ESTIMATION OF MULTIPLE ORIENTATIONS IN MULTI-DIMENSIONAL SIGNALS

Cicero Mota[1,2], *Til Aach*[2], *Ingo Stuke*[2], *and Erhardt Barth*[1]

[1]Institute for Neuro- and Bioinformatics
[2]Institute for Signal Processing
University of Lübeck
Ratzeburger Allee 160, 23538 Lübeck, Germany

ABSTRACT

We present a solution to the general problem of estimating multiple orientations in multi-dimensional signals. The solution is divided in a linear part that provides the *mixed-orientation space* (MOS), and a non-linear part that gives the actual orientation spaces. We show that the angle between two overlaid orientations is an invariant that can be derived from the MOS without solving the non-linear part, and that all other invariants are generated by this angle. Results obtained for synthetic images illustrate that the above invariant is a useful image feature for various applications such as pattern recognition and texture segmentation.

1. INTRODUCTION

Estimation of local orientation has many applications in multidimensional signal processing, e.g., edge detection [4, 15], motion estimation [12, 10, 8, 3], pattern analysis [4, 9, 3] non-isotropic filtering for image enhancement [1, 6], and directional interpolation [7]. Multiple orientations appear in non-opaque imagery like X-ray, Ultrasound, Magnetic Resonance, Computed Tomography, and Positron Emission Tomography and in opaque image structures like corners, crossings, bifurcations, and textures. We present a general approach for the estimation of multiple orientations. This concept is suitable for the estimation of multiple orientations in signals of any dimension, although images, three- and four-dimensional volumes are the most common occurrences of such signals. The approach is based on our solutions for the estimation of transparent motions and multiple orientations in images as well as new insights into the structures of signals suffering from *generalized aperture problems* [11]. The concept generalizes and unifies previous approaches for the estimation of single orientation in gray-scale signals [3, 6], orientation and motion in color images [15, 5] and multiple transparent motion in gray-scale images [12]. Unlike single orientation estimation, our approach to multiple orientations is carried out in two steps: a linear step where the MOS is estimated followed by a non-linear step where the *orientation spaces are separated.* When the orientation spaces are unidimensional (lines), which is always the case for 2D images, the MOS can be represented by a tensor whose components are the so-called *mixed-orientation parameters* (MOP). We show that all scalar invariants of this tensor are functions of the angle between the two lines. This angle can be computed directly from the MOP without performing the separation of the orientations. The set of all such angles across the signal is invariant to rotations, translations and scaling of the signal and, therefore, provides a feature space appropriate for tasks like pattern recognition, tracking, image registration, etc.

Work supported by the Deutsche Forschungsgemeinschaft under Ba 1176/7-2.

2. THEORETICAL RESULTS

Let a mapping $\mathbf{f} : \mathbb{R}^p \to \mathbb{R}^q$ denote a multivariate, multispectral signal. The most important cases are 'color' images and movies, i.e., $p = 2, 3$ and $q = 1, 2, 3, 4$ but the actual values of p, q play no important role in the following analysis.

2.1. Single orientation

We say that $\mathbf{f}(\mathbf{x})$ is oriented in the (open) region Ω if there is a subspace $E \subset \mathbb{R}^p$ such that

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) = \mathbf{f}(\mathbf{x}) \quad \text{for all } \mathbf{x}, \mathbf{v};\ \mathbf{x}, \mathbf{x}+\mathbf{v} \in \Omega,\ \mathbf{v} \in E. \quad (1)$$

This concept is related to the concepts of linear symmetry [3] and intrinsic dimension [16]. It is equivalent to say that the intrinsic dimension of $\mathbf{f}(\mathbf{x})$ restricted to Ω is $p - \dim(E)$. The goal of orientation estimation is to obtain E. We follow a differential approach introduced in [3] for the estimation of an orientation hyperplane for gray-scale signals. This approach is based on the observation that Eq. (1) implies (and is indeed equivalent to)

$$\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{v}} = \mathbf{0} \quad \text{for all } \mathbf{x} \in \Omega \text{ and } \mathbf{v} \in E. \quad (2)$$

