GEOMETRIC CONSTRUCTION OF THE CAUSTIC CURVES FOR CATADIOPTRIC SENSORS

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ABSTRACT

Most of the catadioptric cameras rely on the single viewpoint constraint that is hardly fullfilled. There exists many works on non single viewpoint catadioptric sensors satisfying specific resolutions. In such configurations, the computation of the caustic curve becomes essential if precision and flexibility are aimed. Existing solutions are unfortunately too specific to a class of curves and need heavy precomputations. This paper presents a flexible geometric construction of the caustic surface of a catadioptric sensor that can be applied to the plane curves and under a tighter assumption, to space surfaces. The method holds for smooth mirror shapes that are surfaces of revolution. Tests and experimental results substantiate the possibilities of the approach.

1. INTRODUCTION

The caustic curves are an optical phenomenon studied since Huygens and Hamilton. They are the envelope of the reflected or diffracted light. Most of the existing vision systems are designed in order to achieve the convergence of the incident rays of light at a single point called 'effective viewpoint'.

Such a configuration of sensors can be seen as a degenerated form of the caustic reduced to a single point. The catadioptric sensors are divided into two categories, the ones fullfilling the single viewpoint constraint (SVC) where the caustic is reduced to a point and the none SVC that need the computation of the caustic. The single viewpoint constraint [1] provides easier geometric systems and allows the generation of correct perspective images. However it requires a high precision assembly of the devices that is hardly fullfilled practically [2] and deals with the problem of uniformity of resolution. The problem of designing a catadioptric sensor that results in improved uniformity of resolution compared to the conventional sensor has been studied by several approaches [3, 4]. These solutions rely on the resolution of differential equations that are in most cases resolved numerically providing a set of sampled points.

There are many ways to compute the caustic of a smooth curve, generally they are too specific to a finite class of curves and/or to a particular position of the light source [5]. More complex methods based on the illumination computation (the flux-flow model) are studied in geometric optics [6]. They are highly applied in computer graphics when a realistic scene rendering is required [7]. A method for determining the locus of the caustic is derived from this flux-flow model and a work of analysis and tests are carried out on conic shape mirrors in [8], in order to extract the optical properties of the sensor.

In this paper we introduce a method allowing the computation of the caustic based on a simple geometric constructions as related in [9]. The interest of the approach is its moderate computational load and its great flexibility toward unspecified smooth curves. We will see that its extension to the third dimension is feasible if surfaces have an axis of revolution. In such a case we show that the problem can be handled as a planar one in the incident plane. Finally experimental results carried out on a non explicit equation mirror are presented.

2. THE CAUSTIC CURVE: DEFINITION AND CONSTRUCTION

The catadioptric sensor that does not comply with the single viewpoint constraint, require the knowledge of the caustic surface if one expects to calibrate it. Defined as the envelope of the reflected rays, the caustic curve gives the direction of any incident ray captured by the camera.

In this section, we present in detail two methods applied to

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the caustic curve computation for systems combining mirror and linear camera. The first method derives from the flux-flow computation detailed in [6]. Swaminathan *et al.* used this technique on conical based catadioptric sensors. A detailed analysis and relevant results are obtained.
The second method is based only on geometrical properties of the mirror curve. Caustic surface point is determined by approximating locally the curve by a conic where both the light source and the caustic point are foci of this conic.

2.1. Jacobian method

The vanishing constraint on the Jacobian is applied to the whole class of conical mirrors in [8], though it can be applied for any regular curves.
We define $\mathbf{N}$, $\mathbf{V_i}$ and $\mathbf{V_r}$ as respectively the normal, the incident and the reflected unit vectors, at the point P of the mirror M. Those three vectors are functions of the point P, then if M is parametrised by t, they are functions of t. depend on the parameter t.
According to the reflexion laws, we have:

$$\mathbf{V_r} - \mathbf{V_i} = 2(\mathbf{V_r}.\mathbf{N})\mathbf{N} \Rightarrow \mathbf{V_i} = \mathbf{V_r} - 2(\mathbf{V_r}.\mathbf{N})\mathbf{N}$$

