ACCELERATION OF CSRBF-BASED IMAGE RECONSTRUCTION BY WAVELET DOMAIN PRECONDITIONING

Luis A. Diago, Masaki Kitago and Ichiro Hagiwara

Department of Mechanical Sciences and Engineering,
Graduate School of Science and Engineering,
Tokyo Institute of Technology.
2-12-1, O-okayama, Meguroku, Tokyo, 152-8852, Japan.
Tel: +81-3-5734-3556, Fax: +81-3-5734-3556
E-mail: diago@mech.titech.ac.jp

ABSTRACT

Radial Basis Functions are popular for interpolating scattered data. In this context, the solution of the System of Linear Algebraic Equations (SLAE) is the most time-consuming operation. Techniques fail with large point sets consisting of more than a few thousands points when direct methods and global support are used.

In this paper we propose the use of wavelets to accelerate the solution of the SLAE that arise from the formulation of the problem of image interpolation from scattered data by means of Compactly-Supported Radial Basis Functions. Examples demonstrate the superiority of the solution in the wavelet domain using preconditioned iterative methods.

1. INTRODUCTION

Radial Basis Functions (RBF) are popular for interpolating scattered data. The theory of RBF for interpolating scattered data is introduced in [1]. Further, Wendland proposed the use of Compactly-Supported Radial Basis Functions (CSRBF) [2]. As it is clear from these papers, the solution of the System of Linear Algebraic Equations (SLAE) is the most time-consuming operation. Techniques fail with large point sets consisting of more than a few thousands points when direct methods and global support are used.

Iterative method has been proposed for RBF interpolation with large data sets [3]. Other approach [4] uses algorithmic advances involving domain decomposition and fast multipole methods to reduce the computational cost to $\mathcal{O}(N \log N)$, where N is the number of points to be interpolated. Recently, the main attention is centered on partition of unity (PU) methods [5].

As it is shown in [3, 4, 5] iterative methods are still far from be useful in all situations, so direct methods are still considered as a candidate for SLAE solution in order to get reliable results. We intend to overcome these problems of iterative methods improving its convergence behavior for the problem of image interpolation from scattered data using CSRBF.

A method for the reconstruction of band limited images and the approximation of arbitrary images from nonuniform sampling values have been developed [6]. Other works show that high-fidelity image reconstruction is possible from selected set of regular and irregular samples using splines models [7, 8].

A computationally efficient multiresolution method is developed in [9] for solving the SLAE using uniform B-splines. Further, multiwavelet-like basis functions have been used as generalized piecewise-linear multiple generators [10]. The case of two basis function is studied in details, but to use the method with other basis functions the determinant of the SLAE matrix should become a delay in order to be implemented as a FIR filtering process. The essential assumption for this algorithm to work is the invertibility of an approximate band matrix which is not always ensured.

Instead of using uniform B-Splines, we propose the use of wavelets to accelerate the solution of SLAE that arise from the formulation of the problem of image reconstruction from nonuniform samples by means of CSRBF. Examples of image interpolation from scattered data by means of CSRBF demonstrate the superiority of the solution in the wavelet domain using iterative method.

In section 2 we shortly introduce RBF, and image reconstruction by means of CSRBF. We propose our method in section 3. Results with different preconditioners in spacial domain are compared with results in wavelet domain in section 4. Finally, we conclude and discuss future works in section 5.

This work was supported by the Japan New Energy and Industrial Technology Development Organization (NEDO) under the Grant No.028115

2. THEORY AND APPLICATION

2.1. Radial basis functions

An RBF representation of a signal is a function of the form

$$s(\mathbf{x}) = p(\mathbf{x}) + \sum_{i=1}^{N} \lambda_i \phi(\|\mathbf{x} - \mathbf{x}_i\|), \tag{1}$$

where

- s is the radial basis function representation,
- p is a low degree polynomial,
- λ_i's are the RBF coefficients,
- ϕ is a radially symmetric real valued function called basis function and
- $\mathbf{x}_i$'s are the RBF centers.

