PATH-BASED MORPHOLOGICAL OPENINGS

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ABSTRACT

A well-known problem in image analysis is the extraction of thin and elongated features such as edges, fractures, fibres, vessels, etc. In many cases, such features are not linear but curved rather than straight. A well-known morphological tool for the extraction of linear features is the opening by straight line segments of a certain length. In this paper we will extend this class of morphological filters with openings with narrow structuring elements which are not necessarily straight line segments, but which can be connected paths defined by an adjacency relation.

1. INTRODUCTION

Assume one is given an image in which thin and elongated structures are present which are of interest to the user. If one filters such an image with the goal to remove noise, one is likely to remove, or at least distort, such thin structures. To deal this with this problem, researchers in mathematical morphology have developed algorithms for the computation of openings and closings with line segments [1]. Such algorithms perform well as long as the features of interest are composed of *straight* lines, but tend to fail if they are curved. Another interesting family of morphological filters that may be able to deal with thin structures are the so-called connected operators, in particular the area opening and closing [2]. The drawback of this latter class, however, is that it does not distinguish between thin elongated shapes and other shapes with a similar area.

In this paper we follow the approach first suggested by Buckley and Talbot [3]. Rather than restricting ourselves to one particular direction for the structuring element, we will work with paths that are given by an adjacency relation on the image domain, and our structuring elements will be paths of a given length L. More generally, we will deal with the situation that a given number of points k, where $k \leq L$, from the path are missing. The incorporation of such a tolerance in the definition of our filter appears very useful to preserve elongated structures affected by noise.

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The paper is organised as follows. First we present a brief reminder on mathematical morphology. In Section 3 we introduce the concept of an adjacency on the image domain, and explain how this can be used to define paths. The path-based openings are then defined in Section 4, and in Section 5 we give a general decomposition of this class openings. This decomposition provides an algorithm that enables one to compute the opening in a rather efficient way. We give some simulations in Section 6. Concluding remarks are made in Section 7.

2. MATHEMATICAL MORPHOLOGY

In this section we recall some notions from mathematical morphology that will be used in the sequel. We distinguish between the space of binary images, or sets, $\mathcal{P}(E) = 2^E$, and the space of grey-scale functions $\text{Fun}(E, T) = T^E$, where E is the image domain and T is the set of grey-values. We assume that T is a partially ordered set. In the digital case, E is usually a finite rectangular window in $\mathbb{Z}^2$ and T a finite set $\{0, 1, \ldots, 2^m - 1\}$ for some m, typically $m = 8$ or $m = 16$. Note that both image spaces carry a natural partial ordering [4], namely set inclusion for $\mathcal{P}(E)$ and the pointwise partial ordering induced by T for $\text{Fun}(E, T)$. Both $\mathcal{P}(E)$ and $\text{Fun}(E, T)$ fit within the general complete lattice framework of mathematical morphology. Within this algebraic framework, two common operations are the *infimum*, denoted by '$\wedge$', and *supremum*, denoted by '$\vee$'; see [4]. Throughout this paper we assume that the reader is familiar with the meaning of these operations. If not, he or she may want to substitute 'maximum' for 'supremum' and 'minimum' for 'infimum'.

Let $A : E \to \mathcal{P}(E)$ be the location-dependent structuring element, i.e., $A(x)$ is the structuring element at x. The mapping

$$\delta(X) = \bigcup_{x \in X} A(x)\,, \tag{1}$$

is a dilation on $\mathcal{P}(E)$ in the sense that it distributes over ar-

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bitrary unions [4]. If, for example, E is a rectangular window in $\mathbb{Z}^2$, and $A \subseteq \mathbb{Z}^2$ is a structuring element, we might choose $A(x) = A_x \cap E$, for $x \in E$. In other words, $A(x)$ is obtained by translating a fixed structuring element A and restricting to points inside the window E. This choice also provides a means to handle the 'border problem' that is due to the fact that we can only dispose of that part of the image that lies inside E. The *reciprocal dilation* $\breve{\delta}$ is defined by [4]

$$y \in \breve{\delta}(\{x\}) \iff x \in \delta(\{y\}) .$$

Knowledge of a dilation on the atomic sets $\{x\}$ suffices to extend it to $\mathcal{P}(E)$ by means of (1) with $A(x) = \delta(\{x\})$.