Eq. (2), in turn, is equivalent to

$$\int_{\Omega} \left| \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{v}} \right|^2 \mathrm{d}\Omega = 0. \tag{3}$$

To evaluate Eq. (3), let $\mathbf{x} = (x_1, \dots, x_p)^T$ and $\mathbf{v} = (v_1, \dots, v_p)^T$; thus, $\partial \mathbf{f}(\mathbf{x})/\partial \mathbf{v} = \sum_1^p v_j \mathbf{f}_{x_j}(\mathbf{x})$, where $\mathbf{f}_{x_j}(\mathbf{x})$ is a short notation for $\partial \mathbf{f}(\mathbf{x})/\partial x_j$. Then,

$$\left| \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{v}} \right|^2 = \sum_{i,j} v_i v_j \mathbf{f}_{x_i}(\mathbf{x}) \cdot \mathbf{f}_{x_j}(\mathbf{x}). \tag{4}$$

Eq. (3) can be rewritten as

$$\mathbf{v}^T \mathbf{J} \mathbf{v} = 0, \tag{5}$$

where $\mathbf{J} = \mathbf{J}(\mathbf{f})$ is given by

$$\mathbf{J} = \int \begin{bmatrix} \mathbf{f}_{x_1}(\mathbf{x}) \cdot \mathbf{f}_{x_1}(\mathbf{x}) & \cdots & \mathbf{f}_{x_1}(\mathbf{x}) \cdot \mathbf{f}_{x_p}(\mathbf{x}) \\ \vdots & & \vdots \\ \mathbf{f}_{x_p}(\mathbf{x}) \cdot \mathbf{f}_{x_1}(\mathbf{x}) & \cdots & \mathbf{f}_{x_p}(\mathbf{x}) \cdot \mathbf{f}_{x_p}(\mathbf{x}) \end{bmatrix} \mathrm{d}\Omega. \tag{6}$$

Alternatively, if $\mathbf{f} = (f_1, \dots, f_q)^T$, we have $\partial \mathbf{f}(\mathbf{x})/\partial \mathbf{v} = (\nabla f_1 \cdot \mathbf{v}, \dots, \nabla f_n \cdot \mathbf{v})^T$, and consequently,

$$\left| \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{v}} \right|^2 = \mathbf{v}^T \big(\sum_1^n \nabla f_k \nabla f_k^T \big) \mathbf{v} \tag{7}$$

which implies (see [15] for a similar result obtained for color images)

$$\mathbf{J}(\mathbf{f}) = \sum_1^q \mathbf{J}(f_k). \tag{8}$$

Since $\mathbf{J}$ is symmetric and non-negative, Eq. (5) is equivalent to

$$\mathbf{J}\mathbf{v} = \mathbf{0} \tag{9}$$

which means that E is the null-eigenspace of $\mathbf{J}$. In practice, due to noise and other possible distortions, E is estimated as the subspace spanning the set of solutions of

$$\begin{aligned} &\min\ \mathbf{v}^T \mathbf{J} \mathbf{v} \\ &\text{s.t. } |\mathbf{v}|^2 = 1, \end{aligned} \tag{10}$$

i.e., E is estimated as the eigenspace associated to the smallest eigenvalues of $\mathbf{J}$. Confidence for the estimation can thus be derived from the eigenvalues (or, equivalently, scalar invariants) of $\mathbf{J}$ [12].

2.2. Multiple orientations

We say that $\mathbf{f}(\mathbf{x})$ is multiple oriented in Ω if it is the additive superposition of single-oriented signals, i. e.,

$$\mathbf{f}(\mathbf{x}) = \sum_1^N \mathbf{g}_n(\mathbf{x}) \tag{11}$$

and each $\mathbf{g}_n$ is single oriented. Note, however, that such a decomposition is not unique: $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{u} \cdot \mathbf{x}, \mathbf{v} \cdot \mathbf{x}) = \mathbf{g}_1(\mathbf{u} \cdot \mathbf{x}) + \mathbf{g}_2(\mathbf{v} \cdot \mathbf{x})$ can be seen as a single signal oriented along a line or, alternatively, as the superposition of two signals oriented along planes.

