OPTIMAL MULTIRESOLUTION POLYGONAL APPROXIMATION

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ABSTRACT

We propose optimal and near-optimal algorithm for multiresolution polygonal approximation of digital curves. The solution with minimum number of segments is constructed as the shortest path in a weighted graph where the weights are recursively defined as the number of segments of all embedded layers.

1 INTRODUCTION

Optimal approximation of a digital curve with minimum number of segments can be solved as a shortest path in the graph constructed on the vertices of the curve [1,2] or dynamic programming approach [3,4].

Now consider the problem of multiresolution polygonal approximation of digital curves (Fig. 1). The multiresolution polygonal approximation is used for progressive encoding and transmission of digital shapes, for scalable representation and coding of vector maps [5-11]. Solutions to the problem can be obtained by heuristic algorithms [5-10].

An optimal algorithm was introduced in [11] for multiresolution shape approximation with minimum distortion for the given constraint on the rates. The solution is found for the *binary* partition tree: the next resolution level is generated by possibly inserting a new approximation point between every pair of consecutive points in the current layer.

We introduce optimal algorithm for multiresolution polygonal approximation without any constraints on the type of partition tree. The number of parts in a segment for the next resolution level depends on the local properties of the curve and approximation error bounds for the level. The recursive algorithm is based on searching for the shortest paths in the weighted graphs, where cost of some path in the graph is total number of approximation segments for all resolution levels. A fast near-optimal algorithm based on this approach is also considered.

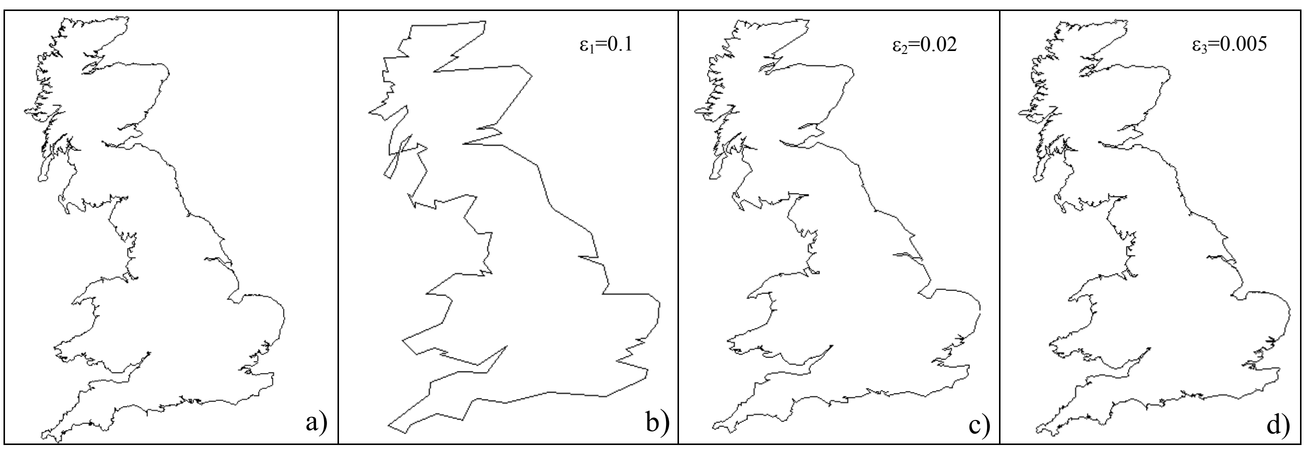

Figure 1. Optimal multiresolution approximation: a) the original test shape "Britain": N=10,910; b) result for level r=1, the error tolerance ε_1=0.1; the corresponding number of segments is M_1=117; c) result for level r=2; ε_2=0.02; M_2=625; d) result for level r=3: ε_3=0.005; M_3=2477. The total numbers of segments is M=3219. The number of levels is R=3.

2. PROBLEM FORMULATION

The sequence of polygonal curves $\{Q_1, Q_2, \ldots, Q_R\}$ is called multiresolution approximation of the N-vertex input curve P, if the set of curves Q_r satisfies two conditions:

Condition 1: A polygonal curve Q_r is an approximation of the curve P for the error tolerance ε_r, where r is the number of resolution level: $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_R$. Set of vertices of curves $\{Q_1, Q_2, \ldots, Q_R\}$ is subset of vertices of the input curve P.

Condition 2: Set of vertices of curve Q_r is subset of vertices of curve Q_{r+1}.