The representation consists of a weighted sum of a radially symmetric basis function ϕ located at the centers $\mathbf{x}_i$ and a low degree polynomial p. Given a set of N points $\mathbf{x}_i$ and values f_i, the process of finding an interpolating RBF, s, such that,

$$s(\mathbf{x}_i) = f_i, \quad i = 1, 2, \ldots N \tag{2}$$

is called fitting. The fitted RBF is defined by the λ_i, the coefficients of the basis function in the summation, together with the coefficients of the polynomial term $p(\mathbf{x})$. The symbol $\|\cdot\|$ denotes the euclidean norm of a vector.

If we let $\{p_1, \ldots, p_l\}$ be a monomial basis for polynomials of the degree of p, and $\mathbf{c} = (c_1, \ldots, c_l)^{\mathrm{T}}$ be the coefficients that give $p(\mathbf{x})$ in terms of this basis, then the interpolation conditions (2), can be rewritten in matrix form as a system of linear equations,

$$\begin{pmatrix} \mathbf{\Phi} & \mathbf{P} \\ \mathbf{P}^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}, \tag{3}$$

where

$$\begin{aligned} \Phi_{i,j} &= \phi(\|\mathbf{x}_i - \mathbf{x}_j\|), & i &= 1, \ldots, N, \\ & & j &= 1, \ldots, N, \\ P_{i,j} &= p_j(\mathbf{x}_i), & i &= 1, \ldots, N, \\ & & j &= 1, \ldots, l \end{aligned}$$

Solving the linear system (3) determines $\boldsymbol{\lambda}$ and $\mathbf{c}$, and hence $s(\mathbf{x})$. In this paper, the problem of image interpolation from scattered data obtained by using feature extraction techniques is solved using CSRBF.

2.2. Image reconstruction by CSRBF

For the selection of scattered data to be interpolated, wavelet transform is used [11]. After thresholding details coefficients, coordinates of scattered data are selected from approximation and thresholded details coefficients. Position of approximation coefficients are used to properly reconstruct the parts of the image where there are no nearby edge points. Coordinates of all scattered data points are normalized according to the size of the original image to carry out the image reconstruction by CSRBF. A compactly supported radial basis function $\phi(r)$ of the form

$$\phi(r) = \begin{cases} (1 - \frac{r}{r_0})^2 & r < r_0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

is used, where $r = \|\mathbf{x}_i - \mathbf{x}_j\|$ and r_0 is the global support radius.

With the normalized set of scattered points we build the matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{\Phi} & \mathbf{P} \\ \mathbf{P}^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \tag{5}$$

of the system (3) and red, green and blue colors at each sampled point are interpolated solving the system

$$\mathbf{A}\boldsymbol{\chi} = \mathbf{b} \tag{6}$$

where $\boldsymbol{\chi} = (\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{c}^{\mathrm{T}})^{\mathrm{T}}$ is the solution of the SLAE and $\mathbf{b} = (\mathbf{f}^{\mathrm{T}}\mathbf{0}^{\mathrm{T}})^{\mathrm{T}}$ has the color values to be interpolated padded with zeros. The polynomial degree $l = 3$ of $p(\mathbf{x})$ with monomial basis $\{1, x, y\}$ is used in section 4.

The solution of the system (6) by direct methods [12] is computationally expensive and rapidly becomes impractical as N, the number of points to be interpolated, becomes larger than a few thousands. The main objective of this work is to improve the solution of the system (6) by preconditioned iterative methods. Further, when r_0, the support radius, increase and the system becomes dense our iterative technique is extended to the wavelet domain.