Assuming the point P to be the point of reflection on M, if we note P_c the associated caustic point then P_c satisfies:

$$P_c = P + r\mathbf{V_i}$$

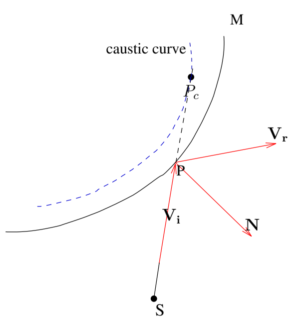


Fig. 1. Catacaustic or caustic by reflexion: the dashed curve shows the locus of the reflected rays envelope.

r is a parameter and since P and $\mathbf{V_i}$ depend on t, P_c is a function of (t, r). We can write the jacobian of P_c as:

$$J(P_c) = \begin{vmatrix} \frac{\partial P_z}{\partial t} & \frac{\partial P_z}{\partial r} \\ \frac{\partial P_\gamma}{\partial t} & \frac{\partial P_\gamma}{\partial r} \end{vmatrix} = \begin{vmatrix} \dot{P}_z + r\dot{V}_{iz} & V_{iz} \\ \dot{P}_\gamma + r\dot{V}_{i\gamma} & V_{i\gamma} \end{vmatrix} \quad (1)$$

$J(P_c)$ must vanish, thus we solve the equation $J(P_c) = 0$ for r:

$$r = \frac{\dot{P}_z V_{i\gamma} - \dot{P}_\gamma V_{iz}}{V_{iz}\dot{V}_{i\gamma} - V_{i\gamma}\dot{V}_{iz}} \quad (2)$$

r is a function of t, then we can have the parametrised form of the caustic.
An in-depth studies have been carried out on the class of conic curves in [8]. The same method is extended to three dimensional curves. M is then a smooth surface and its caustic surface relatively to the light source can be determined by solving a three by three matrix determinant.
This method is adapted to cases where mirrors profile's equation is known and requires cumbersome calculus especially if the mirror is complex, plus no analytical solution is guaranteed. The most generic case is when the reflective surface is defined only by a set of sampled points. Then numerical computations are needed and the vanishing condition on the Jacobian can be hard to satisfy. Hence, we present here another technique of the caustic curves computation, where only local mathematical properties of M are taken into account.
The caustic of a curve M, is function of the source light S. The basic idea is to consider a conic where S is placed on one of its foci. A simple physical consideration shows that any ray emitted by S should converge on the other focus F. It is proved in [9] that for any P on M, there is only one caustic with properties mentioned above so that F is the caustic point of M, relative to S, at P.

2.2. Geometrical construction

definition 1:
Considering a regular(i.e smooth) curve M, a light source S and a point P of M, we construct Q as the symetric of P, relative to the tangent to M at P. The line (QP) is the reflected ray.(see Fig. 2)
When P describes M, Q describes a curve W where (QP)

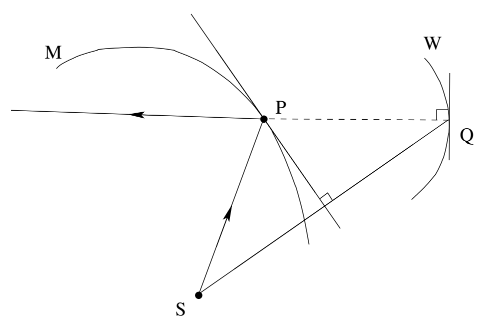


Fig. 2. Illustration of the orthotomic W of the curve M, relative to S. W is interpreted as the wavefront of the reflected wave.

is normal to it at Q. W is known as the orthotomic of M, relatively to S. The physical interpretation of W is the wavefront of the reflected wave.
It is equivalent to define the caustic curve C of M, relative

to S as the evolute of W (i.e. the locus of its centers of curvature) or the envelope of the reflected rays.

definition 2: Given two regular curves f and g of class C^n, with a common tangent at a common point P, taken as $[0\,0]^t$ and the abscissa axis as tangent. Then this point is an n-order point of contact if:

$$\begin{cases} f^{(k)}(0) = g^{(k)}(0) = 0 & \text{if} \quad 0 \leq k < 2 \\ f^{(k)}(0) = g^{(k)}(0) & \text{if} \quad 2 \leq k \leq n-1 \\ f^{(n)}(0) \neq g^{(n)}(0) & \end{cases}$$

There is only one conic $\mathcal{C}$ of at least a 3-point contact with M at P, where S and F are the foci. F is the caustic point of M at P, with respect to S.

