APPROXIMATION OF IMAGES BY BASIS FUNCTIONS FOR MULTIPLE REGION SEGMENTATION WITH LEVEL SETS

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ABSTRACT

Active contours and level sets provide a solid formal framework for image segmention. The problem, stated as the minimization of a functional containing terms of conformity to data and regularization, is solved by curve evolution implemented via level set partial differential equations (PDEs). The purpose of this study is to investigate approximation by basis functions as a model for image representation in segmentation by level set PDEs. This model is mathematically yielding, affords more generality than current piecewise constant and Gaussian models, and can be just as efficient as the most general piecewise smooth model. We state the problem using this model to measure conformity of segmentation to data. The resulting functional is minimized via level set evolution PDEs. Experimental results are shown to demonstrate the formulation.

1. INTRODUCTION

Image segmentation is a fundamental problem with numerous useful applications in image processing and computer vision. Active contours [1] and level sets [2] have led to a new class of algorithms which have succeeded in segmenting difficult images. Several algorithms have been developped for image segmentation [3–8]. These algorithms follow a well posed problem stated according to a variational formulation where assumptions and constraints are transparent. The solution follows the minimization by curve evolution of an objective functional containing terms of conformity to data and regularization. Curve evolution is realized via level set PDEs. A level set implementation has several well known advantages over the classical snake active contour implementation. In particular, it is numerically stable and accounts for variations in the topology of evolving curves.

Most studies on image segmentation by level set PDEs have used general formulations of the problem to concentrate on issues such as evolution equations that lead to a partition of the image domain, implementation efficiency, and application to inputs other than grey-level images such as disparity fields, motion fields, vectorial images, or images of a particular domain of application, such as medical imaging, to incorporate specific, a priori information about the data. The problem of image representation has been secondary in these studies, understandably so, although a good representation is just as important as a good formulation. The image models most often used are the piecewise constant model [4] and the Gaussian model [7]. More representative models have been used in two recent studies, namely the piecewise smooth model in a two-region level set implementation of the Mumford-Shah algorithm [9] and histograms computed over the regions of segmentation [10].

The purpose of this study is to investigate approximation by basis functions as a model for image representation, and its use in segmentation by level set PDEs. This general parametric model is most relevant for several reasons. It is mathematically yielding, affords more generality than current piecewise constant and Gaussian models, is more economical and can be just as efficient as the most general piecewise smooth model. Also, it can serve better important applications of segmentation where parametric models are efficient representations, in, for instance, region-based image reconstruction, region-based coding, and motion-based image partitioning. We state the problem using this model to measure conformity of segmentation to data. The resulting functional is minimized by a greedy algorithm via level set evolution PDEs. Examples are given to demonstrate the algorithm.

The remainder of the paper is organized as follows. In Section 2 we present a formal description of the problem for two-region segmentation. The approximation and regularization terms of the energy functional are presented and the evolution equations are derived. Section 3 presents an extension to multiple regions. In Section 4 we show experimental results that validate the approach. Section 5 contains a conclusion.

2. TWO-REGION SEGMENTATION

2.1. Formulation

For simplicity, we first formulate the two-region segmentation problem. A generalization to multiple regions will be given in Section 3.

Let $\mathbf{I} : \Omega \to \mathbb{R}$ be an image function defined on $\Omega \subset \mathbb{R}^2$. Let $\vec{\gamma} : [0, 1] \to \Omega$ be a simple closed planar curve parameterized by arc parameter $s \in [0, 1]$, $\mathbf{R}_{\vec{\gamma}}$ the region enclosed by $\vec{\gamma}$, and $\mathcal{R} = \{\mathbf{R}_1 = \mathbf{R}_{\vec{\gamma}}, \mathbf{R}_2 = \mathbf{R}^c_{\vec{\gamma}}\}$ the corresponding partition of the image domain. Finally, let $\mathbf{\Theta} = \mathrm{span}\{\theta_i(\mathbf{x})\}$ be the space spanned by basis functions $\{\theta_i(\mathbf{x})\}_{i=1,...,M}$. The goal is to determine $\vec{\gamma}$ and $\boldsymbol{\alpha}_1 = (\alpha_{11}, ..., \alpha_{1M})^T, \boldsymbol{\alpha}_2 = (\alpha_{21}, ..., \alpha_{2M})^T$ that minimize the

