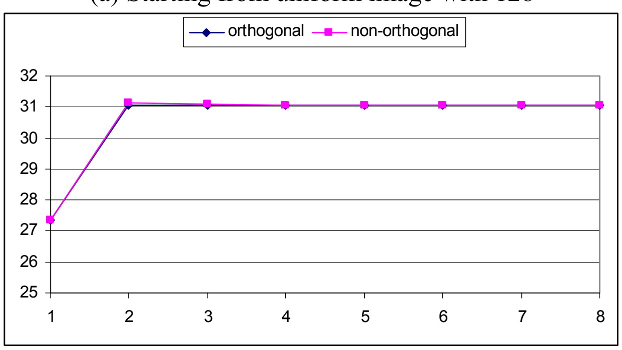
$$R-R_{new}^{(n)}=R-R_{old}^{(n)} \text{ or } R_{new}^{(n)}=R_{old}^{(n)}$$

Figure 2 shows PSNR vs. iteration times for 256x256 Lena image using the orthogonal and non-orthogonal decoding algorithm. The Lena image is encoded with $2^2\times2^2$ range block and $2^{2+1}\times2^{2+1}$ domain block (single level), and the fractal codes are obtained by full search of global domain block pool, and initial iteration image is uniform image with 128 or the range-averaged image. Fig. 2(a) shows that PSNR of the orthogonal decoding algorithm is always larger than PSNR of the non-orthogonal decoding algorithm if the initial image is the uniform image. The experimental result is in compliance with Theorem 1. Fig. 2 (b) shows the decoding algorithm produces the same PNSRs if the initial image is the range-averaged image, and the experimental result is in compliance with Theorem 2. Experimental results also shows the orthogonal decoding algorithm exactly reaches its attractor in two iterations if the initial image is the range-averaged image, and an additional iteration is needed if the initial image is uniform image with 128. The result is in compliance with Property 2 [11].

In summary, Theorem 1 provides a simple and clear proof of Property 2 [11]. Combining Theorem 1 and 2 provides another proof of the theorem presented by Moon et al [12] that the range-averaged image is the optimal initial image for the non-orthogonal decoding algorithm.



(a) Starting from uniform image with 128



(b) Starting from the range-averaged image

Figure 2: PSNR vs. iteration times using the orthogonal and non-orthogonal decoding algorithm for 256x256 Lena image

4. CONCLUSIONS

In this paper, we prove that the orthogonal decoding algorithm converges faster than the non-orthogonal algorithm. We also prove two decoding algorithms produce the same iteration series, if the initial image is the range-averaged image and the step size equals the side-length of range block.

REFERENCES

[1] M. Barnsley and A. D. Sloan, "A Better Way to Compress Images," *Byte*, vol. 13, pp.215-233, 1988.

[2] A. E. Jacquin, "Image Coding Based on a Fractal Theory of Iterated Contractive Image Transform", *IEEE Trans. on IP*, vol. 1, no.1, pp. 18-30, 1992.

[3] E. W. Jacobs, Y. Fisher, and R.D.Boss, "Image Compression: A Study of Iterated Transform Method," *Signal Processing*, vol. 29, no. 3, pp. 251-263, Dec. 1992.

[4] Y. Fisher (Editor), *Fractal Image Compression: Theory and Application*, Springer-Verlag, 1995.

[5] F. Davoine, M. Antonini, J.-M Chassery, and M. Barlaud, "Fractal image compression based on Delaunay triangulation and vector quantization" , *IEEE Trans. on IP*, vol. 5, pp. 338 –346, Feb. 1996.

[6] K. Belloulata and J. Konrad, "Fractal image compression with region-based functionality, " *IEEE Trans. on IP*, vol. 11, pp. 351-362, Apr. 2002.

[7] D. Saupe, "Fractal Image Compression Via Nearest Neighbor Search", *Conf. Proc. NATO ASI Fractal Image Encoding and Analysis*, Trondheim, July 1995.

[8] B. Bani-eqbal, "Enhancing the Speed of Fractal Image Compression," *Optical Engineering*, vol. 34, no. 6, pp.1705-1710, June 1995.

[9] C. Tong and M. Pi, "Fast Fractal Image compression using Adaptive Search" *IEEE Trans. on IP*, vol.10, no.9, pp.1269-1277, Sep. 2001.

[10] T. K. Truong, J. H. Jeng, I. S. Reed, P. C. Lee and A. Q. Li, "A Fast Encoding Algorithm for Fractal Image Compression Using the DCT Inner Product", *IEEE Trans. on IP*, vol. 9, no. 4, pp.529-535, Apr. 2000.

[11] G. E. Øien and S. Lepsøy, "A Class of Fractal Image coders with Fast Decoder Convergence," *Fractal Image Compression: Theory and Application*, Springer-Verlag, 1995 (in Chapter 8).

[12] Y. H. Moon, H. S. Kim, and J. H. Kim, "A Fast Fractal Decoding Algorithm Based on the Selection of an Initial Image", *IEEE Trans. on IP*, vol. 9, no. 5, pp.941-945, May 2000.

[13] B. Wohlberg and G. Jager, "A Review of the Fractal Image Coding Literature," *IEEE Trans. on IP*, vol. 8, no. 12, pp.1716-1729, 1999.

[14] M. Pi, A. Basu, and M. Mandal, "A new decoding algorithm based on range block mean and contrast scaling", *Proc. ICIP 2003*, Barcelona, Spain, September 14-17, 2003.