UNIQUENESS OF BLUR MEASURE

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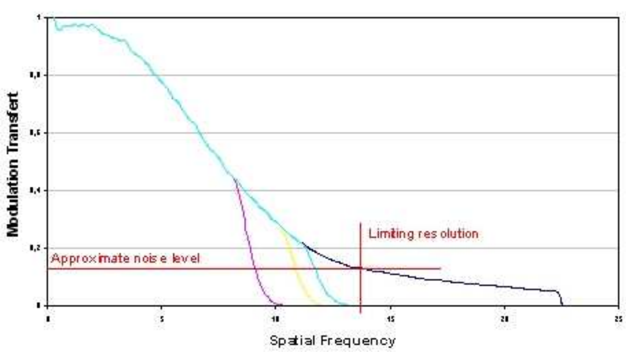
ABSTRACT

After discussing usual approaches to measuring blur, we show theoretically that there is essentially a unique way to quantify blur by a single number and we confirm the usefulness of that measure by some experiment on a natural image.

1. INTRODUCTION

The blurring produced by an imaging chain is a key component of its evaluation. It is therefore important to be able to quantify it (i) by a single number; which is (ii) related to the perceived level of blur; (iii) additive in the sense that the blur of the imaging chain should be the sum of the blurs introduced by each component. Having a single number is necessary to compare different imaging chains. The additivity allows one to understand, e.g., which component of the imaging chain is responsible for most of the blur, and so to optimize an imaging chain, say under cost constraints.



line/mm assuming sensor 24x36 mm^2

Figure 1. MTF's differing only for large ω.

The two following measures are widely used. The first one is the **Modulation Transfer Function** or MTF (see, e.g., [5]):

$$\mathrm{MTF}(\omega) = \frac{\int |\hat{f}_0(\omega\cos\theta, \omega\sin\theta)|\, d\theta}{\int |\hat{f}(\omega\cos\theta, \omega\sin\theta)|\, d\theta}$$

where $\hat{f}_0$ is the Fourier transform of the true image and $\hat{f}$ is that of the observed image. A perfect imaging chain would have $\mathrm{MTF}(\omega) \equiv 1$, but real imaging chains attenuate high frequencies so that $\mathrm{MTF}(\omega) \to 0$ as $\omega \to \infty$.

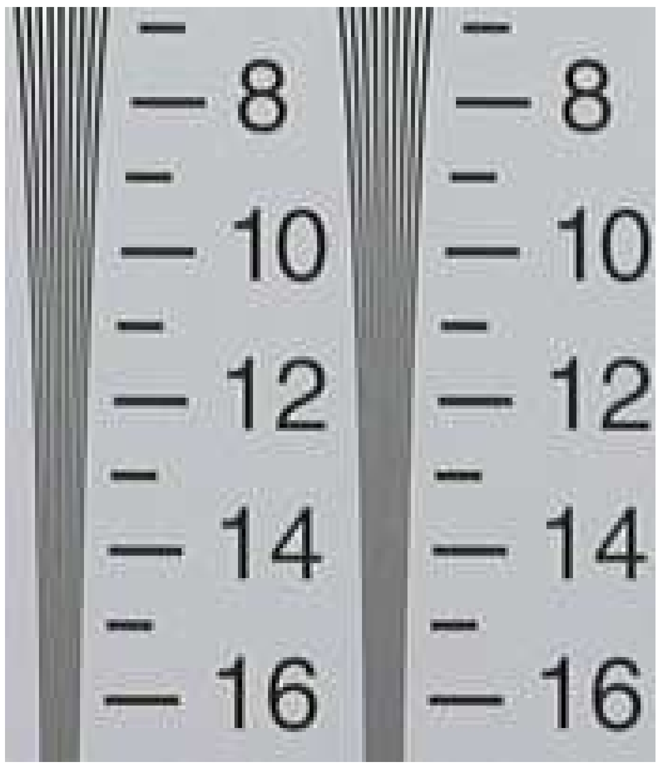


Figure 2. The different MTFs of figure 2 applied to the same image.