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following functional:

$$\mathcal{E}(\vec{\gamma}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = \frac{1}{2} \int_{\mathbf{R}_{\vec{\gamma}}} \left(\mathbf{I}(\mathbf{x}) - \sum_{j=1}^{M} \alpha_{1j} \theta_j(\mathbf{x}) \right)^2 d\mathbf{x} \quad (1)$$
$$+ \quad \frac{1}{2} \int_{\mathbf{R}^c_{\vec{\gamma}}} \left(\mathbf{I}(\mathbf{x}) - \sum_{j=1}^{M} \alpha_{2j} \theta_j(\mathbf{x}) \right)^2 d\mathbf{x} + \frac{1}{2} \lambda \oint_{\vec{\gamma}} ds$$

where λ is a positive real constant. The first two terms on the right-hand side of (1) are an approximation error measuring the segmentation conformity to data, and the third term, proportional to the length of $\vec{\gamma}$, is a regularization term for smooth region boundary.

2.2. Minimization

The necessary conditions for a minimum of (1) are:

$$\begin{aligned} \frac{\partial \mathcal{E}(\vec{\gamma}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)}{\partial \alpha_{1i}} &= 0 \quad i = 1, ..., M \\ \frac{\partial \mathcal{E}(\vec{\gamma}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)}{\partial \alpha_{2i}} &= 0 \quad i = 1, ..., M \\ \frac{\partial \mathcal{E}(\vec{\gamma}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)}{\partial \vec{\gamma}} &= 0 \end{aligned} \quad (2)$$

For $i = 1, ..., M$, we have:

$$\frac{\partial \mathcal{E}(\vec{\gamma}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)}{\partial \alpha_{1i}} = \quad (3)$$
$$\sum_{j=1}^{M} \left(\int_{\mathbf{R}_{\vec{\gamma}}} \theta_j(\mathbf{x}) \theta_i(\mathbf{x}) d\mathbf{x} \right) \alpha_{1j} - \int_{\mathbf{R}_{\vec{\gamma}}} \mathbf{I}(\mathbf{x}) \theta_i(\mathbf{x}) d\mathbf{x}$$

If we let $\mathbf{A}(\mathbf{R}_{\vec{\gamma}})$ be the $M \times M$ matrix with elements:

$$a_{ij} = \int_{\mathbf{R}_{\vec{\gamma}}} \theta_j(\mathbf{x}) \theta_i(\mathbf{x}) d\mathbf{x} \quad (4)$$

and $\mathbf{b}(\mathbf{R}_{\vec{\gamma}})$ the vector with elements:

$$b_i = \int_{\mathbf{R}_{\vec{\gamma}}} \mathbf{I}(\mathbf{x}) \theta_i(\mathbf{x}) d\mathbf{x} \quad (5)$$

then equations (3) for $i = 1, ..., M$ can be written in matrix form as follows:

$$\frac{\partial \mathcal{E}(\vec{\gamma}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)}{\partial \boldsymbol{\alpha}_1} = -\mathbf{A}(\mathbf{R}_{\vec{\gamma}}) \boldsymbol{\alpha}_1 + \mathbf{b}(\mathbf{R}_{\vec{\gamma}}) \quad (6)$$

Similarly, for $\boldsymbol{\alpha}_2$ we would obtain:

$$\frac{\partial \mathcal{E}(\vec{\gamma}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)}{\partial \boldsymbol{\alpha}_1} = -\mathbf{A}(\mathbf{R}^c_{\vec{\gamma}}) \boldsymbol{\alpha}_1 + \mathbf{b}(\mathbf{R}^c_{\vec{\gamma}}) \quad (7)$$

Therefore, the necessary conditions for a minimum of (1) with respect to $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are:

$$\begin{aligned} \mathbf{A}(\mathbf{R}_{\vec{\gamma}}) \boldsymbol{\alpha}_1 &= \mathbf{b}(\mathbf{R}_{\vec{\gamma}}) \\ \mathbf{A}(\mathbf{R}^c_{\vec{\gamma}}) \boldsymbol{\alpha}_2 &= \mathbf{b}(\mathbf{R}^c_{\vec{\gamma}}) \end{aligned} \quad (8)$$