A mapping α on $\mathcal{P}(E)$ or $\text{Fun}(E,T)$ is called an *opening* if its increasing (or monotone), anti-extensive, and idempotent. The latter means that $\alpha^2 = \alpha$. Anti-extensivity on $\mathcal{P}(E)$ means that $\alpha(X) \subseteq X$ for $X \subseteq E$, and anti-extensivity on $\text{Fun}(E,T)$ means that $\alpha(f) \leq f$ for every function $f \in \text{Fun}(E,T)$. It is well-known [4] that a supremum of openings is again an opening. This fact will be used explicitly in Section 6, where we build a general path-based opening as a supremum of path-based openings using only paths in a given direction.

3. ADJACENCY AND PATHS

Assume that the image domain E is endowed with a binary *adjacency* relation $x \mapsto y$, meaning that that there is an edge going from x to y. In general, the relation '$\mapsto$' is non-symmetric, which means that the graph given by the vertices E and the adjacency relation $\mapsto$ is a *directed graph*. If $x \mapsto y$, we call y a successor of x and x a predecessor of y. Using the adjacency relation we can define a dilation on $\mathcal{P}(E)$ by putting

$$\delta(\{x\}) = \{y \in E \mid x \mapsto y\} .$$

In other words, the dilation of a subset $X \subseteq E$ comprises all points which have a predecessor in X. These concepts are illustrated in Fig. 1. Here b_1, b_2, b_3 are successors of a and $\delta(\{a\}) = \{b_1, b_2, b_3\}$. Furthermore, a_1, a_2, a_3 are the predecessors of b and $\breve{\delta}(\{b\}) = \{a_1, a_2, a_3\}$. The L-tuple $\boldsymbol{a} = (a_1, a_2, \ldots, a_L)$ is called a *δ-path of length L* if $a_k \mapsto a_{k+1}$, or equivalently, if

$$a_{k+1} \in \delta(\{a_k\}) , \text{ for } k = 1, 2, \ldots, L-1 .$$

Note that $\boldsymbol{a} = (a_1, a_2, \ldots, a_L)$ is a δ-path of length L if and only if the reverse path $\breve{\boldsymbol{a}} = (a_L, a_{L-1}, \ldots, a_1)$ is a $\breve{\delta}$-path of length L. Given a path $\boldsymbol{a}$ in E, we denote by $\sigma(\boldsymbol{a})$ the set of its elements:

$$\sigma(a_1, a_2, \ldots, a_L) = \{a_1, a_2, \ldots, a_L\} .$$

We denote the set of all δ-paths of length L by Π_L and the set of all $\breve{\delta}$-paths of length L by $\breve{\Pi}_L$. The set of δ-paths of

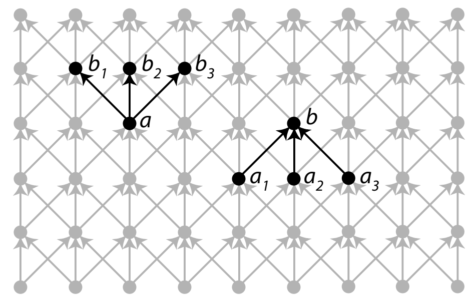


Fig. 1. *b_1, b_2, b_3 are successors of a and a_1, a_2, a_3 are the predecessors of b.*

length L contained in a subset X of E is denoted by $\Pi_L(X)$, i.e.,

$$\Pi_L(X) = \{\boldsymbol{a} \in \Pi_L \mid \sigma(\boldsymbol{a}) \subseteq X\} ,$$

and the $\breve{\delta}$-paths of length L in X by $\breve{\Pi}_L(X)$. Furthermore, we define $\Pi_L^k(X)$ as the collection of length-L paths in E which contain at most k points outside X:

$$\Pi_L^k(X) = \{\boldsymbol{a} \in \Pi_L \mid |\sigma(\boldsymbol{a}) \cap X^{\mathrm{c}}| \leq k\} .$$

