A Para-Pseudo Inverse Based Method for Reconstruction of Filter Bank Frame-Expanded Signals from Erasures

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***Abstract*—Packet losses due to congestion or buffer overflows is a common problem in packet switched networks. Current network protocols manage this problem by retransmitting the lost packets. However, the delay due to the retransmission of the lost packets may be unacceptable for many real-time applications. Recent focus to resolve this problem is to recover the lost data from the received packets using some error control coding scheme. In this context, signal representation using frames has got attention and has been studied in [1], [2], [4], and [6]. Oversampled transforms and oversampled filter banks have been considered as joint-source channel codes and methods for reconstructing from erasures is studied in these articles. However, for oversampled filter banks, the reconstruction methods based on operating the pseudo-inverse based on the entire signal length are computationally complex and those based on reconstructing the erasures are not optimal as far as reconstruction mean square error is concerned.**

In this paper, we propose a method for reconstruction from erasures using a synthesis filter bank which functions as a pseudo-inverse. Hence, the scheme minimizes the reconstruction mean square error. Further, the method is computationally efficient, because it does not operate the pseudo-inverse corresponding to the entire signal vector. The synthesis filter bank, which obviously depends on the erasure pattern implements the pseudo-inverse at a practical computational cost. Some typical bursty erasure patterns which permit existence of a FIR synthesis filter banks are studied. The theoretical results are validated for bursty erasure patterns by simulations using image data.

EDICS: Multiple Descriptive Coding, Joint Source-Channel Coding, Oversampled Filter Banks, Frame Expansion, Pseudo-Inverse.

I. Introduction

Packet switched communication networks which transport multimedia data over delivery infrastructures offering no QoS guarantee are creating challenging problems in the area of coding. Packet losses are inevitable due to various causes such as network congestion and buffer overflows at intermediate nodes. The receiving node either asks for retransmission of the lost packets in TCP or does nothing about the lost packets in the user datagram protocol (UDP). The former approach requires delay, which is inappropriate for many real-time applications, whereas the latter approach degrades the quality of the received data severely. In recent works, DFT codes [4] and oversampled filter banks (OFBs) [1], [2], [6] have been considered for joint source-channel coding to provide robustness to packet losses in such networks.

In this paper we propose a novel reconstruction scheme for recovering signals which are represented using OFBs from erasures. In [6], algorithms based on reconstructing the erased values from the received data were proposed. These methods are not equivalent to operation of the pseudo-inverse on the received signal and are hence sub-optimal. Direct application of the pseudo-inverse is a computationally expensive operation, since it involves inverting a matrix whose dimensions are proportional to the signal length. Instead, here we propose the construction of a synthesis filter bank (SFB) which depends on the erasure pattern. The operation of this SFB on the received signal is equivalent to operating the pseudo-inverse on the received signal at a feasible computational cost.