We proceed now to derive the necessary condition for a minimum of (1) with respect to $\vec{\gamma}$. We start with the first term on the right-hand side of (1). Using the definition of $\mathbf{A}(\mathbf{R}_{\vec{\gamma}})$ and $\mathbf{b}(\mathbf{R}_{\vec{\gamma}})$ and the first equation in (8), this first term can be written as:

$$\frac{1}{2} \int_{\mathbf{R}_1} \left(\mathbf{I}(\mathbf{x}) - \sum_{j=1}^{M} \alpha_{1j} \theta_j(\mathbf{x}) \right)^2 d\mathbf{x} = \quad (9)$$
$$= \quad \frac{1}{2} \int_{\mathbf{R}_{\vec{\gamma}}} (\mathbf{I}(\mathbf{x}))^2 \, d\mathbf{x} - \frac{1}{2} \boldsymbol{\alpha}_1^T \mathbf{b}(\mathbf{R}_{\vec{\gamma}})$$

Functional derivation with respect to $\vec{\gamma}$ of the integral on the right-hand side of (9) [11], and of the second term using the fact that $\boldsymbol{\alpha}_1 = \mathbf{A}^{-1}(\mathbf{R}_{\vec{\gamma}}) \mathbf{b}(\mathbf{R}_{\vec{\gamma}})$ leads to:

$$\frac{1}{2} \frac{\partial}{\partial \vec{\gamma}} \int_{\mathbf{R}_{\vec{\gamma}}} \left(\mathbf{I}(\mathbf{x}) - \sum_{j=1}^{M} \alpha_{1j} \theta_j(\mathbf{x}) \right)^2 d\mathbf{x} = \quad (10)$$
$$= \quad \frac{1}{2} \left(\mathbf{I}(\mathbf{x}) - \sum_{j=1}^{M} \alpha_{1j} \theta_j(\mathbf{x}) \right)^2 \vec{n}(\mathbf{x})$$

where $\vec{n}$ is the external unit normal function to $\vec{\gamma}$. A similar expression can be obtained for the second term of the objective function (1). Inclusion of the functional derivative of the third term gives:

$$\frac{\partial \mathcal{E}(\vec{\gamma}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)}{\partial \vec{\gamma}} = \frac{1}{2} \left(\left(\mathbf{I}(\mathbf{x}) - \sum_{j=1}^{M} \alpha_{1j} \theta_j(\mathbf{x}) \right)^2 \right. \quad (11)$$
$$\left. - \quad \left(\mathbf{I}(\mathbf{x}) - \sum_{j=1}^{M} \alpha_{2j} \theta_j(\mathbf{x}) \right)^2 \right) \vec{n}(\mathbf{x}) + \frac{1}{2} \lambda \kappa(\mathbf{x}) \vec{n}(\mathbf{x})$$

where κ is the mean curvature function of $\vec{\gamma}$. The necessary condition for a minimum of (1) with respect to $\vec{\gamma}$ follows setting to zero the right-hand side of (11).

2.3. Algorithm

Following the necessary conditions for a minimum, we minimize (1) by a greedy algorithm which, after initialization of $\vec{\gamma}$, iterates two alternating steps, one to solve the linear systems (8) for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$, the other to perform gradient descent on $\mathcal{E}$ with respect to $\vec{\gamma}$, i.e., in the first step we solve:

$$\begin{aligned} \mathbf{A}(\mathbf{R}_{\vec{\gamma}}) \boldsymbol{\alpha}_1 &= \mathbf{b}(\mathbf{R}_{\vec{\gamma}}) \\ \mathbf{A}(\mathbf{R}^c_{\vec{\gamma}}) \boldsymbol{\alpha}_2 &= \mathbf{b}(\mathbf{R}^c_{\vec{\gamma}}) \end{aligned}$$

and in the second step we make $\vec{\gamma}$ evolve according to the descent equation:

$$\frac{d\vec{\gamma}}{dt} = -\frac{1}{2} \left(\left(\mathbf{I} - \sum_{j=1}^{M} \alpha_{1j} \theta_j \right)^2 + \left(\mathbf{I} - \sum_{j=1}^{M} \alpha_{2j} \theta_j \right)^2 - \lambda \kappa \right) \vec{n}$$