The MTF certainly contains all relevant information. Also, the MTF of the concatenation of chains is the product of their MTFs. However it is a whole function where one would like a single number. The MTF clearly contains information irrelevant to blur quantification. For instance, one

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can modify the tail of a MTF with very little perceptual impact, see Figures 1 and 2. Contrarily to a widespread belief (see, e.g., [4]), blur is primarily **not** related to high frequency attenenuation and we shall confirm this both by our theoretical analysis and by experiment on a natural image.
Remark. **Spot diagrams** [5] much used in optics are theoretically very close to MTFs so the above comments apply to them.

The second well-known measure with widespread use (for instance in the evaluation of cameras) is the **limiting resolution** (see, e.g., [5]), that is, one looks for the smallest visible details or, in other terms, for the highest spatial frequency visible in the produced image. This is a single number with a clear meaning. However, it is not at all additive and, in line with the fact discussed above that preservation of high frequencies is not so relevant on a perceptual level, Figure 3 shows that the same limiting resolution can occur in imaging chains having very different perceptual blurring.

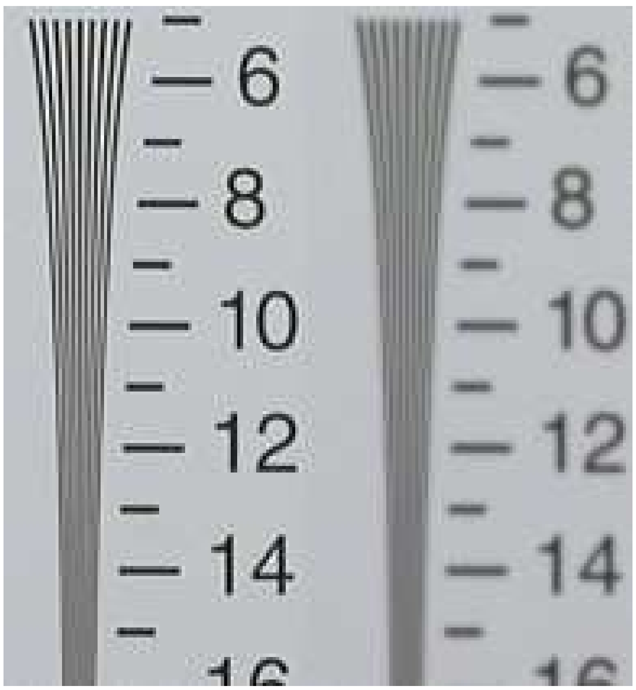

Figure 3. Different perceptual blurrings with the same limiting resolution (ISO-12233 chart, detail).

We therefore feel that there is a need of a good way to quantify blur for the evaluation of image quantity.

We begin by explaining that blur measures of imaging chains must derive from blur measures of images and then gives a list of expected properties and see that it must coincides with the variance for Gaussian blur (section 2.2). We then check that the variance is indeed a blur measure (section 2.3) before proving that it is the unique one (section 2.4). Finally we present some experiments (section 3) and conclude (section 4).

2. AXIOMATICS OF BLUR MEASURES

An imaging chain in this paper is represented by a continuous, linear, translation-invariant operator on images, i.e., it is of the form $f \mapsto K * f$ for some kernel K. The kernels are assumed to be L^2 nonnegative function K over the plane with finite second order moments and the normalization $\int K\,dxdy = 1$ and centering $\int xK\,dxdy = \int yK\,dxdy = 0$. The set of kernels will be denoted by $\mathcal{K}$.

We denote the Gaussian kernels by

$$G_{\sigma_{xx}^2,\sigma_{yy}^2}(x,y) := \frac{1}{2\pi\sigma_{xx}\sigma_{yy}} e^{-x^2/2\sigma_{xx}^2 - y^2/2\sigma_{yy}^2}.$$

We denote $G_{\sigma^2,\sigma^2} =: G_{\sigma^2}$.

A **blur measure** will be any real function $\mathcal{B}(K)$, $K \in \mathcal{K}$, with the following properties:

- (i) **Additivity** for each $g \in \mathcal{K}$, for all $f \in \mathcal{K}$
  $\mathcal{B}(f * g) = \mathcal{B}(f) + \mathcal{B}(g)$;
- (ii) **Scaling** there is a positive function ϕ such that for each $f \in \mathcal{K}$, $\lambda > 0$, writing $f_\lambda(x,y) := \lambda f(\lambda x, \lambda y)$:
  $\mathcal{B}(f_\lambda) = \phi(\lambda)\mathcal{B}(f)$;
- (iii) **Normalization** $\mathcal{B}(G_1) = 1$;
- (iv) **Stability** $\mathcal{B}$ is continuous w.r.t. the norm[1]:
  $\|f\|_{\mathcal{K}} := \iint |f|^2 + (x^2+y^2)|f|\,dxdy$;
- (v) **Invariance** if H is an isometry of the plane, then
  $\mathcal{B}(f \circ H) = \mathcal{B}(f)$.