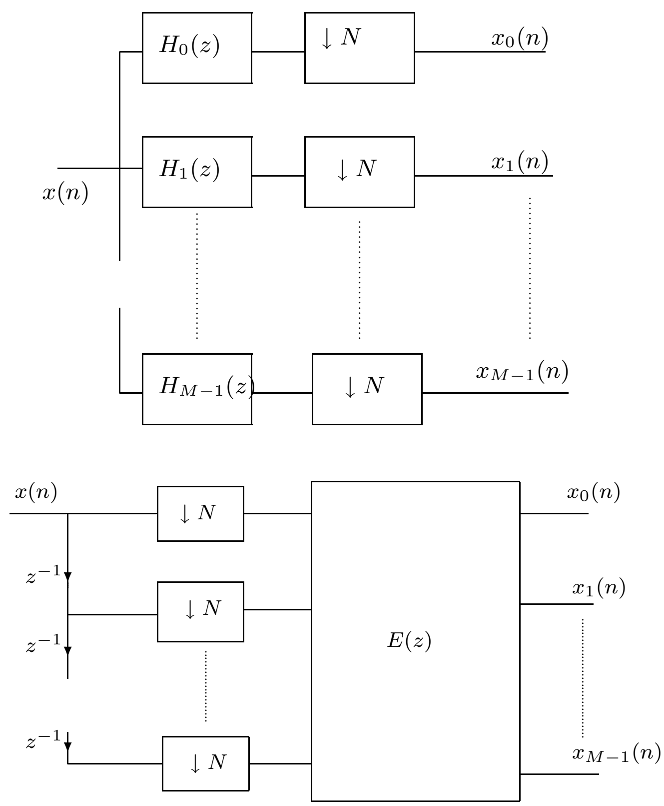

Fig. 1. (a) M-channel oversampled filter bank (top) (b) Polyphase representation (bottom).

For some erasure patterns, a FIR SFB exists. Some other erasure patterns do not permit existence of a SFB and some other patterns permit the existence of an IIR SFB. We first study some typical erasure patterns which permit the existence of a FIR SFB. For periodic erasure patterns, permitting a FIR SFB, the SFB operation is practically implementable. For aperiodic erasure patterns, the SFB is the pseudo-inverse and hence impractical to implement. For a particular case of bursty erasure pattern, we design the SFB and validate the theoretical results by simulations using image signals. From experiments, we observe that for the largest bursty erasure pattern which can be reconstructed, the reconstructed image is still perceptible.

II. M- Channel Oversampled Filter Banks and Frames

Consider the M-channel OFB, shown in Fig. 1.a which has downsamplers with factor N, where N is a positive integer less than M. In this article, we will carry out the analysis for 2-channel OFB with $N = 1$.

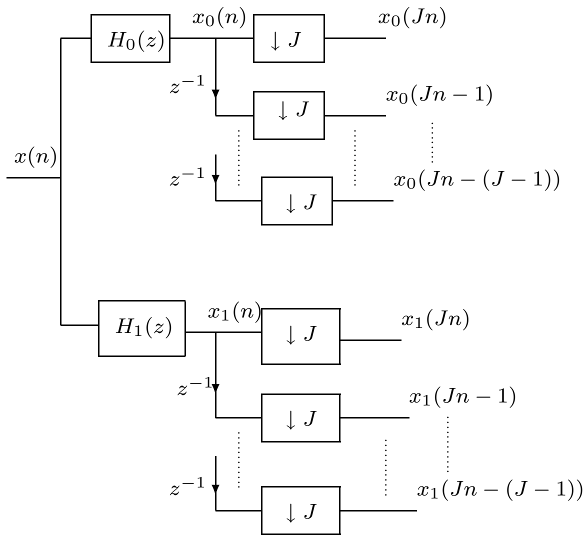

Fig. 2. J-polyphase representation of subband components.

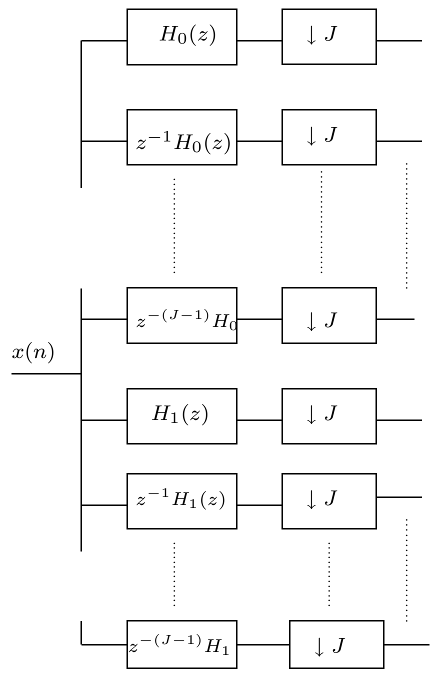

Fig. 3. $2J$-channel equivalent filter bank.