3. WAVELET DOMAIN SOLUTION

3.1. Background

Discrete wavelet transform (DWT) approximation have been used successfully to precondition linear systems that come from elliptic PDE and BEM problems [13, 14, 15, 16]. The advantage of applying wavelet technique is to increase the number of zeros in the matrix $\mathbf{A}$ so that the system can be solved more efficiently by iterative technique.

Variants of algebraic preconditioners developed in [13] may not work at all for the simple reason that the matrix of CSRBF SLAE has zeros in the diagonal. If the irregularity of the original matrix is limited to a relatively small known set of rows or columns, the acceleration of the iterative process can be achieved by a mixed approach in which only the smooth submatrix is transformed [15]. Wavelet compression techniques can also be combined with Kronecker product approximation [16]. The recent work in [14] has attempted to develop a direct solution method based on the non-standard form (NS-form) that requires a careful choice of a threshold. None of these methods have been applied to the problem of image reconstruction by CSRBF.

3.2. Preconditioning

Given an orthogonal wavelet function in the continuous space, there exists an orthogonal matrix $\mathbf{W}$ that transforms vectors from the standard basis to the wavelet basis. If the vector $\mathbf{b}$ of (6) has smoothly varying values (with possibly local singularities), its wavelet representation is $\widetilde{\mathbf{b}} = \mathbf{Wb}$. We can also represent two dimension transforms of the matrix $\mathbf{A}$ in the standard basis by $\mathbf{W}$ as $\widetilde{\mathbf{A}} = \mathbf{WAW}^{\mathrm{T}}$. Then, to solve SLAE in wavelet domain we change equation (6) to

$$\mathbf{WAW}^{\mathrm{T}}\mathbf{W}\boldsymbol{\chi} = \mathbf{Wb}, \tag{7}$$

or

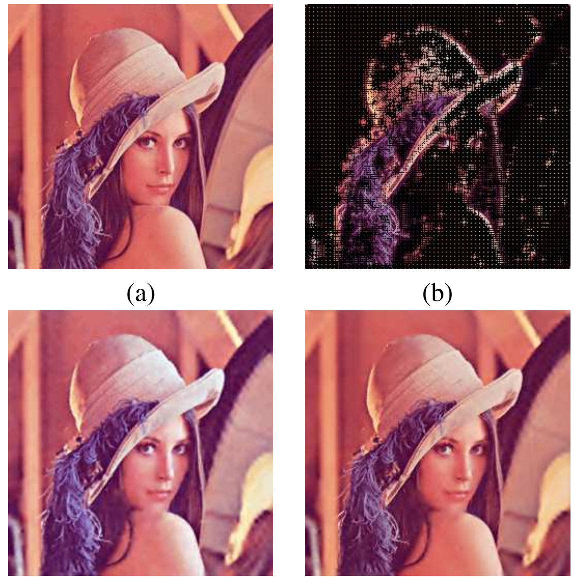
$$\widetilde{\mathbf{A}}\widetilde{\mathbf{x}} = \widetilde{\mathbf{b}}, \tag{8}$$

where $\widetilde{\mathbf{x}} = \mathbf{W}\boldsymbol{\chi}$ is the wavelet representation of the solution. After solving the system (8) by iterative preconditioning technique from [17] the solution of SLAE is obtained by inverse DWT. Daubachies 4 wavelet filters are used in the experiments with the lifting scheme implementation. Numerical simulations show better results with Incomplete LU factorization preconditioner (ILU) and generalized minimum residuals (GMRES) iterative method in the wavelet domain (DWT).

4. NUMERICAL SIMULATIONS

A PC Pentium 4 CPU with 1300MHz, 768MB RAM and Microsoft Visual C++ 6.0 was used for numerical simulations. Figure 1 shows the superiority of CSRBF over Multilevel B-spline Approximation (MBA) [7] for Lena reconstruction from scattered data. Sampled points are shown in Figure 1b. Reconstructed images using MBA and CSRBF are shown in figures 1c and 1d respectively with their Peak Signal to Noise Ratio (PSNR).