For the smooth curve M, we considere its cartesian equation:

$$M : y = f(x) \tag{3}$$

where $P = [x\ y]^t \in M$. The curvature of a M at P is given by:

$$k = \frac{f''(x)}{\sqrt{1 + f'(x)^2}^3} \tag{4}$$

With regard to these definitions, we can deduce that M and $\mathcal{C}$ have the same curvature at P. If k is known, we are able to build the caustic C independendly of W.

For more details and proofs of these affirmations, reader should refer to [9]. We give here the geometrical construction of the focus F, with respect to the conic $\mathcal{C}$ complying with the properties decribed above. Figure 3 illustrates the geometrical construction detailed below.

- Compute O, center of curvature of M at P, according to $r = \frac{1}{k}$, radius of curvature at P.
- Project orthogonaly O to (SP) at u.
  Project orthogonaly u to (PO) at v.
  (Sv) is the principal axis of C.
- Place F on (Sv) so that (OP) is bisectrix of $\widehat{SPF}$.

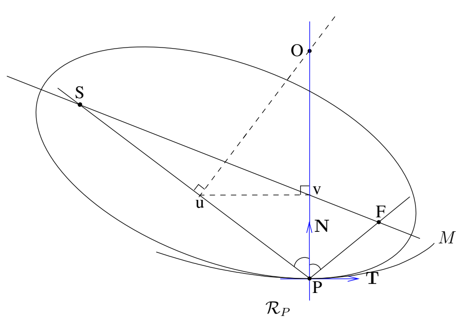


Fig. 3. Geometric construction of the caustic point expressed in the local Frenet's coordinate system $\mathcal{R}_\mathcal{P}$

Depending of k, $\mathcal{C}$ can be an ellipse, an hyperbola or a parabola. For the first two, F is at finite distance of S ($\mathcal{C}$ has a center) and $k \neq 0$. If $k = 0$, F is at infinity and $\mathcal{C}$ is a parabola.

We consider here only the case where S is at a finite distance from M. If S is placed at infinity or projected through an telecentric camera, the incident rays are parallel and the definition of the caustic is slightly different. We can write down the coordinates of F in the local Frenet's coordinate system R_p if we express analytically the cartesian equation of each line of Fig (3).

$$F = \begin{pmatrix} -\frac{y_s^2 x_s |r|}{2y_s(x_s^2+y_s^2) - y_s^2|r|} \\ \\ \frac{y_s^2|r|}{2(x_s^2+y_s^2) - y_s|r|} \end{pmatrix} \tag{5}$$

The generic expression of the coordinates of F depends only on the source S and the curve M through r.

2.3. Extension to the third dimension

According to the Snell's law of reflexion, the incident and the reflected rays are coplanar and define the plane of incidence Π_P. Since the caustic point associated to S and the point P belongs to (PQ), one can assume to apply the geometric construction to M if M is the intersection of the three dimensions surface $\mathcal{M}$ and Π_P.

Unfortunately, the normal $\mathbf{N}$ to $\mathcal{M}$ at P is not necessarily the normal $\mathbf{n}$ to M at P. However, reader can easily prove that if $\mathcal{M}$ has an axis of revolution such that S is placed on it, then **N=n** and the geometric construction can be applied to planes.

3. EXPERIMENTAL RESULTS

The geometrical construction is illustrated with a smooth curve example. As we can see in section 2.2, only Eq. (4) is specific to M, the curvature at P implies only the first two derivatives at P. Hence, if the profile of M is given only as a set of sampled points, the algorithm can handle it if the sampling step is small enough. We consider the most general case we have to face: the reflector is given only by a set of sampled points, no explicit equation is known. The curvature at each point is numerically computed, providing a numerical estimation of the caustic.

The mirror tested has a symmetric axis and the camera is placed arbitrarily on it. We computed the caustic curve relative to this configuration (see Fig. 4). The catadioptric sensor has been fully calibrated by combining the method similar to [2] and the caustic curve. Given a set of points taken from a scene captured by this sensor, we reproject the rays on the floor in order to justify our method of construction.