A. Para-Pseudo Inverse

Consider the M-channel OFB shown in Fig. 1.a, with $M > N$. Using the N polyphase component representation of the analysis filters, $H_k(z) = \sum_{l=0}^{N-1} z^{-l} E_{kl}(z),\ \ 0 \le k \le M-1$, the analysis filter bank can be equivalently expressed as shown in Fig. 1.b, where the matrix $\boldsymbol{E}(z)$ is given by

$$\boldsymbol{E}(z) = \begin{pmatrix} E_{00}(z) & E_{01}(z) & E_{0,N-1}(z) \\ E_{10}(z) & E_{11}(z) & E_{1,N-1}(z) \\ & \ddots & \\ E_{M-1,0}(z) & E_{M-1,1}(z) & E_{M-1,N-1}(z) \end{pmatrix}$$

It has been shown in [5] that the para-pseudo inverse of $\boldsymbol{E}(z)$ given by $\boldsymbol{R}(z) = (\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z))^{-1}\tilde{\boldsymbol{E}}(z)$ is one of the SFB. The SFB in this case is not unique. However, if the subband components are quantized, then the para-pseudo inverse is the SFB which minimizes the reconstruction mean square error. Note that invertibility of $(\tilde{\boldsymbol{E}}(z)\boldsymbol{E}(z))$ is a necessary and sufficient condition for existence of a SFB and hence perfect reconstruction in the absence of quantization.

Consider the 2-channel OFB shown in Fig. 2. The subband components $x_0(n)$ and $x_1(n)$ are quantized and the quantized sequences are interleaved into one output sequence, say $y(n)$; hence $y(2n) = x_0(n)$ and $y(2n+1) = x_1(n)$. The sequence $y(n)$ is then input to an erasure channel. At the channel output, certain samples of $y(n)$ are lost. The lost positions correspond to erasures. We consider the problem of reconstruction of $x(n)$ from erasures. First, to simplify the problem, we consider reconstruction under no quantization. Then, we consider quantization of the subbands and devise techniques which will ensure that the reconstruction mean square error is minimized.

III. Construction of Erasure-Pattern Dependent Synthesis Filter Bank

In earlier works [1], [2], [6], the problem of reconstruction from erasures when OFBs are used has been studied. In [6], the approach is to first reconstruct the erased values from the received values using the linear dependency between the erased and the received components. However, this method is sub-optimal when the subband components are quantized. This is because the methods are not equivalent to operating the pseudo-inverse and hence do not guarantee minimization of the reconstruction mean square error. As seen, these methods lead to amplification of quantization noise, especially so, if the erasure pattern is bursty in nature.

In this section, we propose a different method for signal reconstruction from erasures. We reconstruct the signal directly from the received component without resorting to reconstructing the erased values. We study conditions on existence of an appropriate SFB corresponding to the erasure pattern which will permit perfect reconstruction in the absence of quantization. Further, since this SFB also performs a pseudo-inverse operation, it is optimal in terms of minimizing the reconstruction mean square error under quantization. We carry out the analysis for orthogonal 2-channel filter bank, wherein both the filters are of the same length, say L. The analysis can be easily extended for biorthogonal M-channel filter banks, for $M > 2$.

Erasure patterns can be either periodic or aperiodic. By a periodic erasure pattern is meant that if we consider the K-block representation of $y(n)$, then the erasure positions are the same within every block. The smallest positive even integer $K = 2J$ for which this holds is called the period of the erasure pattern. For a periodic erasure pattern, consider the J-polyphase representation of the subband components as shown in Fig. 2. Consider the equivalent $2J$-channel filter bank shown in Fig. 3. Erasures correspond to loss of some of the subband components of the $2J$-channel filter bank shown in Fig. 3. Hence the effective filter bank which operates on the signal $x(n)$ to give us the erased signal is a filter bank which can be obtained from the filter bank of Fig. 3 by removing those channels which correspond to the erased values. If we have m erasures in a $2J$-block, consider the equivalent filter bank obtained by removing the channels corresponding to the erased components. This analysis filter bank (which we call as equivalent filter bank) is a $(2J-m)$-channel filter bank. If $m < J$ the equivalent filter bank is an OFB. If $m = J$, the equivalent filter bank is a critically sampled one. It is possible to reconstruct $x(n)$ if and only if a SFB exists for this $(2J-m)$-channel equivalent analysis filter bank.