Now we formulate the multiresolution *min-# problem*: a given polygonal curve P, approximate it by a sequence of multiresolution polygonal curves $\{Q_1, Q_2, \ldots, Q_R\}$ with the minimum total number of segments M so that the approximation error with measure L_∞ for all levels does not exceed the corresponding error tolerance ε_r:

$$M = \min_{\Psi} \{\sum_{r=1}^{R} M_r\},$$

subject to $\{d_{\max}(P, Q_r) \le \varepsilon_r : r = 1, 2, ..., R\}$,

where $\Psi=\Psi(Q_1, Q_2, \ldots, Q_R)$ is the partition tree for the scalable approximation of the curve P; M_r is the number of segments of the curve Q_r, and $d_{\max}(P, Q_r)$ is approximation error (maximum deviation) for level r.

3. KNOWN SOLUTIONS

Solutions to the problem can be obtained by heuristic algorithms. In the *Split* [12] approach, an iterative procedure splits the input curve into smaller and smaller segments until the maximum deviation is smaller than a given error tolerance ε_r. The algorithm has been used for multiresolution approximation in [6-10].

In the *Merge* approach [13, 14], the approximation is performed by sequential elimination of the vertices with the smallest cost function value (deviation), and the two adjacent segments are merged into one segment. The approximation curve Q_r is obtained by eliminating vertices from the curve Q_{r+1} until the maximum deviation is smaller than the given error bound ε_r.

The quality of heuristic algorithm for *min-# problem* is measured by *efficiency*: $E=(M_r^{(opt)}/M_r)\times 100\%$, where M is the number of segments for algorithm under study; the segments number $M^{(opt)}$ is defined by optimal approximation of the curve P [15].

In [11] was introduced algorithm for multiresolution shape approximation with minimum distortion for the given constraint on the rates. The solution is found for the *binary* partition tree: the next resolution level is generated by possibly inserting a new approximation point between every pair of consecutive points in the current layer. This approach is justified for convex/concave curves. In the general case, when curve is oscillating along the approximation line, there are no restrictions on the number of parts to be generated for the next embedded layer, no restrictions on the type of partition tree.

```
(M) = MR_min_num(P:curve, r,n1,n2:integer)
FOR n = n1  TO  n2  DO
   C(n)←∞
   FOR j = n-i TO n1  DO
      IF e(j,n) ∈ G(P,εr) THEN
        IF r = R  THEN
           Weight←1;
        ELSE
           Weight←1+MR_min_num(P,r+1,j,n);
        ENDIF
        IF C(j) + Weight < C(n) THEN
           C(n)←C(j) + Weight;
           A(n)←j;
        ENDIF
      ENDIF
   ENDFOR
ENDFOR
M←C(n2)
RETURN M
```

Figure 2. Recursive algorithm for the shortest paths in the weighted graph $G(P,\varepsilon)$ with the introduced recursive cost function C.

4. OPTIMAL ALGORITHM

Optimal single-resolution approximation of curve with minimum numbers of segments can be constructed as follows: (a) for the curve P construct a feasibility graph $G=G(P, \varepsilon)$ with the given error tolerance ε; (b) find the shortest path in the weighted graph G. The complexity of the algorithm is $O(N^2)$ [2].

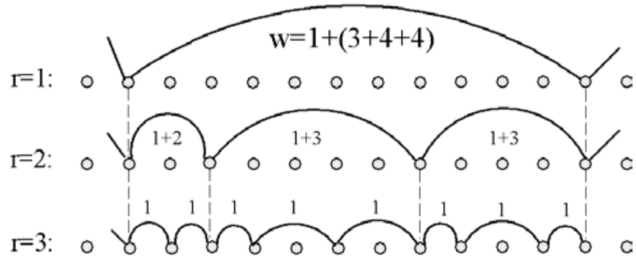

Figure 3. Weight w of edge $e(i,j)$ at a level r is the number of approximation segments for the segment (i,j) including those for embedded levels r+1, r+2, *etc*.

The algorithm for a single-resolution approximation can be generalized to the multiresolution approximation. The solution is found as the shortest path in the weighted graph $G_1=G(P, \varepsilon_1)$. The main idea of the algorithm is to include the number of segments for the embedded levels $r+1, r+2, \ldots, R$ into the weight of the edge for level r (Fig. 3). The weight of an edge for the last level R is equal to one: $w=1$.

So, the cost function C of a path in graph G_1 is the total number of segments in approximation curves $Q_1, Q_2, \ldots, Q_R$. The shortest path in the weighted graph G_1 gives us solution of multiresolution approximation problem with the minimum number of segments. Pseudo-code of the recursive algorithm for the shortest path with introduced cost function C is presented in Fig. 2.