Let E_n be the orientation space of $\mathbf{g}_n$, then the goal of multiple orientation estimation is to obtain $E_1, \dots, E_N$ given $\mathbf{f}(\mathbf{x})$. Note that neither N nor $\mathbf{g}_1, \dots, \mathbf{g}_N$ are known in advance. We restrict the following to $N = 2$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors such that $\partial \mathbf{g}_1(\mathbf{x})/\partial \mathbf{u} = \partial \mathbf{g}_2(\mathbf{x})/\partial \mathbf{v} = \mathbf{0}$, we have

$$\frac{\partial^2 \mathbf{f}(\mathbf{x})}{\partial \mathbf{u} \partial \mathbf{v}} = \mathbf{0} \tag{12}$$

which expands to

$$\sum_{i \le j} c_{ij} \mathbf{f}_{x_i x_j}(\mathbf{x}) = \mathbf{0}, \tag{13}$$

where

$$c_{ij} = \begin{cases} u_j v_j & \text{if } i = j \\ u_i v_j + u_j v_i & \text{otherwise.} \end{cases} \tag{14}$$

Proceeding in a way similar to the estimation of single orientations, we take the square and integrate Eq. (13), to obtain

$$\mathbf{c}^T \mathbf{J}_2 \mathbf{c} = 0, \tag{15}$$

where $\mathbf{J}_2 = \mathbf{J_2}(\mathbf{f})$ is given by

$$\mathbf{J}_2 = \int_{\Omega} \Big[\mathbf{f}_{x_i x_j}(\mathbf{x}) \cdot \mathbf{f}_{x_{i'} x_{j'}}(\mathbf{x}) \Big]_{i \le j,\, i' \le j'} \mathrm{d}\Omega \tag{16}$$

and $\mathbf{c} = \mathbf{c}(\mathbf{u}, \mathbf{v}) = (c_{ij})_{i \le j}^T$ is the MOP vector. As in the single orientation case, Eq. (15) is solved in a least square sense as the subspace $\mathcal{S}_2$ spanning the solutions of

$$\begin{aligned} &\min\ \mathbf{s}^T \mathbf{J}_2 \mathbf{s} \\ &\text{s.t. } |\mathbf{s}|^2 = 1. \end{aligned} \tag{17}$$

Note that the unknown $\mathbf{c}$ in Eq. (15) has changed to $\mathbf{s} = (s_{ij})_{i \le j}^T$ in the above equation because not all solutions of Eq. (17) can account for Eq. (14). $\mathcal{S}_2$ is the subspace spanning the set $\mathcal{C} = \{\mathbf{c}(\mathbf{u}, \mathbf{v});\ \mathbf{u} \in E_1, \mathbf{v} \in E_2\}$. We call $\mathcal{S}_2$ the MOS. It provides a implicit although complete representation of the orientations of the components of $\mathbf{f}$: E_1, E_2. Note that Eq. (8) is also valid for $\mathbf{J}_2$.

2.3. The mixed-orientation space

Since $\mathcal{C}$ is not a subspace, we need a rule to decide which elements of $\mathcal{S}_2$ belong to $\mathcal{C}$. For this, we note that the entries of $\mathbf{C}_s = \frac{1}{2}(\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u})$ and $\mathbf{c}$ are the same up to the factor $1/2$ in the off-diagonal entries of $\mathbf{C}_s$. The following applies to $\mathbf{C}_s$:

Proposition 1 *Let* $\mathbf{u}, \mathbf{v}$ *be unit vectors and* θ *the angle between them. Then* $\mathbf{C}_s = \frac{1}{2}(\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u})$ *has only two non-zero eigenvalues* $\lambda_1 = \cos^2 \frac{\theta}{2}$ *and* $\lambda_2 = -\sin^2 \frac{\theta}{2}$ *with corresponding eigenvectors* $\mathbf{u} + \mathbf{v}$ *and* $\mathbf{u} - \mathbf{v}$.