In Figure 2, results with ILU-GMRES is shown in comparison with algorithm in [12] (DIRECT) for standard domain. Value of global support radius is expressed in parenthesis. The number of zeros in the matrix $\mathbf{A}$ of (5) is between 90% ($r_0 = 0.2$) and 99% ($r_0 = 0.05$). The higher the radius the denser the system. As same PSNR are obtained for $r_0 = 0.13$ and $r_0 = 0.2$ with Lena reconstruction, the value of $r_0 = 0.13$ is enough. Time reduction is obtained with $r_0 = 0.05$ and $r_0 = 0.13$ with ILU-GMRES but for $r_0 = 0.2$ the system is denser and ILU preconditioner in the wavelet domain (ILU-DWT-GMRES) is needed to outperform DIRECT method.



(a) (b) (c) (d)

Fig. 1. Lena reconstruction: (a)Original (256 X 256), (b) 15148 Sampled points (c) MBA (PSNR = 26.08 [dB]) (d) CSRBF (PSNR = 32.04 [dB])

5. CONCLUSIONS

We have proposed the use of wavelets to accelerate the solution of the SLAE that arise from the formulation of the problem of image interpolation from scattered data by means of Compactly-Supported Radial Basis Functions. Examples demonstrate the superiority of the solution in the wavelet domain using preconditioned iterative methods.

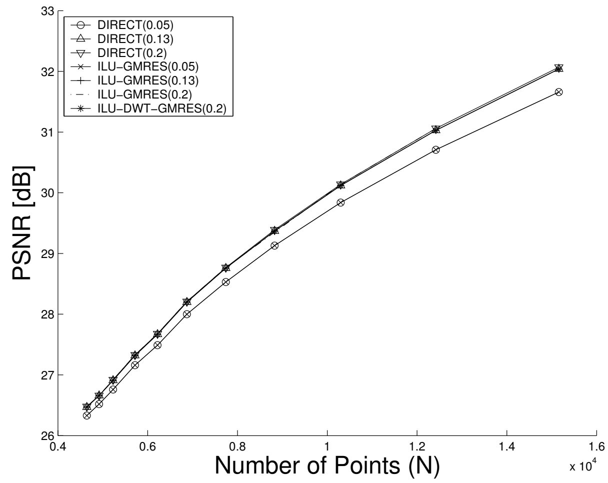
As the matrix of the SLAE is symmetric, the efficiency of the method can be improved with the use of minimal residuals method (MINRES) and incomplete Cholesky factorization (IC) preconditioner [18].

6. REFERENCES

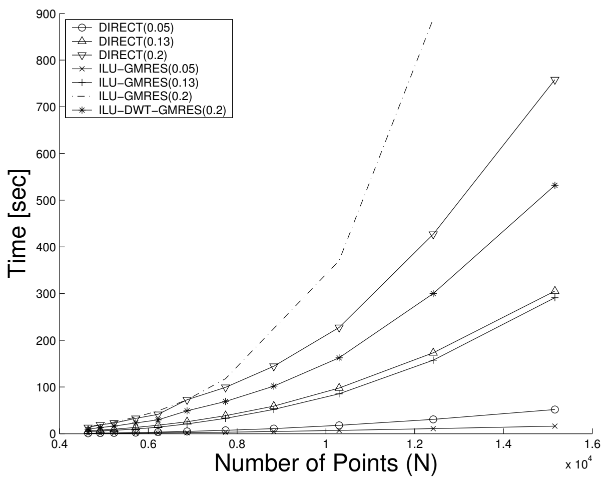
[1] V. Savchenko, A. Pasko, O. G. Okunev, and T. L. Kunii, "Function representation of solids reconstructed from scattered surface points and contours," *Computer Graphics Forum*, vol. 14, no. 4, pp. 181–188, May 1995.