2.1. Blur contribution of gaussian kernels

As $G_{\alpha^2+\beta^2} = G_{\alpha^2} * G_{\beta^2}$, we have by the additivity (i)

$$m \cdot \mathcal{B}(G_{(n/m)\sigma^2}) = n \cdot \mathcal{B}(G_{\sigma^2})$$

for arbitrary integers $n, m \geq 1$. Using the continuity property (iv) and the normalization (iii) , we obtain:

$$\mathcal{B}(G_{\sigma^2}) = \sigma^2.$$

On the other hand, $\mathcal{B}(G_{\alpha^2/2,\beta^2/2}) = \mathcal{B}(G_{\beta^2/2,\alpha^2/2})$ by property (v), hence:

$$\begin{aligned}
\mathcal{B}(G_{\alpha^2,\beta^2}) &= \mathcal{B}(G_{\alpha^2/2,\beta^2/2} * G_{\alpha^2/2,\beta^2/2}) \\
&= \mathcal{B}(G_{\alpha^2/2,\beta^2/2}) + \mathcal{B}(G_{\alpha^2/2,\beta^2/2}) \\
&= \mathcal{B}(G_{\alpha^2/2,\beta^2/2}) + \mathcal{B}(G_{\beta^2/2,\alpha^2/2}) \\
&= \mathcal{B}(G_{\alpha^2/2,\beta^2/2} * G_{\beta^2/2,\alpha^2/2}) = \mathcal{B}(G_{(\alpha^2+\beta^2)/2}).
\end{aligned}$$

Finally:

$$\mathcal{B}(G_{\sigma_{xx}^2,\sigma_{yy}^2}) = \frac{1}{2}(\sigma_{xx}^2 + \sigma_{yy}^2). \qquad (1)$$

2.2. A solution

By eq. (1), the blur measure of a Gaussian kernel is equal to its variance. Hence an obvious candidate for the blur measure is just the variance (not of f, but of the position w.r.t. the distribution f):

$$\mathcal{B}_*(f) = \frac{1}{2}\iint (x^2+y^2) f(x,y)\,dxdy. \qquad (2)$$

Recall that the image is normalized, so the general formula is one half of:

$$\int (x^2+y^2)\frac{f}{Z}\,dxdy - \left(\int x\frac{f}{Z}\,dxdy\right)^2 - \left(\int y\frac{f}{Z}\,dxdy\right)^2$$

with $Z = \int f\,dxdy$.

Remark. $\mathcal{B}_*(f)$ is homogeneous to a surface.

One can check that $\mathcal{B}_*(f)$ is indeed a blur measure. Properties (ii)-(v) are immediate. The additivity property (i) follows from the formula [6]:

$$\int f(x).(g*h)(x)\,dx = \iint f(x+y)g(x)h(y)\,dxdy.$$

Remark. The variance $\mathcal{B}_*$ is **not** continuous w.r.t. the L^2-norm. In fact our norm is the weakest possible. One can wonder whether it is *too* weak. In fact the above proof shows that the blur measure must be homogeneous of degree 2 and using this fact one can extend our result to much **stronger topologies** including the usual one on rapidly decreasing C^∞ functions [6]. Thus our result does not seem to depend too much on the more technical continuity condition (iv).

2.3. Uniqueness

In this section $\mathcal{B}$ is an arbitrary blur measure. We are going to show that $\mathcal{B}(f) = \mathcal{B}_*(f)$ the variance w.r.t. f.