If the erasure pattern is aperiodic, then the polyphase component matrix $\boldsymbol{E}(z)$ is a matrix with constant entries (not a function of z). This matrix admits a pseudo-inverse iff the rows of the equivalent transform coder constitute a frame.

In the next section we show that for specific periodic erasure patterns, the equivalent analysis filter bank admits FIR SFBs. For notational convenience, the analysis in the next section pertains to 2-channel orthogonal filter banks with filters having same length. The same analysis can be extended (with appropriate modifications) for biorthogonal filter banks with filters $h_0(n)$ and $h_1(n)$ having different lengths.

A. Study of Erasure Patterns permitting FIR Synthesis Filter Bank

Since the subband components are packetized and we experience packet losses, there is a high possibility that the erasure pattern will exhibit some periodicity. Consider periodic erasure patterns with periodicity $4L-2$. Without loss of generality, we consider erasure patterns with erasures commencing with the lowpass component. In the absence of quantization, we show that it is always possible to perfectly reconstruct the signal, (i.e. there exists a SFB) if the number of erasures in a block of $4L-2$ symbols is less than or equal to $2L-1$.

Consider the analysis filter bank with the $2L-1$ polyphase component representation of the subband components. The $4L-2$ analysis filter bank has a polyphase component matrix $\boldsymbol{E}(z)$ which is given by

$$\boldsymbol{E}(z) = \begin{pmatrix} h_0(0) & h_0(1) & \dots & 0 & \dots & 0 \\ h_1(0) & h_1(1) & \dots & 0 & \dots & 0 \\ 0 & h_0(0) & \dots & h_0(L-1) & \dots & \\ 0 & h_1(0) & \dots & h_1(L-1) & \dots & \\ & & & \ddots & & \\ 0 & 0 & \dots & h_0(1) & \dots & h_0(L-1) \\ 0 & 0 & \dots & h_1(1) & \dots & h_1(L-1) \\ z^{-1}h_0(L-1) & 0 & \dots & h_0(0) & \dots & h_0(L-2) \\ z^{-1}h_1(L-1) & 0 & \dots & h_1(0) & \dots & h_1(L-2) \\ & & & \ddots & & \\ z^{-1}h_0(1) & z^{-1}h_0(2) & \dots & & \dots & h_0(0) \\ z^{-1}h_1(1) & z^{-1}h_1(2) & \dots & & \dots & h_1(0) \end{pmatrix}.$$

Theorem 1. Any burst of length $2L-1$ corresponding to the polyphase matrix given above permits a SFB with FIR filters.

Proof. We first show that if the last $2L-1$ components are erased, the result holds.

If the last $2L-1$ components are erased, the equivalent polyphase component matrix is given by

$$\boldsymbol{E}_r = \begin{pmatrix} h_0(0) & h_0(1) & \dots & h_0(L-1) & \dots & 0 \\ h_1(0) & h_1(1) & \dots & h_1(L-1) & \dots & 0 \\ 0 & h_0(0) & \dots & h_0(L-2) & \dots & 0 \\ 0 & h_1(0) & \dots & h_1(L-2) & \dots & 0 \\ & & \ddots & & & \\ 0 & 0 & \dots & h_1(1) & \dots & 0 \\ 0 & 0 & \dots & h_0(0) & \dots & h_0(L-1) \end{pmatrix}. \quad (1)$$