The optimal path for the level $r=1$ can be restored using an array A of the parent nodes. To be able to restore optimal paths for other levels $r \geq 2$ we have to keep data for all segments (i, j) of the curve P involved in the processing at all levels. This results in $O(RN^2)$ space complexity.

We reduce the space demands at the negligible additional processing time. At first, we perform recursive multiresolution approximation of P for the levels $[1, R]$, and restore optimal approximation Q_1 only for the 1st level. The obtained approximation Q_1 defines the partition Ψ_1 of the curve P for the level $r=1$, which cannot be changed at the level $r=2$ because of condition 2. So, we can apply the recursive algorithm to the segments of P individually for the levels $[2, R]$, and restore approximation Q_2 which gives us partition Ψ_2 of P. The procedure is repeated until all the levels are processed. The main source of computation originates from processing for the levels $[1, R]$, because of full depth of recursion

Solution of single-resolution *min-# problem* with the error measure L_∞ can be obtained with the algorithm for the shortest path in graph G in $O(N^2)$ time if the approximation errors are calculated in advance and stored in 2-D array of $O(N^2)$ size. Using of the precalculated data allows us to reduce the time complexity of the multiresolution algorithm at the cost of $O(N^2)$ space. To reduce time and space demands we perform the preliminary single-resolution approximation for $r = 1$ to fill the 2-D table of approximation errors only for the segments involved to the processing.

5. FAST NEAR-OPTIMAL-ALGORITHM

The main drawback of the introduced algorithm for multiresolution polygonal approximation is its high time complexity. A faster near-optimal algorithm can be constructed on the basis of the optimal algorithm as follows.

The main idea is to divide the R levels $L=[1, R]$ by some level p into two parts: $L_1=[1, p-1]$ and $L_2=[p, R]$. At the first step, we perform approximation by the recursive algorithm for levels $r \in L_2$. The processing time for the levels $r \in L_2$ is much smaller than for the whole range $L=[1, R]$ because of a smaller number of levels to be recursively processed, and a smaller size of segments to be approximated.

At the second step, we perform approximation for the levels L_1 with the optimal algorithm using a set of vertices of Q_p as admissible nodes for the graph $G_1=G(Q_p, \varepsilon_1)$. The feasibility graph G_1 is constructed on the vertices of Q_p instead of the input curve P to satisfy the Condition 2 for multiresolution approximation curves $Q_1, Q_2, \ldots, Q_p$. Processing time for the levels L_1 is also smaller in comparison with those for levels L for two reasons: the depth of recursion is smaller than for L, and search space in the feasibility graph $G(Q_p, \varepsilon_1)$ is reduced in comparison with graph $G(P, \varepsilon_1)$ for the original algorithm.

6. RESULTS AND DISCISSIONS

To test the optimal algorithm for multiresolution *min-# problem*, the 10,910-vertex test shape “Britain” is approximated for three resolution levels ($R = 3$) with the following error tolerances: $\varepsilon_r = 0.1$, 0.02, and 0.005 (Fig. 1). The number of segments for different resolution levels and the total number of segments are represented in Tab. 1. The total processing time is about 1 hour. Calculation of 2-D table of approximation errors with preliminary single-resolution approximation takes about 3 seconds. The results for optimal single-resolution approximation are provided to estimate how does the condition 2 affect on the total number of segments.

Efficiency of the solution obtained with heuristic algorithms for the test shape "Britain" is E=75%. With fast near-optimal algorithm based on the introduced optimal algorithm the processing time was reduced from 1 hour to T=4 seconds with efficiency E=99.7%.

7. CONCLUSIONS

The optimal algorithm for multiresolution *min-#* polygonal approximation of digital curves is developed. Solution is found recursively as the shortest graph where weights are defined as summa of line segments for all embedded resolution levels. The fast near-optimal algorithm is considered to reduce processing time for large input.

Table 1. The number of segments, efficiency and processing time for the 10,910-vertex test shape "Britain" for heuristic methods (Merge and Split), optimal single-resolution (SR) and multiresolution (MR) algorithms, and fast near-optimal multiresolution algorithm with *p*=2; *R*=3.

	ε_r	**Optimal SR**	**Merge-based**	**Split-based**	**Optimal MR**	**Near-optimal MR**
r=1	0.1	104	168	202	117	132
r=2	0.02	575	855	882	625	622
r=3	0.005	2456	3263	3206	2477	2476
M		3135	4286	4290	3219	3230
Efficiency		––	75.1%	75.0%	100%	99.7%
Time (s)		3.4	23.6	3.2	3540.0	4.1

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