Let $\hat{f} = \mathrm{TF}(f)$ be the Fourier transform of an image $f \in \mathcal{K}$. Let $\hat{I}$ be $\{\mathrm{TF}(f) : f \in \mathcal{K}\}$. The functions $\hat{f} \in \hat{I}$ are L^2, twice differentiable at $(0,0)$ and admit the following finite norm:

$$\|\hat{f}\|_{\hat{I}} := \|f\|_{\mathcal{K}} = \iint |\hat{f}|^2\,dxdy - \frac{\partial^2 \hat{f}}{\partial x^2}(0,0) - \frac{\partial^2 \hat{f}}{\partial y^2}(0,0)$$

Observe that $-\frac{\partial^2 \hat{f}}{\partial x^2}(0,0) = \iint x^2 f(x,y)\,dxdy$ and similarly for the other second order derivatives. For $\hat{f} \in \hat{I}$, we define: $\hat{B}(\hat{f}) = \mathcal{B}(\mathrm{TF}^{-1}(\hat{f}))$, $\sigma_{xx}^2 := \frac{\partial^2 \hat{f}}{\partial x^2}(0,0)$ and similarly for σ_{xy}^2 and σ_{yy}^2.

The value of the blur for Gaussian kernels implies that scaling property (ii) can only hold with $\phi(\lambda) = \lambda^{-2}$. Hence:

$$\hat{B}(\widehat{f_{n^{1/2}}}) = \frac{1}{n}\hat{B}(\hat{f}). \qquad (3)$$

As $\hat{f}.\hat{g} = \widehat{f*g}$, the additivity property (i) gives, for any integer $n \geq 0$, $\hat{B}(\hat{f}^n) = n\hat{B}(\hat{f})$. Together with (3), this gives:

$$\hat{B}(\hat{f}) = \hat{B}\left(\left(\widehat{f_{n^{1/2}}}\right)^n\right).$$

Claim. If $\hat{g}(\omega_x,\omega_y) = e^{-\sigma_{xx}^2\omega_x^2/2 - \sigma_{xy}^2\omega_x\omega_y - \sigma_{yy}^2\omega_y^2/2}$, then

$$\|(\widehat{f_{n^{1/2}}})^n - \hat{g}\|_{\hat{I}} \to 0 \qquad \text{as } n \to \infty.$$

Remark. The proof of this claim will be essentially that of the Central Limit Theorem [7].

Before proving the claim, we show that it implies that $\mathcal{B}(f) = \mathcal{B}_*(f)$. The $\|\cdot\|_{\hat{I}}$-continuity of $\hat{B}$ implies $\hat{B}(\hat{f}) = \hat{B}(\hat{g})$. Hence $\hat{B}(\hat{f})$ depends only on the matrix M of the second order derivatives at the origin of $\hat{f}$. Using the invariance by isometry of $\mathcal{B}$ (property (v) in the requirements), we see that $\hat{B}(\hat{f})$ can in fact only depend on the two eigenvalues of M. Comparing with the Gaussian case, eq. (1), we see that $\hat{B}(\hat{f})$ must in fact be their average. Hence

$$\mathcal{B}(f) = \frac{1}{2}\iint (x^2+y^2) f(x)\,dxdy$$

as announced. It remains to prove the claim. To begin with,

$$\hat{F}_n(x,y) := \widehat{f_{n^{1/2}}}(x,y)^n = \hat{f}(x/n^{1/2}, y/n^{1/2})^n.$$

Thus, $\frac{\partial^2 \hat{F}}{\partial x^2}(0,0) = \frac{\partial^2 \hat{f}}{\partial x^2}(0,0) = \frac{\partial^2 \hat{g}}{\partial x^2}(0,0)$ and similarly for the other second order derivatives. Hence it is enough to bound $\|\hat{F}_n - \hat{g}\|_{L^2}$. Let $\epsilon > 0$ be arbitrarily small. As $\log\hat{f}$ is twice differentiable at $(0,0)$, for $\delta := n^{-1/4}$ small enough, for $x^2+y^2 < n\delta^2$, $|n\log\hat{f}(x/n^{1/2}, y/n^{1/2}) - \sigma_{xx}^2 x^2 - ...| < \epsilon\delta^2$. Therefore, Δ_t being the disk of radius $t\sqrt{n}$,

$$\begin{aligned}\iint_{\Delta_\delta} |\hat{F}_n - \hat{g}|^2\,dxdy &\leq \iint_{\Delta_\delta} |\hat{g}|^2 (e^{\epsilon\delta^2} - 1)\,dxdy \\ &\leq \pi n\delta^2 \cdot \mathrm{const} \cdot \epsilon\delta^2 \leq \mathrm{const}\cdot\epsilon.\end{aligned}$$