We first show that the $2L-1 \times 2L-1$ matrix $\boldsymbol{E}_r$ is invertible. Note that the z-transform of any odd row, $(2k-1)$-th row, $1 \leq k \leq L$, is given by $z^{-(k-1)}H_0(z)$ and the z-transform of any even row, $2k$-th row, $1 \leq k \leq L-1$, is given by $z^{-(k-1)}H_1(z)$. If $\boldsymbol{E}_r$ is not invertible, then there exists at least one row which can be expressed as a linear combination of some of the other rows. Without loss of generality, consider that the first row can be written as a linear combination of other rows. Then, the z-transform of the first row can be written as a corresponding linear combination of the z-transform of the other rows; namely, there exists scalars a_i, $1 \leq i \leq L-1$ and b_i, $0 \leq i \leq L-2$, such that

$$H_0(z) = \sum_{i=1}^{L-1} a_i z^{-i} H_0(z) + \sum_{i=0}^{L-2} b_i z^{-i} H_1(z), \quad (2)$$

$$\implies \frac{H_0(z)}{H_1(z)} = \frac{\sum_{i=0}^{L-2} b_i z^{-i}}{1 - \sum_{i=1}^{L-1} a_i z^{-i}}, \quad (3)$$

$$\implies \frac{H_0(z)}{H_1(z)} = \frac{\sum_{i=0}^{L-2} b_i z^{-i}}{\sum_{i=0}^{L-1} a_i z^{-i}}, \text{ where } a_0 = 1. \quad (4)$$

Since $H_0(z)$ and $H_1(z)$ are relatively prime polynomials and $\frac{H_0(z)}{H_1(z)}$ is expressed as a ratio of two polynomials in z^{-1}, (4) implies that $h_0(L-1)$ equals zero, which is a contradiction [7]. Hence, $\boldsymbol{E}_r$ is invertible. Further, since $\boldsymbol{E}_r$ is not a function of z, the SFB is a FIR filter bank.

This proves the existence of FIR SFB for one specific bursty erasure pattern of length $2L-1$. For other bursty erasure patterns of length $2L-1$, the proof is on similar lines. For erasure patterns for which the equivalent polyphase component matrix is as below

$$\boldsymbol{E}_r = \begin{pmatrix} h_0(0) & h_0(1) & \dots & h_0(L-2) & h_0(L-1) & \dots & 0 \\ h_1(0) & h_1(1) & \dots & h_1(L-2) & h_1(L-1) & \dots & 0 \\ 0 & h_0(0) & \dots & h_0(L-3) & h_0(L-2) & \dots & \\ 0 & h_1(0) & \dots & h_1(L-3) & h_1(L-2) & \dots & \\ & & \ddots & & & & \\ 0 & 0 & \dots & h_0(0) & h_0(1) & \dots & 0 \\ 0 & 0 & \dots & h_1(0) & h_1(1) & \dots & 0 \\ z^{-1}h_1(1) & z^{-1}h_1(2) & \dots & \dots & 0 & \dots & h_1(0) \end{pmatrix}, \quad (5)$$

we can consider a filter bank with the highpass and the lowpass components flipped. The $\boldsymbol{E}_r$ as in (5) can then be reduced to the form as in (1) for this new filter bank. The invertibility of $\boldsymbol{E}_r$ given in (5) then follows from the invertibility of the matrix in (1).

We state a few more results without explicitly proving them.

Lemma 1. In a window of size $4N, N > L$ if the number of bursty erasures is $2N$ or more, there exists no SFB permitting perfect reconstruction.

Proof. It is easy to show that for these cases, the equivalent polyphase component matrix $\boldsymbol{E}_r(z)$ is singular.

Lemma 2. For $N = L$, there exists certain bursty erasure patterns for which the equivalent polyphase matrix $\boldsymbol{E}_r$ is non-singular and infact permits FIR SFB. This bursty pattern of length $2L$ is the one which starts with the highpass component. The proof of invertibility of $\boldsymbol{E}_r$ for this bursty pattern is on similar lines as the proof of invertibility of $\boldsymbol{E}_r$ in Theorem 1.