2.4. Level set implementation

Let $\vec{\gamma}$ be embedded as the zero level-set of a function $\phi : \mathbb{R}^2 \to \mathbb{R}$. One can show [2] that if the evolution of $\vec{\gamma}$ is described by the equation:

$$\frac{d\vec{\gamma}(s,t)}{dt} = F(\vec{\gamma}(s,t), t) \vec{n}(s,t) \quad (12)$$

where F is a real-valued function defined on $\mathbb{R}^2 \times \mathbb{R}^+$, then the evolution equation of function ϕ, with the convention that $\phi > 0$ inside the zero level-set, is:

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} = F(\mathbf{x}, t) \|\vec{\nabla} \phi(\mathbf{x}, t)\| \quad (13)$$

In our case, the evolution equation for function ϕ is:

$$\frac{\partial \phi(\mathbf{x},t)}{\partial t} = \frac{1}{2}\left(e_1(\mathbf{x}) - e_2(\mathbf{x}) + \lambda\kappa(\mathbf{x})\right) \|\vec{\nabla}\phi(\mathbf{x},t)\| \qquad (14)$$

where the data approximation term e_i is defined by:

$$e_i(\mathbf{x}) = \left(\mathbf{I}(\mathbf{x}) - \sum_{j=1}^{M} \alpha_{ij}\theta_j(\mathbf{x})\right)^2 \qquad (15)$$

and the curvature function κ is given in terms of the level set function by:

$$\kappa(\mathbf{x}) = \operatorname{div}\left(\frac{\nabla\phi(\mathbf{x})}{|\nabla\phi(\mathbf{x})|}\right) \qquad (16)$$

3. EXTENSION TO MULTIPLE REGIONS

Let $\vec{\gamma}_i|_{i=1,\dots,N-1}$ be a family of simple closed planar curves, their interior defining regions $\mathbf{R}_i|_{i=1,\dots,N-1}$. Region $\mathbf{R}_N$ will be formed by the intersection of the exteriors of all curves:

$$\mathbf{R}_N = \bigcap_{i=1}^{N-1} \mathbf{R}_i^c \qquad (17)$$

We follow our view of segmentation as regularized clustering of image intensity values [12] and define $N-1$ energy functionals, each involving two regions, namely, the interior of a curve and its complement:

$$\mathcal{E}_\Omega(\vec{\gamma}_i|\mathbf{I}) = \frac{1}{2}\int_{\mathbf{R}_i} e_i(\mathbf{x})\,d\mathbf{x} + \frac{1}{2}\int_{\mathbf{R}_i^c} \psi_i(\mathbf{x})\,d\mathbf{x} + \frac{1}{2}\lambda\oint_{\vec{\gamma}_i} ds$$

where e_i is defined as in (15), and $\psi_i(\mathbf{x}) = \min_{j\neq i}(e_j(\mathbf{x}))$. Segmentation of the image then results from the following set of simultaneous minimizations:

$$\widetilde{\mathbf{R}}_i = \arg\min_{\mathbf{R}_i}\ \left(\mathcal{E}_\Omega(\vec{\gamma}_i|\mathbf{I})\right), \qquad i \in [1, N-1] \qquad (19)$$

with the Nth region defined by (17). The evolution equations of curves γ_i, $i = 1, \dots, N-1$, for the minimizations (19) are given by:

$$\frac{d\vec{\gamma}_i}{dt}(\mathbf{x})|_{\mathbf{x}\in\vec{\gamma}_i} = -\left(e_i(\mathbf{x}) - \psi_i(\mathbf{x}) + \lambda\kappa_i(\mathbf{x})\right)\vec{n}_i(\mathbf{x}) \qquad (20)$$

Provided that the curves do not intersect initially, these evolution equations for the curves produce a partition of the image domain at convergence, the intensity within each region of the partition described by the coefficients of its approximation by basis functions. The level set equations corresponding to (20) are given by:

$$\frac{\partial \phi_i(\mathbf{x},t)}{\partial t} = -\left(e_i(\mathbf{x}) - \psi_i(\mathbf{x}) + \lambda\kappa_i\right)\|\vec{\nabla}\phi_i(\mathbf{x},t)\| \qquad (21)$$

To implement the level set equations, one must define extention velocities [2]. For instance, the extention velocity at a point is the velocity at the point closest to it on the evolving curve. Extention velocities can also be defined so that the level set function is at all times the distance function from the evolving curve. Both of these definitions, often implemented via narrow banding, require the initial curves intersect the regions they segment. This is important when a region has unconnected components. An alternative robust to initialization, which we use in our experiments, extends the expression of velocity on the evolving curve to the image domain [5, 13].