If $\mathbf{A}$ is a matrix and $p(\lambda) = |\lambda \mathbf{I} - \mathbf{A}| = \sum_{i=0}^{p} S_i \lambda^{p-i}$ is its characteristic polynomial, the coefficients S_i are the basic invariants (symmetric polynomials of the eigenvalues) of $\mathbf{A}$ and analytically expressed in terms of the entries in $\mathbf{A}$. Prop. 1 allows to conclude that for $\mathbf{C}_s$

$$S_1 = \cos\theta, \quad S_2 = -\frac{\sin^2\theta}{4}, \quad S_i = 0 \quad \text{for } i > 2. \tag{18}$$

2.3.1. Characterization of the mixed-orientation parameters

To decide which elements $\mathbf{s} \in \mathcal{S}_2$ can be written in the form $\mathbf{s} = \mathbf{c}(\mathbf{u}, \mathbf{v})$, we look at Eq. (15) in the Fourier domain where it becomes

$$\mathbf{u} \cdot \boldsymbol{\omega}\, \mathbf{v} \cdot \boldsymbol{\omega}\, \mathbf{F}(\boldsymbol{\omega}) = \mathbf{0}. \tag{19}$$

$\mathbf{F} = (F_1, \ldots, F_q)^T$, F_k is the Fourier representation of f_k and $\boldsymbol{\omega}$ is the transformed variable. Eq. (19) shows that $\mathbf{F}(\boldsymbol{\omega})$ is supported by two subspaces of the Fourier domain whose orthogonal complements are E_1 and E_2. The spectral decomposition of matrices allows to conclude that $\mathbf{s} = \mathbf{c}(\mathbf{u}, \mathbf{v})$ if and only if the matrix $\mathbf{S} = (s'_{ij})$ with

$$s'_{ij} = \begin{cases} s_{jj} & \text{if } i = j \\ \frac{s_{ij}}{2} & \text{if } i < j \\ \frac{s_{ji}}{2} & \text{if } i > j \end{cases} \tag{20}$$

has basic invariants satisfying

$$S_2 < 0, \quad S_i = 0 \quad \text{for } i > 2. \tag{21}$$

2.3.2. Separation of the mixed-motion parameters

The separation of the orientation spaces E_1, E_2 given $\mathcal{S}_2$ involves solving the equation

$$\mathbf{c}(\mathbf{u}, \mathbf{v}) = \mathbf{s} \tag{22}$$

for $\mathbf{u}, \mathbf{v}$ given $\mathbf{s} \in \mathcal{S}_2$. The vector $\mathbf{s}$ is restricted by Eq. (21). Separation of the orientation spaces seems a non-trivial problem but for the case $\dim E_k = 1$. In the absence of aperture problems, the separation of two transparent motion vectors by eigenvalue analysis has been proposed by Shizawa and Mase [13]. An analytical method for the separation of N transparent motion vectors has been presented in [12]. For the important case of images, the only non-trivial value for $\dim E_k$ is *one* and different approaches can be used for the separation of the orientations parameters [2, 14]. The case $\dim E_k = 1$ is simpler because it implies $\dim \mathcal{S}_2 = 1$, and thus Eq. (21) is either true or false for every element in $\mathcal{S}_2$. Thus, for two unidimensional orientations, $\mathbf{J}_2$ has a non-singular zero eigenvalue and its null eigenvector must satisfy Eq. (21). Under the previous conditions, Prop. 1 applies and the orientations can be recovered from the relationships

$$\begin{aligned} \mathbf{u} &= \cos\frac{\theta}{2}\,\mathbf{e}_1 + \sin\frac{\theta}{2}\mathbf{e}_2 \\ \mathbf{v} &= \cos\frac{\theta}{2}\mathbf{e}_1 - \sin\frac{\theta}{2}\mathbf{e}_2 \end{aligned} \tag{23}$$

where $\mathbf{e}_n$ is the unit eigenvector for λ_n, $n = 1, 2$. Due to Prop. 1, $\cos\frac{\theta}{2} = \sqrt{\lambda_1}$ and $\sin\frac{\theta}{2} = \sqrt{-\lambda_2}$. However, the eigenvalues analysis can be performed analytically by the use of Eq. (18) and basic trigonometric identities.