Note that this definition only makes sense for $0 \leq k \leq L$, and that

$$\Pi_L(X) = \Pi_L^0(X) \subseteq \Pi_L^1(X) \subseteq \cdots \subseteq \Pi_L^L(X) = \Pi_L .$$

A path in $\Pi_L^k(X)$ is called an *incomplete path*.

4. PATH-BASED OPENINGS

In this and the following section, all definitions will be given for the binary image space $\mathcal{P}(E)$. But the results can be generalised to the space of grey-scale images $\text{Fun}(E,T)$ by means of the thresholding theorem for flat morphological operators [4, Chapter 11].

Define the operator α_L^k on $\mathcal{P}(E)$ by

$$\alpha_L^k(X) = \bigcup \{\sigma(\boldsymbol{a}) \cap X \mid \boldsymbol{a} \in \Pi_L^k(X)\} .$$

Thus, $\alpha_L^k(X)$ contains all points in X that lie on incomplete paths of length L with at most k points outside X. It is easy to verify that α_L^k is an opening on $\mathcal{P}(E)$, and we call it the *incomplete path-opening*. It is obvious that

$$\alpha_L^0 \leq \alpha_L^1 \leq \cdots \leq \alpha_L^L ,$$

and that

$$\alpha_L^L(X) = \{x \in X \mid \Lambda(x) \geq L\} ,$$

where $\Lambda(x)$ is length of the longest path in E that contains x. In most cases $\Lambda(x)$ is constant on E. In Fig. 2 we depict

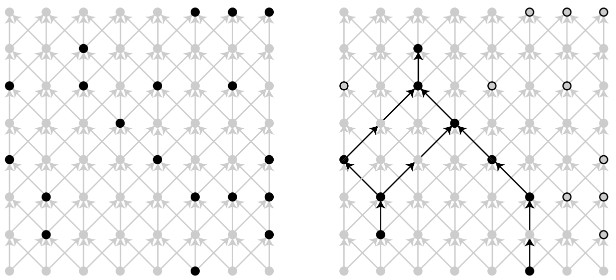


Fig. 2. *A set X (left) and its opening $\alpha_6^1(X)$ (right). The grey points with thick boundaries have been discarded by the opening. Note that $\Pi_6^1(X)$ contains four paths in this case, two of which substantially overlap each other.*

the opening α_6^1 of a set X.

The opening α_L^0, henceforth denoted by α_L, extracts the δ-paths of length L contained in X. It is evident that the family $\{\alpha_L\}$ forms a morphological granulometry in the sense that $\alpha_L \leq \alpha_{L-1}$, which implies that

$$\alpha_K \alpha_L = \alpha_L \alpha_K = \alpha_L \text{ if } L \geq K\,.$$

5. DECOMPOSITION

In this section we give a decomposition of the openings α_L^k which provides also an algorithm that allows efficient computation of the openings.

Define the operator

$$\psi_L^k(X) = \{a_1 \mid \boldsymbol{a} \in \Pi_L^k(X)\}\,,\ 0 \leq k \leq L\,.$$

Obviously

$$\psi_L^0(X) = F_L \tag{2}$$

where $F_L \subseteq E$ contains all points in E which are the begin-point of a path with length L in E. Henceforth we use the convention that $\psi_L^k \equiv \emptyset$ if $k < 0$.