4. EXPERIMENTAL RESULTS

The first example, which uses a synthetic image allows us to verify the formulation and its implementation. It also illustrates the impact of using a basis functions image model rather than commonly used piecewise constant and Gaussian models. Fig. 1a shows a 3D rendering of the original image, consisting of background with planar variation of intensity, a square region also with planar variation of intensity, and another square region with intensities forming a paraboloid. Fig. 1b shows the original image with the two initial curves. Fig. 1c shows an intermediate step in the evolution of the two curves, and Fig. 1d shows the final positions of the curves. The segmentation is accurate and the parameters of the regions have been accurately estimated. To offer a point of comparison, we

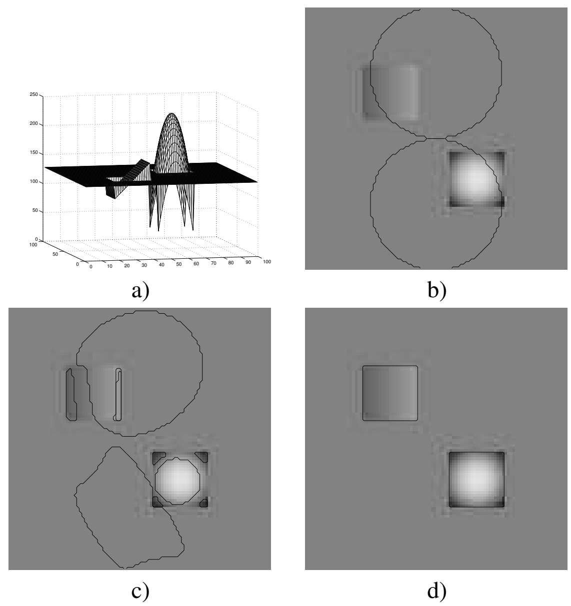
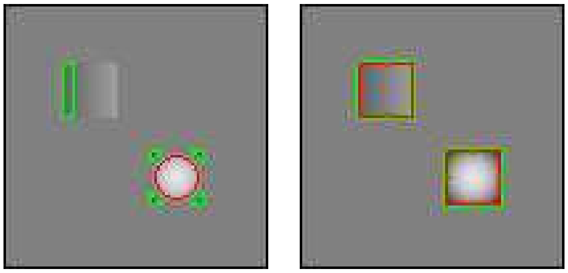


Fig. 1. Segmentation of a synthetic image: a) 3D view of intensity variations, b) original image with initialization, c) a step in evolution, d) final segmentation.

segmented the image using the piecewise constant model (Fig. 2a) and with the Gussian model (Fig. 2b). Both models failed in segmenting the image properly. With the piecewise constant model, the segmentation is erroneous and fragmented. With the Gaussian model both square regions are segmented as one region, and one region is lost.

The second example uses a real image consisting of two occluding spheres on a noisy background. Fig. 3a shows the original image with the two initial curves. Fig. 3b shows an intermediate step in the evolution of the two curves, and Fig. 3c shows the final positions of the curves. In Fig. 3d and Fig. 3e are shown the segmented regions. The segmentation is almost perfect. The estimated parameters $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ reproduced the original image with almost no error (Fig. 3f). There is no visible difference between the reconstructed image and the original one. We are currently conducting a comprehensive experimentation on the method.

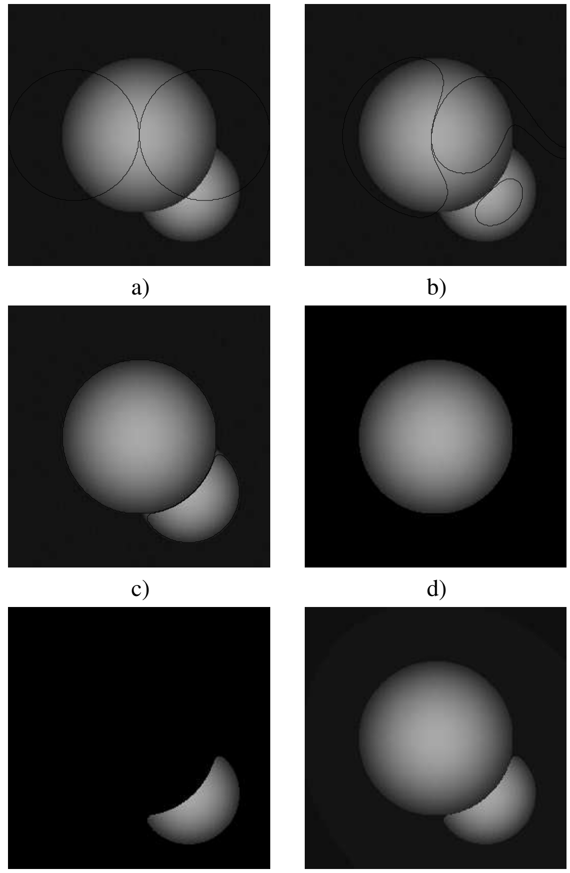


a) b)