The rest of the L^2 norm is:

$$\begin{aligned}\iint_{\mathbf{R}^2\setminus\Delta_\delta} |\hat{F}_n - \hat{g}|^2\,dxdy &\leq \iint_{\mathbf{R}^2\setminus\Delta_\delta} |\hat{F}_n|^2\,dxdy + \\ &+ \iint_{\mathbf{R}^2\setminus\Delta_\delta} |\hat{g}|^2\,dxdy.\end{aligned}$$

On the complement of Δ_δ, $x^2+y^2 \geq n^{1/2}$. Hence, the second term is of the order $e^{-\mathrm{const}\cdot n^{1/2}} \leq \mathrm{const}\cdot\epsilon$.

Let $\delta_1 > 0$, $C_1 < \infty$ and $\eta_1 > 0$ be such that for $x^2+y^2 < \delta_1^2$, $|\hat{f}(x,y)| \leq C_1 e^{-\eta_1(x^2+y^2)}$. Now, $|\hat{f}|$ has a strict global maximum equal to 1 at $(0,0)$ as it is the Fourier transform of a nonnegative function with integral normalized to 1. Hence, $\sup_{\mathbf{R}^2\setminus\Delta_{\delta_1}} |\hat{F}_n|^{1/n} =: \kappa_1 < 1$. Thus, the first integral above is

$$\iint_{\Delta_{\delta_1}\setminus\Delta_\delta} |\hat{F}_n|^2 \, dxdy + \iint_{\mathbf{R}^2\setminus\Delta_{\delta_1}} |\hat{F}_n|^2 \, dxdy$$
$$\leq \pi\delta_1^2 n \cdot C_1 e^{-\eta_1\sqrt{n}} + n\kappa_1^n \iint_{\mathbf{R}^2} |\hat{f}(X,Y)|^2 \, dXdY$$

where $(X,Y) = (x,y)/n^{1/2}$. This tends to zero as $n \to \infty$, proving the claim.

3. EXPERIMENTS

A striking implication of our result, is that blur is a low frequency phenomenon. To check this, we have taken a natural image and i) decreased the energy at low frequencies (Figure 4); ii) removed the high frequencies (Figure 5). It is the tampering with **low frequencies** which affects the perceptual blur[2] as our analysis suggested.

We have computed directly the evolution of the variance of position in several images after repeated convolution with blurring kernels. We have observed that numerical round-off errors can affect adversely these direct estimations, calling for more sophisticated procedures. On the other hand, when the kernel f has been estimated from other sources, it is easy to measure the blur by computing $\frac{1}{2}\left(\frac{\partial^2 \log \hat{f}}{\partial x^2} + \frac{\partial^2 \log \hat{f}}{\partial y^2}\right)$ at $(x,y) = (0,0)$. We note that to be able to compare different imaging chains it is necessary to *normalize* the blur measure by declaring, for instance, that the image has a standard size, e.g., $24 \times 36 mm^2$: one therefore multiplies the blur measure in squared pixels by $864/N$, if N is the number of pixels in the output of the imaging chain. One can also normalize with respect to viewing condition, if these are given, by convering the blur measure from a surface in pixels to a solid angle.

Figure 4. A natural image with low frequencies decreased.

4. CONCLUSION

In this paper we showed that once one writes down a minimum list of reasonable mathematical properties that a blur measure should possess, there is a **unique** theoretical solution: the *variance* of the positions in the image. One might be surprised to see blur as a low frequency phenomenon but this conclusion cannot be avoided if the blur measure is to be a measure of the size of the spot diagram, say a function of some moments of the corresponding distribution.

Figure 5. A natural image with high frequencies removed

This variance had of course been considered in many related problems, e.g., blind deconvolution or depth estimates from blur (see for instance [2, 3] and the references therein) — or showed to be strongly correlated with other seemingly distant blur assessments both objective and subjective [4]. But our result is independent of any assumption on the blur kernel (which we do not need to estimate and which maybe far from Gaussian) and, out of its abstract axiomatization, has given rise to tools which have shown their practical usefulness[3]

5. REFERENCES

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[2] Of course, removing the high frequencies creates many artifacts.

[3] Notably through the DxO Analyzer software which is used by a number of retailers and photograph magazines.