We will now express ψ_{L+1}^{k+1} in terms of ψ_L^{k+1} and ψ_L^k. Observe that $x \in \psi_{L+1}^{k+1}(X)$ if there exists $\boldsymbol{a} = (a_1, \ldots, a_L)$ such that $(x, a_1, \ldots, a_L) \in \Pi_{L+1}$ and either $x \in X$ and $\boldsymbol{a} \in \psi_L^{k+1}(X)$ or $\boldsymbol{a} \in \psi_L^k(X)$. We have shown that

$$\psi_{L+1}^{k+1} = (\mathsf{id} \wedge \breve{\delta}\psi_L^{k+1}) \vee \breve{\delta}\psi_L^k\,. \tag{3}$$

With the recursive formula in (3) and the initialisation in (2), we are able to compute each one of the operators ψ_L^k.

Now consider a point $x \in \alpha_L^k(X)$, where $0 \leq k \leq L - 1$. Thus $x \in X$ and there is a path $\boldsymbol{a} \in \Pi_L$ with $x \in \sigma(\boldsymbol{a})$ such that $|\sigma(\boldsymbol{a}) \cap X^c| \leq k$. Assume that $a_l = x$. Now $\boldsymbol{a}$ is the concatenation [1] of the sequences $\boldsymbol{b} = (a_1, \ldots, a_{l-1}, x)$ and $\boldsymbol{c} = (x, a_{l+1}, \ldots, a_L)$. Define $j = |\sigma(\boldsymbol{b}) \cap X^c|$ which implies $|\sigma(\boldsymbol{c}) \cap X^c| \leq k - j$. We conclude that

$$x \in \breve{\psi}_l^j(X) \cap \psi_{L-l+1}^{k-j}(X)\,.$$

Since the length of $\boldsymbol{b}$ is l and $x \in X$ we have $0 \leq j \leq l - 1$. Similarly since the length of $\boldsymbol{c}$ is $L + 1 - l$ we have $0 \leq k - j \leq L - l$. Together these imply that $0 \leq j \leq k$ and

$$j + 1 \leq l \leq L + j - k$$

and we conclude that

$$\alpha_L^k = \mathsf{id} \wedge \bigvee_{j=0}^{k} \bigvee_{l=j+1}^{L+j-k} \left(\breve{\psi}_l^j \wedge \psi_{L-l+1}^{k-j}\right) \tag{4}$$

for $0 \leq k \leq L - 1$.

The parameter k can be regarded as a kind of *tolerance parameter*. In practice, k will be (very) small, say 0,1 or 2.

6. SIMULATIONS

So far we have restricted ourselves to an image domain E with a fixed adjacency relation. This adjacency relation is used to build a family of path-based openings α_L^k. In Figures 1-2 the adjacency yields a northward directed grid, and henceforth we will denote the corresponding family of path-based openings by $[\alpha\!\uparrow]_L^k$, rather than α_L^k. In a similar fashion, we can endow E with an eastward, a north-eastward, and a north-westward adjacency, resulting in the grids depicted in Fig. 3. These give rise to path-based openings

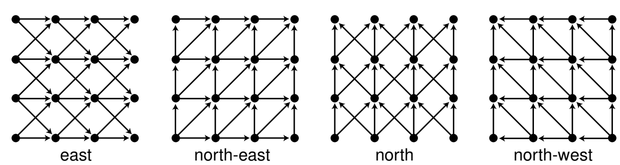


Fig. 3. *Four different adjacencies.*

$[\alpha\!\rightarrow]_L^k$, $[\alpha\!\nearrow]_L^k$, $[\alpha\!\nwarrow]_L^k$, respectively. The supremum of these openings

$$\alpha_L^k = [\alpha\!\rightarrow]_L^k \vee [\alpha\!\nearrow]_L^k \vee [\alpha\!\uparrow]_L^k \vee [\alpha\!\nwarrow]_L^k$$

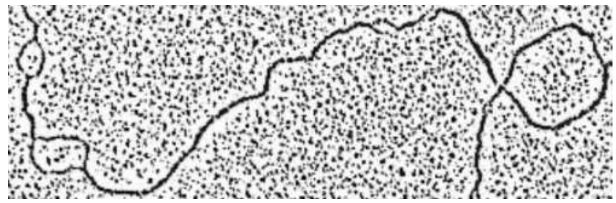
is also an opening [4]. In the binary case $\alpha_L^k(X)$ is the union of all points in X that lie on a path of length L in either of the 4 directions, which contains at most k points outside X.