Fig. 2. Segmentation of a synthetic image: a) piecewise constant model, b) Gaussian model

5. CONCLUSION

We investigated approximation by basis functions as a model for image representation in segmentation by level set PDEs. This model is mathematically yielding, affords more generality than current piecewise constant and Gaussian models, and can be just as efficient as the most general piecewise smooth model. We gave examples to demonstrate the resulting algorithm. Several applications of basis function approximation of images are possible, including coding-oriented segmentation and joint motion estimation and segmentation, two projects we are currently undertaking.



a) b) c) d) e) f)

Fig. 3. Segmentation of a real image: a) original image with initialization, b) a step in evolution, c) final segmentation, d) one of the segmented regions, e) the other segmented region f) reconstructed image from estimated regions and parameters.

6. REFERENCES

[1] M. Kass, A. Witkin, and D. Terzopoulos, "Snakes: Active contour models," *Intern. J. Comput. Vis.*, vol. 1, pp. 321–331, 1987.

[2] J. Sethian, *Level Set Methods and Fast Marching Methods*, Cambridge University Press, 2 edition, 1999.

[3] C. Samson, L. Blanc-Féraud, G. Aubert, and J. Zerubia, "A level set method for image classification," in *Proc. Int. Conf. Scale-Space Theories in Computer Vision*, Corfu, Greece, Sept. 1999, pp. 306–317.

[4] T. Chan and L. Vese, "An active contour model without edges," in *Proc. Int. Conf. Scale-Space Theories in Computer Vision*, Corfu, Greece, 1999, pp. 141–151.

[5] A.-R. Mansouri and J. Konrad, "Multiple motion segmentation with level sets," *IEEE Trans. Image Processing*, , no. 2, pp. 201–220, Feb. 2003.

[6] N. Paragios and R. Deriche, "Coupled geodesic active regions for image segmentation: A level set approach," in *Proc. European Conf. Computer Vision*, Dublin, Ireland, June 2000, pp. 224–240.

[7] M. Rousson and R. Deriche, "A variational framework for active and adaptative segmentation of vector valued images," in *Proc. IEEE Workshop on Motion and Video Computing*, Orlando, FL, Dec. 2002.

[8] A. Yezzi, A. Tsai, and A. Willsky, "A fully global approach to image segmentation via coupled curve evolution equations," *J. Vis. Commun. Image Represent.*, vol. 13, no. 1, pp. 195–216, Mar. 2002.

[9] A. Tsai, A. Yezzi, and A. Willsky, "A curve evolution approach to smoothing and segmentation using the mumford-shah functional," in *Proc. Conf. on Comp. Vision and Pattern Recognition*, Hilton Head, SC, USA, June 2000, vol. 1, pp. 1119–1124.

[10] G. Aubert, M. Barlaud, O. Faugeras, and S. Jehan-Besson, "Image segmentation using active contours: Calculus of variations or shape gradients?," *SIAM Journal on Applied Mathematics*, vol. 63, no. 6, pp. 2128–2154, 2003.

[11] S. Zhu and A. Yuille, "Region competition: Unifiying snakes, region growing, and bayes /MDL for multiband image segmentation," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 18, no. 9, pp. 884–900, Sept. 1996.

[12] C. Vázquez, A. Mitiche, and I. Ben Ayed, "Segmentation of vectorial images by a global curve evolution method," in *Proc. Reconnaissance des Formes et Intelligence Artificielle, RFIA-04*, Toulouse, France, Jan. 2004, to appear.

[13] A. Mitiche, R. Feghali, and A. Mansouri, "Motion tracking as spatio-temporal motion boundary detection," *Journal of Robotics and Autonomous Systems*, vol. 43, pp. 39–50, 2003.