[2] H. Wendland, "Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree," *Advances in Computational Mathematics*, vol. 4, pp. 389–396, 1995.

[3] R. K. Beatson, J. B. Cherrie, and C. T. Mouat, "Fast fitting of radial basis functions: Methods based on preconditioned GMRES iteration," *Advances in Computational Mathematics*, vol. 11, pp. 253–270, 1999.



(a) PSNR



(b) Time

Fig. 2. Lena reconstruction in standard and wavelet domain

[4] J. C. Carr, R. K. Beatson, J. B. Cherrie, T. J. Mitchell, W. R. Fright, B. C. McCallum, and T. R. Evans, "Reconstruction and representation of 3D objects with radial basis functions," in *Computer Graphics*. ACM SIGGRAPH, May 2001, pp. 67–76.

[5] Y. Ohtake, A. Belyaev, M. Alexa, G. Turk, and H. Seidel, "Multi-level partition of unity implicits," in *Computer Graphics*. ACM SIGGRAPH, 2003, pp. 27–31.

[6] T. Strohmer, "Computationally attractive reconstruction of bandlimited images from irregular samples," *IEEE. Trans. on Image Processing*, vol. 6, no. 4, pp. 540–548, 1997.

[7] S. Lee, G. Wolberg, and S. Y. Shin, "Scattered data interpolation with multilevel B-splines," *IEEE. Trans. Visualization and Computer Graphics*, vol. 3, no. 3, pp. 1–17, 1997.

[8] C. Vázquez, E. Dubois, and J. Konrad, "Reconstruction of irregularly-sampled images by regularization in spline spaces," in *Proceedings of the 2002 IEEE International Conference on Image Processing (ICIP'02)*, 2002, vol. III, pp. 405–408.

[9] M. Arigovindan, M. Sühling, P. Hunziker, and M. Unser, "Multigrid image reconstruction from arbitrarily spaced samples," in *Proceedings of the 2002 IEEE International Conference on Image Processing (ICIP'02)*, 2002, vol. III, pp. 381–384.

[10] K. Ichige, T. Blu, and M. Unser, "Multiwavelet-like bases for high quality image interpolation," in *SPIE Conference on Mathematical Imaging: Wavelet Applications in Signal and Image Processing X*, August 2003.

[11] W. Sweldens, "The lifting scheme: A construction of second generation wavelets," *SIAM J. Math. Appl.*, vol. 29, no. 2, pp. 511–546, 1998.

[12] N. Kojekine, I. Hagiwara, and V. Savchenko, "Software tools using CSRBFs for processing scattered data," *Computers & Graphics*, vol. 27, no. 2, pp. 311–319, 2003.

[13] T. F. Chan and K. Chen, "On Two Variants of an Algebraic Wavelet Preconditioner," Research Report 02-02, University of California, Group in Computational and Applied Mathematics, 2002.

[14] D. Gines, G. Beylkin, and J. Dunn, "LU factorization of non-standard forms and direct multiresolution solvers," *Applied and Computational Harmonic Analysis*, vol. 5, pp. 156–201, 1998.

[15] P. Gonzalez, J. C. Cabaleiro, and T. F. Pena, "Iterative solution of large linear systems with non-smooth submatrices using partial wavelet transforms and split-matrix matrix-vector multiplication," *Int. J. Numer. Meth. Eng.*, pp. 1–27, 2002.

[16] J. M. Ford and E. E. Tyrtyshnikov, "Solving linear systems using wavelet compression combined with Kronecker product approximation," Numerical Algorithms (submitted), 2003.

[17] J. Dongarra, A. Lumsdaine, R. Pozo, and K. Remington, "A sparse matrix library in c++ for high performance architectures," in *Second Object Oriented Numerics Conference*, 1992, pp. 214–218, See also URL `http://math.nist.gov/iml++`.

[18] Y. Saad, *Iterative Methods for Sparse Linear Systems*, PWS publishing, New York, 1996.