Since it is difficult to characterize all erasure patterns, we try to answer some of the following questions which arise quite naturally. The erasure patterns to consider are those for which the equivalent filter bank is either oversampled or critically sampled.

- If the highpass component is completely erased, then does an FIR SFB always exist?
  Consider the simple two-tap Haar Wavelet basis. The two filters $h_0(n)$ and $h_1(n)$ are given by

$$h_0(0) = \frac{1}{\sqrt{2}}, h_0(1) = \frac{1}{\sqrt{2}},$$
$$h_1(0) = \frac{1}{\sqrt{2}}, h_1(1) = -\frac{1}{\sqrt{2}}.$$

If we consider the $4L-2$ polyphase matrix for this case, it can be written as

$$\boldsymbol{E}(z)=\frac{1}{\sqrt{2}}\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ z^{-1} & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ -z^{-1} & 0 & 1 \end{array}\right).$$

If the highpass component is completely erased, then the equivalent polyphase component matrix is given by

$$\boldsymbol{E}_r(z)=\frac{1}{\sqrt{2}}\left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ z^{-1} & 0 & 1 \end{array}\right).$$

The determinant of this matrix is $\frac{1}{2\sqrt{2}}(1+z^{-1})$ and hence the SFB consists of IIR filters. Infact, perfect reconstruction in this case is not possible. All the analysis filters have a spectral null at $\omega=\pi$ and hence this frequency component in the signal is irrecoverably lost. Reconstruction using IIR filter banks will give rise to quantization noise amplification around the poles of the filter responses. The same conclusions hold if the lowpass component is completely erased.

- Do non-bursty erasure patterns (other than total loss of lowpass or highpass components) always permit FIR SFB?
  There exist non-bursty erasure patterns which do not permit FIR SFB. For the Haar wavelet filter bank, the erasure pattern which gives rise to the following equivalent polyphase component matrix

$$\boldsymbol{E}_r(z)=\frac{1}{\sqrt{2}}\left(\begin{array}{ccc} z^{-1} & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{array}\right),$$

does not permit an FIR SFB, since the determinant of $\boldsymbol{E}_r(z)$ is $\frac{1}{2\sqrt{2}}(1+z^{-1})$.

IV. Simulation Results

Simulations were performed using an orthogonal filter bank with filter lengths $L=16$. We have considered a simple image coding system wherein the image is decomposed into 7 subbands using a 2-channel wavelet transform. The lowest frequency subband is then input to the 2-channel oversampled filter bank. Hence erasure reconstruction is possible only for this subband component. This is done to offer unequal error protection since this component is more important as compared to the other subband components. The quantized 128×128 size subimages corresponding to the subband decomposition of these components are packetized such that every packet contains half of the components corresponding to the lowpass signal and half corresponding to the highpass component. For 31 consecutive erasures with equivalent analysis filter bank as in (1), we constructed a SFB which is FIR and input the received signal to this SFB. Fig. 4 shows the reconstructed image for 31 consecutive erasures, with PSNR value obtained as -26.90.

V. Conclusions

In this work, we considered the construction of SFBs to reconstruct oversampled signals from erasures. The SFB obviously depends on the erasure pattern and the set of SFBs for all possible periodic erasure patterns can be stored at the receiver. The operation of the SFB is equivalent to performing the pseudo-inverse corresponding the the equivalent

Fig. 4. Reconstructed images with tree-structured OFB: with no erasure (left), Burst of 31 erasures starting with the lowpass component (right).

frame operator and hence minimizes the reconstructed mean square error when the subband components are quantized. Some typical bursty erasure patterns which permit existence of a FIR synthesis filter banks were studied. The theoretical results were validated for bursty erasure patterns by simulations using image data. Compared to previously known techniques, this method is optimal and practically implementable due to its low computational cost.

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