In Figure 4 we show the effect of a path closing rather than an opening because of the better contrast in the printing process; note however that the same conclusions would apply for the opening[2]. Fig. 4(a) is the original 500×160 image of DNA (the long thin structure) observed in a scanning

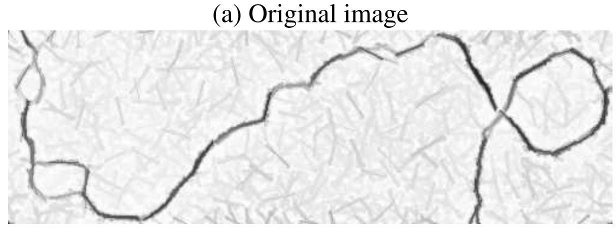
[1] In such a concatenation, the overlap point x is taken only once.

[2] Recall that the closing of an image is the same as the opening of its negative.

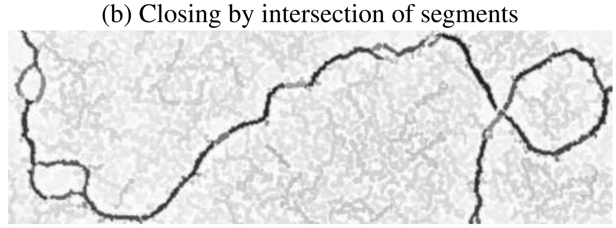
electron microscope. The objective is to separate the DNA from the noisy background, and we use two types of closings as a pre-processing filtering. Fig. 4(b) is the result of applying a closing resulting from the minimum (infimum) of 44 different structural closings [4] where the structuring element is a line segment with length 23 pixels, each in a different direction, approximately uniformly oriented (subject to the digital grid). As can be seen the background is mostly filtered out by this closing but so are various parts of the DNA. Fig. 4(c) is the path closing with path length $L = 33$ and $k = 0$ resulting by taking the minimum of the four closings as explained above. As can be seen, the path closing better preserves the shape of the object of interest than the closing with line segments. Fig. 4(d) is the incomplete path closing with $L = 33$ and $k = 2$. Even more of the DNA is preserved while more background noise is present also, especially close to the DNA strand.



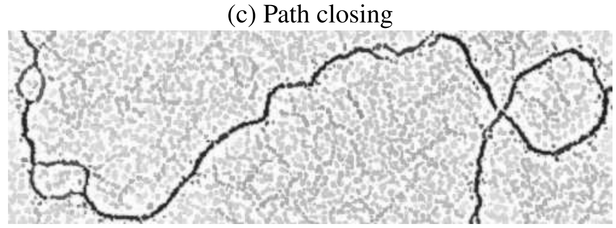
(a) Original image



(b) Closing by intersection of segments



(c) Path closing



(d) Incomplete path closing ($k = 2$)

Fig. 4. *Top to bottom: Original image, closing by intersection of line segments, and complete and incomplete path closings.*

7. CONCLUSIONS AND FUTURE WORK

In this paper we have introduced a new family of morphological openings. These openings are tailor-made to preserve thin and elongated structures which are not necessarily straight but are allowed to be curved, following one or more given adjacency relations. We have given a recursive decomposition of these openings which enables their efficient computation. Using a duality argument, one can use the same techniques to compute the corresponding morphological closing.

8. REFERENCES

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