As said before, $\dim \mathcal{S}_N > 1$ never occurs in 2D-images, where except for flat components one can expect only unidimensional orientations spaces. Nevertheless, the problem already appears in the processing of transparencies in movies where two components can suffer from the *aperture problem* in the same local neighborhood. A categorization of the oriented patterns appearing in the components of $\mathbf{f}(\mathbf{x})$ based in the rank of $\mathbf{J}_N$ has appeared in [11].

2.3.3. Invariants of the mixed-orientation parameters

Being an eigenvector of $\mathbf{J}_2$, $\mathbf{c}$ is an invariant of $\mathbf{f}(\mathbf{x})$. From the relationship between $\mathbf{c}$ and $\mathbf{C}_s$, it is a tensor invariant. Thus, it is interesting to see what scalar invariants of $\mathbf{f}(\mathbf{x})$ are coded in $\mathbf{c}$. Prop. 1 has this answer. All scalar invariants of $\mathbf{c}$ are generated by the angle θ between the orientations. In fact, any scalar invariant can be written as a function $g(\lambda_1, \ldots, \lambda_p)$ of the eigenvalues of $\mathbf{C}_s$ and consequently as a function $g(\theta)$.

3. RESULTS

We implemented an hierarchical algorithm that first tries to fit a single orientation model and next a double orientation model to the signal. Model selection is based on the invariants of $\mathbf{J}_N$ according to [12]: $H > \epsilon_N$ and $\sqrt[M]{K} < c_N \sqrt[M-1]{S}$, where $M = \text{order}\, \mathbf{J}_N$, $H = \frac{S_1}{M}$, $S = \frac{S_{M-1}}{M}$ and $K = S_M$. Fig. 1 shows a synthetic double-oriented noisy image with orientations differing by $\theta = 45°$ (half-left) and $\theta = 50°$ (half-right). Mean and standard deviation of the estimated angles are $45.08°$, $0.35°$ (50.82, 0.34) for the left half (right half) of the image. In the input image, the angle increases continuously from $45°$ to $50°$ in the transition area between both regions. Parameters for this example are: region Ω of 9×9 pixels, $\epsilon_N = 0.01$, $c_1 = 0.2$, $c_2 = 0.4$.

To deal with opaque structures, such as corners, where the additive superposition model does not exactly hold, we apply a two step procedure: firstly, pixels where both single and double orientation models are rejected are marked;

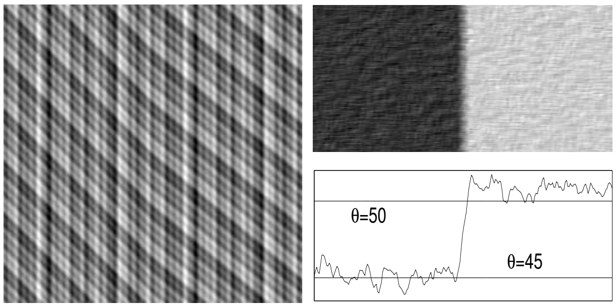

Fig. 1. Left: synthetic double oriented image with additive white Gaussian noise; orientations differ by $\theta = 45°$ (left) and $\theta = 50°$ (right); SNR is 25 dB. Upper-right: estimated values of θ shown as gray-scale intensities. Lower-right: horizontal cut through the upper-right image.

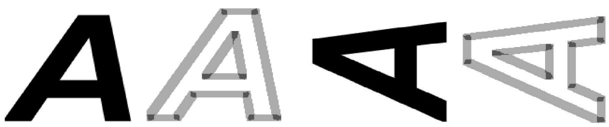

Fig. 2. Middle-left: Segmentation of 'A' showing regions with single (light gray) and double (dark gray) orientations. Right: same for 'A' rotated by $90°$. The mean and standard-deviation for the estimation of the three different angles are $64.6, 0.19$; $31.3, 0.19$; $83.9, 0.21$ for both images; true values are $60, 38, 82$.

next, the size of the integration region Ω is increased, marked pixels are excluded from it, and the algorithm is applied to the marked pixels only. This procedure excludes points that do not meet the additive superposition assumption and, thus, allows for the estimation of two orientations at opaque structures. Fig. 2 shows results for an A-shaped pattern. Parameters are: region Ω of size 7×7 and 21×21 for the first and second step, respectively; $\epsilon = 0.01$; $c_1 = 0.1$; and $c_2 = 0.2$. In all examples, derivatives were simply estimated by finite differences after smoothing the image with a 3×3 Gaussian kernel with standard deviation of 1 pixel.