As we can see in Fig. 5, the geometry of the calibration pattern is acurately reconstructed retrieving the actual metric (the tiles on the floor are squares of 30x30cm). The reconstruction shows that the farer we are from the center of the optical axis, the less acurate we are which is an expected result as the mirror was not computed to fullfill this property.

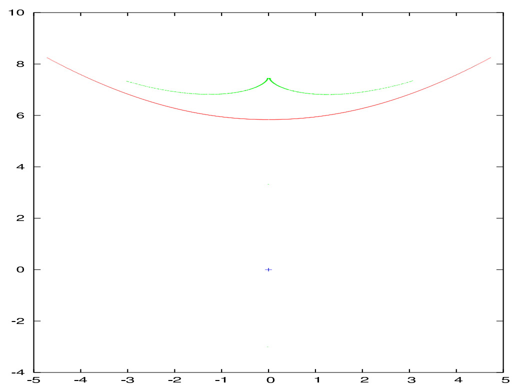
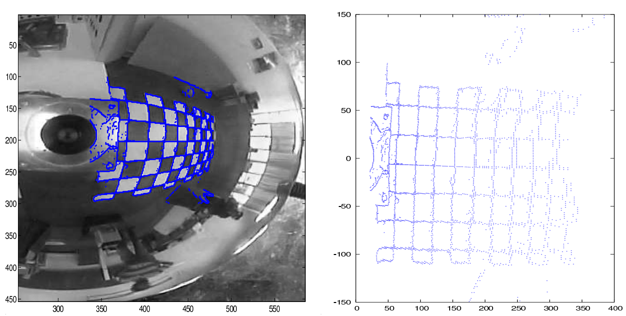


Fig. 4. Caustic curve of the sampled mirror. The camera are placed on the origin of the coordinate system, represented by the cross.



(a) I (b) II

Fig. 5. Scene captured by the sensor. The blue dots are scene points that are reprojected on the floor, illustrated on the left plot.

4. CONCLUSION

This paper presented a geometric construction of caustic curves in the framework of catadioptric cameras. When the single viewpoint constraint cannot be fullfilled, the caustic becomes essential if calibration and reconstruction are needed.
Existing methods imply heavy preprocessing work that can lead to an exact solution of the caustic if the mirror profile is known, however this is not guaranteed. At the contrary, the presented geometric construction is more flexible as it relies its computation only on local properties of the mirror. However, unless the mirror has an axis of revolution with S on it, this construction cannot be extended to 3D shapes. In such a case, alternative methods should be applied: computing the causic directly from the orthtotomic W for example.

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5. REFERENCES

[1] S. Baker and S. K. Nayar, "A theory of catadioptric image formation," in *Proceedings of the ICCV*, 1198, pp. 35–42.

[2] J. Fabrizzio, P. Tarel, and R. Benosman, "Calibration of panoramic catadioptric sensors made easier," in *Proceedings of the IEEE Workshop on Omnidirectionnal Vision*, 2002.

[3] H. Backstein and T. Padjla, "on central cameras: A review," in *Proceedings of Computer Vision Winter Workshop, Ljubljana, Slovenian Pattern Recognition Society*, 2001, pp. 223–233.

[4] J. Conroy and J. Moore, "Resolution invariant surfaces for panoramic vision systems," in *Proceedings of ICCV*, 1999, pp. 392–397.

[5] C. Bellver-Cebreros, E.Gómez-González, and M. Rodríguez-Danta, "Obtention of meridian caustics and catacaustics by means of stigmatic approximating surfaces," *Pure Applied Optic*, vol. 3, pp. 7–16, 1994.

[6] D.G. Burkhard and D.L. Shealy, "Flux density for ray propagation in geometric optics," *Journal of the Optical Society of America*, vol. 63, no. 3, pp. 299–304, Mar. 1973.

[7] D. Mitchell and P. Hanrahan, "Illumination from curved reflectors," in *Proceedings of the SIGGRAPH'92*, July 1992.

[8] R. Swaminathan, M.D. Grossberg, and S.K. Nayar, "Caustics of catadioptric cameras," in *Proceedings of the IEEE International Conference on Computer Vision*, July 2001.

[9] J.W. Bruce, P.J. Giblin, and C.G. Gibson, "On caustics of plane curves," *American Mathematical Monthly*, vol. 88, pp. 651–667, Nov. 1981.