4. CONCLUSIONS

We have shown that the angle between two local orientations in an image is a basic scalar invariant that can be derived from the MOS. The MOS is obtained based on a linearization of the non-linear constraint equations that define the multiple orientations. In addition, we have shown that all other invariants of the signal that could be derived from the MOS, can be expressed in terms of this angle. Results were derived within a more general framework that is not restricted to the case of two-dimensional images and one-dimensional orientation spaces. We have presented analytical solutions for the separations of the orientation spaces of dimension one. The constraints for estimating two oriented subspaces with any codimension were also derived but the numerical problem of solving the non-linear separation of the MOS remains open.

The results that we present are meant to illustrate the potential for applications. The local angle between lines is a useful feature for pattern recognition since it contains more information than corner or junction detectors. The texture example illustrates that even small and hardly visible differences in the local angle, as they can occur in a fabric production, can be reliably segmented by our algorithm.

5. REFERENCES

[1] T. Aach and D. Kunz. Anisotropic spectral magnitude estimation filters for noise reduction and image enhancement. In *Proc. IEEE Int. Conf. Image Processing*, pp. 335–8, 1996.

[2] T. Aach, I. Stuke, C. Mota, and E. Barth. Estimation of multiple local orientations in image signals. In *Proc. IEEE Int. Conf. Acoustics, Speech and Signal Processing*, vol.III, pp. 553–6, 2004.

[3] J. Bigün, G. H. Granlund, and J. Wiklund. Multidimensional orientation estimation with application to texture analysis and optical flow. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 13(8):775–90, 1991.

[4] W. T. Freeman and E. H. Adelson. The design and use of steerable filters. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 13(9):891–906, 1991.

[5] P. Golland and A. M. Bruckstein. Motion from color. *Computer Vision and Image Understanding*, 68(3):346–62, 1997.

[6] G. H. Granlund and H. Knutsson. *Signal Processing for Computer Vision*. Kluwer, 1995.

[7] J. Hladůvka and E. Gröller. Direction-driven shape-based interpolation of volume data. In *Proc. Vision, Modeling, and Visualization*, pp. 113–20, 2001. Aka GmbH.

[8] B. Jähne. Image sequence analysis in environmental and live sciences. In *Pattern Recognition, 25th DAGM Symp.*, vol. 2781 of *LNCS*, pp. 608–17, 2003. Springer.

[9] M. Kass and W. Witkin. Analyzing oriented patterns. *Computer Vision, Graphics, and Image Processing*, 37:362–85, 1987.

[10] R. Mester. Some steps towards a unified motion estimation procedure. In *Proc. 45th IEEE MidWest Symposium on Circuits and Systems*, 2002.

[11] C. Mota, M. Dorr, I. Stuke, and E. Barth. Categorization of Transparent-Motion Patterns Using the Projective Plane. In *International Journal of Computer & Information Science*, 5(2), 2004.

[12] C. Mota, I. Stuke, and E. Barth. Analytic solutions for multiple motions. In *Proc. IEEE Int. Conf. Image Processing*, vol. II, pp. 917–20, 2001.

[13] M. Shizawa and K. Mase. Simultaneous multiple optical flow estimation. In *IEEE Conf. Computer Vision and Pattern Recognition*, vol. I, pp. 274–8, 1990.

[14] I. Stuke, T. Aach, E. Barth, and C. Mota. Analysing superimposed oriented patterns. In *Proc. IEEE Southwest Symp. Image Analysis and Interpretation*, pp. 133–7, 2004.

[15] S. Di Zenzo. A note on the gradient of a multi-image. *Computer Vision, Graphics, and Image Processing*, 33:116–25, 1986.

[16] C. Zetzsche and E. Barth. Fundamental limits of linear filters in the visual processing of two-dimensional signals. *Vision Research*, 30:1111–7, 1990.